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A VARIATIONAL CONVERGENCE PROBLEM WITH ANTIPERIODIC BOUNDARY CONDITIONS

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Abstract: We present a variational convergence approach involving existence of solutions for some classes of evolution inclusions with anti-periodic boundary conditions.

Key words: *anti-periodic, chain rule, maximal monotone, recession, second dual, subdifferential, Young measures*

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1 Introduction

Existence and uniqueness of antiperiodic solution for evolution inclusions generated by the subdifferential of a convex lower semicontinuous even function appeared in a series of papers, see [1, 2, 3, 14, 15, 18, 21, 22] and the references therein. In this paper, we present two epigraphical versions of the mentioned results involving new variational convergence techniques and the stable convergence of Young measures [10]. In section 2, we summarize some basic results of convergence for bounded sequences in $L^1_H([0,T])$ where H is a Hilbert space. In section 3 we state some existence and uniqueness results of anti-periodic solutions for a first order evolution inclusion generated by a subdifferential of a convex lower semicontinuous even function defined on H and its application to a new existence of antiperiodic solutions. Section 4 is devoted to the existence of anti-periodic solutions for a second order evolution inclusion via a variational approach [11, 12] involving the biting convergence, Young measures and the characterization of the second dual of $L^1_H([0,T])$ and other tools.

2 Preliminaries and Background

We introduce some basic notions and results. In this paper, H is a separable Hilbert space. By $L^1_H([0,T])$ we denote the space of all Lebesgue-Bochner integrable H-valued functions defined on $[0,T]$. A sequence (φ_n) of lower semicontinuous functions defined on H *lower epiconverges* to a lower semicontinuous function φ_∞ defined on H if, for every sequence (x_n) in H converging to x, we have $\liminf_n \varphi_n(x_n) \geq \varphi_\infty(x)$. (φ_n) *upper epiconverges* to φ_∞ if, for every $y \in H$, there exists a sequence $(y_n)_n$ in H converging to y such that $\limsup_n \varphi_n(y_n) \leq \varphi_\infty(y)$. If (φ_n) both lower and upper epiconverges to φ_∞, we say that (φ_n) *epiconverges* to φ_∞. These notions are easily extended to normal integrands (see e.g. [13, 23]).

The following result is a particular form of a similar result given in ([11], Proposition 4.1).

Lemma 2.1. *Let H be a Hilbert space. Let φ be a proper convex lower semicontinuous function defined on H with values in $]-\infty, +\infty]$. Let $(u_n)_{n\in\mathbf{N}\cup\{\infty\}}$ be a sequence of measurable mappings from $[0,T]$ into H such that $u_n \to u_\infty$ pointwisely with respect to the norm topology. Assume that $(\zeta_n)_{n\in\mathbf{N}}$ is a sequence in $L^1_H([0,T])$ satisfying*

$$\zeta_n(t) \in \partial\varphi(u_n(t)) \quad \text{a.e.} \quad t \in [0,T]$$

for each $n \in \mathbf{N}$ and $\sigma(L^1_H, L^\infty_H)$ converging to $\zeta_\infty \in L^1_H([0,T])$. Then we have

$$\zeta_\infty(t) \in \partial\varphi(u_\infty(t)) \quad \text{a.e.} \quad t \in [0,T].$$

Proof. We will use Komlós techniques. See [16, 17, 19]. Namely we may assume that (ζ_n) Komlós converges to ζ_∞ and $(|\zeta_n|)$ Komlós converges to $\rho_\infty \in L^1_{\mathbf{R}}([0,T])$, because the sequence (ζ_n) (resp. $(|\zeta_n|)$) is bounded in $L^1_H([0,T])$ (resp. $L^1_{\mathbf{R}}([0,T])$). Accordingly there are a Lebesgue negligible set $\mathcal{M}$ in $[0,T]$ and subsequences $({\zeta'}_m)$, $(|{\zeta'}_m|)$ such that

$$\lim_n \frac{1}{n}\sum_{m=1}^{n} {\zeta'}_m(t) = \zeta_\infty(t),$$
$$\lim_n \frac{1}{n}\sum_{m=1}^{n} |{\zeta'}_m|(t) = \rho_\infty(t),$$

for all $t \in [0,T]\setminus\mathcal{M}$. Let $\varepsilon > 0$ and let $t \in [0,T]\setminus\mathcal{M}$. By lower semicontinuity of φ and pointwise convergence of u_m to u_∞, there is $N_\varepsilon \in \mathbf{N}$ such that $||u_m(t) - u_\infty(t)|| \le \varepsilon$ and that $\varphi(u_m(t)) \ge \varphi(u_\infty(t)) - \varepsilon$ for all $m \ge N_\varepsilon$. Then we have the estimate

$$\varphi(x) \ge \varphi(u_\infty(t)) - \varepsilon + \langle x - u_\infty(t), {\zeta'}_m(t)\rangle - |{\zeta'}^m|(t)\varepsilon$$

for all $x \in H$, using the classical definition of subdifferential in convex analysis and the preceding estimate. Applying the previous Komlós convergences in the last inequality gives

$$\varphi(x) \ge \varphi(u_\infty(t)) - \varepsilon + \langle x - u(t), \zeta_\infty(t)\rangle - \rho_\infty(t)\varepsilon$$

As ε is arbitrary > 0 we finally get

$$\varphi(x) \ge \varphi(u_\infty(t)) + \langle \zeta(t), x - u_\infty(t)\rangle$$

for all $x \in H$. Whence we have $\zeta_\infty(t) \in \partial\varphi(u_\infty(t))$ a.e.. □

Let us recall and summarize another classical closure type lemma. See e.g. [6].

Lemma 2.2. *Let H be a Hilbert space. Let φ be a proper convex lower semicontinuous function defined on H with values in $]-\infty, +\infty]$. Let $(u_n)_{n\in\mathbf{N}} \cup \{\infty\}$ be a sequence in $L^2_H([0,T])$ such that $(u_n)_{n\in\mathbf{N}}$ strongly converges to $u_\infty \in L^2_H([0,T])$. Assume that $(\zeta_n)_{n\in\mathbf{N}}$ is a sequence in $L^2_H([0,T])$ satisfying*

$$\zeta_n(t) \in \partial\varphi(u_n(t)) \quad \text{a.e.} \quad t \in [0,T]$$

for each $n \in \mathbf{N}$ and converging weakly to $\zeta_\infty \in L^2_H([0,T])$. Then we have

$$\zeta_\infty(t) \in \partial\varphi(u_\infty(t)) \quad \text{a.e.} \quad t \in [0,T].$$

Let us recall some facts on Young measures. Let X be a completely regular Suslin space and let $\mathcal{C}^b(X)$ be the space of all bounded continuous functions defined on X. Let $\mathcal{M}^1_+(X)$ be the set of all Borel probability measures on X endowed with the narrow topology. A Young measure $\lambda : [0,T] \to \mathcal{M}^1_+(X)$ is, by definition, a *scalarly Lebesgue-measurable* mapping from $[0,T]$ into $\mathcal{M}^1_+(X)$, that is, for every $f \in \mathcal{C}^b(X)$, the mapping $t \mapsto \langle f, \lambda_t \rangle := \int_X f(x)\, d\lambda_t(x)$ is Lebesgue-measurable on $[0,T]$. A sequence (λ^n) in the space of Young measures $\mathcal{Y}([0,T];\mathcal{M}^1_+(X))$ *stably converges* to a Young measure $\lambda \in \mathcal{Y}([0,T];\mathcal{M}^1_+(X))$ if the following holds

$$\lim_n \int_A [\int_X f(x)\, d\lambda^n_t(x)]\, dt = \int_A [\int_X f(x)\, d\lambda_t(x)]\, dt$$

for every Lebesgue-measurable set $A \subset [0,T]$ and for every $f \in \mathcal{C}^b(X)$.

3 Existence Results Involving Anti-Periodic Boundary Conditions

The following deal with an evolution inclusion generated by subdifferential operators of convex lower semicontinuous functions with anti-periodic boundary conditions and $cwk(H)$-valued upper semicontinuous perturbations, here $cwk(H)$ is the set of all nonempty convex weakly compact subsets of H.

Proposition 3.1. *Assume that $\varphi : H \to]-\infty, +\infty]$ is convex lower semicontinuous, even, with $\varphi(0) = 0$ and $D(\varphi)$ closed and satisfying:*

(a) *for every $r > 0$, $\sup_{x \in D(\varphi)\cap \overline{B}_H(0,r)} |\partial\varphi(x)|_0 < +\infty$,*

(b) *for every $r > 0$, $D(\varphi)\cap\overline{B}_H(0,r)$ is strongly compact in H, shortly $D(\varphi)$ is ball-compact.*

Assume that $F : [0,T] \times H \to cwk(H)$ is upper semicontinuous on $[0,T] \times H$ satifying $|F(t,x)| \leq \alpha(1 + ||x||)$ for all $(t,x) \in [0,T] \times H$ for some positive constant α and $G : [0,T] \times H \to cwk(H)$ is a separately scalarly measurable on $[0,T]$ and separately scalarly upper semicontinuous on H such that $|G(t,x)| \leq \beta$ or all $(t,x) \in [0,T] \times H$ for some positive constant β. Assume further that $F + G$ satisfies the following monotone condition: there exists a positive constant γ such that $\langle x - y, u - v\rangle \geq \gamma||u - v||^2, \forall u, v \in H, \forall x \in F(t,u) + G(t,u), \forall y \in F(t,v) + G(t,v)$ and $\forall t \in [0,T]$. Then there is a unique absolutely continuous T-anti-periodic solution $u : [0,T] \to H$ with $\dot{u} \in L^\infty_H([0,T])$ of the problem

$$(\mathcal{P}) \begin{cases} 0 \in \dot{u}(t) + \partial\varphi(u(t)) + F(t,u(t)) + G(t,u(t)) \\ u(T) = -u(0) \end{cases}$$

Proof. Existence and uniqueness of absolutely continuous solution of the problem

$$(\mathcal{Q}) \begin{cases} 0 \in \dot{u}(t) + \partial\varphi(u(t)) + F(t,u(t)) + G(t,u(t)) \\ u(0) = a \in D(\varphi) \end{cases}$$

follow from ([8], Theorem 3.1). Nevertheless we repeat the uniqueness argument for $(\mathcal{Q})$ because this led to the uniqueness of T-anti-periodic solution for $(\mathcal{P})$. Let u and v be two solutions of $(\mathcal{Q})$ whose existence is ensured by Theorem 3.1 in [8]. There exist two functions h and k in $L^\infty_H([0,T])$ such that for almost all $t \in [0,T]$, we have

$$-\dot{u}(t) - h(t) \in \partial\varphi(u(t)), \tag{3.1}$$

$$-\dot{v}(t) - k(t) \in \partial\varphi(v(t)). \tag{3.2}$$

with

$$h(t) \in F(t, u(t)) + G(t, u(t)) \text{ and } k(t) \in F(t, v(t)) + G(t, v(t)).$$

Further, by our monotone condition on $F + G$,

$$\langle h(t) - k(t), u(t) - v(t)\rangle \geq \gamma ||u(t) - v(t)||^2. \tag{3.3}$$

Then (3.1)—(3.3) and the monotonicity of $\partial\varphi$ entail, for almost all $t \in [0, T]$,

$$\langle \dot{u}(t) + h(t) - \dot{v}(t) - k(t), u(t) - v(t)\rangle \leq 0$$

and hence

$$\begin{aligned} \langle \dot{u}(t) - \dot{v}(t), u(t) - v(t)\rangle &\leq -\langle h(t) - k(t), u(t) - v(t)\rangle \\ &\leq -\gamma ||u(t) - v(t)||^2 \leq 0. \end{aligned} \tag{3.4}$$

From the preceding estimate we see by integrating on $[s, s']$ $(s, s' \in [0, T])$

$$||u(s') - v(s')||^2 \leq ||u(s) - v(s)||^2.$$

Since this inequality is true for $s = 0$, we have $u = v$.

Now let $a, b \in D(\varphi)$ and let u_a (resp. u_b) be the solution of the above problem associated with the initial value a (resp. b). Applying the last inequality in (3.4) by taking $u = u_a$ and $v = u_b$ and integrating

$$\frac{1}{2}||u_a(t) - u_b(t)||^2 \leq \frac{1}{2}||a - b||^2 - \int_0^t \gamma ||u_a(s) - u_b(s)||^2\, ds. \tag{3.5}$$

Now, we finish the proof by checking that $a \mapsto -u_a(T)$ is a strict contraction on the closed convex set $D(\varphi)$, using similar arguments as in ([9], Theorem 5.3). It is enough to show that

$$||u_a(T) - u_b(T)|| < ||a - b||,$$

if $||a - b|| > 0$. By Lemma 5.4 in [9] asserting that, if ψ is a continuous real valued function such that

$$0 \leq \psi(t) \leq \delta - \int_0^t \theta(s)\varphi(s)\, ds,$$

with $\delta > 0$ and $\theta(.) > 0$ Lebesgue-integrable, then $\psi(t) < \delta$, $\forall t \in [0, T]$, so we conclude from (3.5) that

$$||u_a(T) - u_b(T)|| < ||a - b||.$$

Let us consider the mapping $U : a \mapsto -u_a(T)$ from $D(\varphi)$ into $D(\varphi)$ because φ is even. Since this mapping is a (strict) contraction, it has a unique fixed point that is the T-anti-periodic solution of the problem $(\mathcal{P})$. □

Here is an application of the preceding result. For this purpose, we need a useful result.

Lemma 3.2. *Let* $w : [0, T] \to H$ *and* $\dot{w} \in L^2_H([0, T])$ *satisfying:*

$$\begin{aligned} w(t) &= w(0) + \int_0^t \dot{w}(s)ds, \quad t \in [0, T] \\ w(T) &= -w(0). \end{aligned}$$

Then the following inequality hold

$$||w||_{\mathcal{C}_H([0,T])} \le \frac{\sqrt{T}}{2}||\dot{w}||_{L^2_H([0,T])}. \tag{a}$$

Assume further that

$$\dot{w} \in \mathcal{C}_H([0,T]), \quad \dot{w}(T) = -\dot{w}(0).$$

Then the following inequality hold

$$\int_0^T ||w(t)||^2 dt \le \frac{T^2}{\pi^2}\int_0^T ||\dot{w}(t)||^2 dt. \tag{b}$$

Proof. The proof is omitted, see e.g. [5, 7, 18]. Estimate (a) is quoted in several proofs presented here. Estimate (b) is useful when dealing with the uniqueness of solutions of anti-periodic second order inclusions with Lipschitzean perturbations. See the remark 2) of Corollary 4.3.

Here is a useful application.

Corollary 3.3. *Let $w^n : [0,T] \to H$ and $\dot{w}^n \in L^2_H([0,T])$ satisfying:*

$$\begin{aligned} w^n(t) &= w^n(0) + \int_0^t \dot{w}^n(s)ds, \quad t \in [0,T]. \\ w^n(T) &= -w^n(0), \quad \sup_{n\ge 1}||\dot{w}^n||_{L^2_H([0,T])} < +\infty. \end{aligned}$$

Then, up to extracted subsequences, there exist $v^\infty \in L^2_H([0,T])$ and a absolutely continuous mapping $w^\infty : [0,T] \to H$ satisfying

(1) $w^\infty(t) = w^\infty(0) + \int_0^t v^\infty(s)ds, \quad \forall t \in [0,T]$.

(2) $w^\infty(T) = -w^\infty(0)$.

(3) *For every $e \in H$, for every $t \in [0,T]$, $\lim_{n\to\infty}\langle e, w^n(t)\rangle = \langle e, w^\infty(t)\rangle$.*

(4) *For every $h \in L^2_H([0,T])$,*

$$\lim_{n\to\infty}\int_0^T \langle h(t), w^n(t)\rangle dt = \int_0^T \langle h(t), w^\infty(t)\rangle dt.$$

Proof. Applying Lemma 3.2 (a) to w^n gives

$$||w^n||_{\mathcal{C}_H([0,T])} \le \frac{\sqrt{T}}{2}||\dot{w}^n||_{L^2_H([0,T])}.$$

Whence (w^n) is bounded in $\mathcal{C}_H([0,T])$ because $(\dot{w}^n)$ is bounded in $L^2_H([0,T])$. Extracting subsequences we may assume that $(\dot{w}^n)$ converges weakly in $L^2_H([0,T])$ to a function $v^\infty \in L^2_H([0,T])$ and $(w^n(0))$ weakly converges in H to an element $x^\infty \in H$. Let us set

$$w^\infty(t) = x^\infty + \int_0^t v^\infty(s)ds, \forall t \in [0,T].$$

Whence

$$\lim_{n\to\infty}\langle e, w^n(t)\rangle = \langle e, x^\infty\rangle + \langle e, \int_0^t v^\infty(s)ds\rangle$$

for every $e \in H$ and for every $t \in [0,T]$, so that $(w^n(t))$ weakly converges in H to $w^\infty(t)$ for every $t \in [0,T]$. We have $w^\infty(0) = \text{weak-}\lim_{n\to\infty} w^n(0) = x^\infty$. Since $w^n(T) = -w^n(0)$, we also have

$$w^\infty(T) = \text{weak-}\lim_{n\to\infty} w^n(T) = -\text{weak-}\lim_n w^n(0) = -x^\infty = -w^\infty(0).$$

Then w^∞ is absolutely continuous with $\dot{w}^\infty = v$ and satisfies $w^\infty(T) = -w^\infty(0)$. It remains to check (4). For every $h \in L^2_H([0,T])$, we have

$$\int_0^T \langle h(t), w^n(t)\rangle dt = \int_0^T \langle h(t), w^n(0)\rangle dt + \int_0^T \langle h(t), \int_0^t \dot{w}^n(s)ds\rangle dt$$

It is clear that $\lim_{n\to\infty}\langle h(t), w^n(0)\rangle = \langle h(t), w^\infty(0)\rangle$. Hence

$$\lim_{n\to\infty}\int_0^T \langle h(t), w^n(0)\rangle dt = \int_0^T \langle h(t), w^\infty(0)\rangle dt$$

by Lebesgue convergence theorem. Similarly we have

$$\lim_{n\to\infty}\langle h(t), \int_0^t \dot{w}^n(s)ds\rangle = \langle h(t), \int_0^t v^\infty(s)ds\rangle, \quad \forall t \in [0,T].$$

By Holder inequality $||\int_0^t \dot{w}^n(s)ds|| \le \sqrt{T}||\dot{w}^n||_{L^2_H([0,T])} \le M$ for some positive constant M, again by Lebesgue convergence theorem, we see that

$$\lim_{n\to\infty}\int_0^T \langle h(t), \int_0^t \dot{w}^n(s)ds\rangle dt = \int_0^T \langle h(t), \int_0^t v(s)ds\rangle dt$$

thus finishing the proof. □

Proposition 3.4. *Assume that $\varphi : H \to]-\infty, +\infty]$ is convex lower semicontinuous, even, with $\varphi(0) = 0$ and $D(\varphi)$ closed and satisfying:*

(a) *for every $r > 0$, $\sup_{x\in D(\varphi)\cap\overline{B}_H(0,r)} |\partial\varphi(x)|_0 < +\infty$,*

(b) *for every $r > 0$, $D(\varphi)\cap\overline{B}_H(0,r)$ is strongly compact in H, shortly $D(\varphi)$ is ball-compact.*

Let $\gamma > 0$ and $f \in L^2_H([0,T])$. Then the problem

$$(\mathcal{P}_1)\begin{cases} 0 \in \dot{u}(t) + \gamma u(t) + f(t) + \partial\varphi(u(t)) \\ u(T) = -u(0) \end{cases}$$

admits at least a T-anti-periodic absolutely continuous solution $u : [0,T] \to H$ which satisfies $||\dot{u}||_{L^2_H([0,T])} \le ||f||_{L^2_H([0,T])}$.

Proof. Step 1. Assume that $f \in \mathcal{C}_H([0,T])$. It is enough to apply Proposition 3.1 by taking $F(t,x) = \gamma x + f(t)$ and $G(t,x) = 0$ for all $(t,x) \in [0,T]\times H$ to get a unique T-anti-periodic absolutely continuous solution for the problem $(\mathcal{P}_1)$. Indeed we have $\langle \gamma x + f(t) - (\gamma y +$

$f(t)), x - y\rangle = \gamma||x - y||^2, \forall x, y \in H$, and $\forall t \in [0, T]$. Using the classical chain rule formula for lower semicontinuous functions and integrating on $[0, T]$ gives

$$0 = \int_0^T ||\dot{u}(t)||^2 dt + \varphi(u(T)) - \varphi(u(0)) + \int_0^T \langle \gamma u(t) + f(t), \dot{u}(t)\rangle dt.$$

Hence the inequality $||\dot{u}||_{L^2_H([0,T])} \leq ||f||_{L^2_H([0,T])}$ follows by anti-periodicity.

Step 2. Assume that $f \in L^2_H([0, T])$. Let (f_n) be a sequence in $\mathcal{C}_H([0, T])$ converging to f with respect to the topology of the norm of $L^2_H([0, T])$. Let u_{f_n} be the T-anti-periodic absolutely continuous solution of $(\mathcal{P}_1)$ associated with f_n

$$\begin{cases} 0 \in \dot{u}_{f_n}(t) + \gamma u_{f_n}(t) + f_n(t) + \partial\varphi(u_{f_n}(t)) \\ u_{f_n}(T) = -u_{f_n}(0) \end{cases}$$

with $||\dot{u}_{f_n}||_{L^2_H([0,T])} \leq ||f_n||_{L^2_H([0,T])}$. It is clear that $(\dot{u}_{f_n})$ is bounded in $L^2_H([0, T])$. So we may assume that $(\dot{u}_{f_n})$ weakly converges in $L^2_H([0, T])$ to $v \in L^2_H([0, T]$. As $||u_{f_n}||_{\mathcal{C}_H([0,T])} \leq \frac{\sqrt{T}}{2}||\dot{u}_{f_n}||_{L^2_H([0,T])}$ in view of Lemma 3.2 (a), using the ball-compactness assumption and Ascoli theorem, we infer that (u_{f_n}) is relatively compact in $\mathcal{C}_H([0, T])$. Taking account of Corollary 3.3 we may assume that (u_{f_n}) converges uniformly to a T-anti-periodic absolutely continuous function u and $\dot{u}_{f_n}$ weakly converges in $L^2_H([0, T])$ to $\dot{u}$. For simplicity, let $g_n = -\dot{u}_{f_n} - \gamma u_{f_n} - f_n$. Then $g_n(t) \in \partial\varphi(u_{f_n}(t))$ a.e. and (g_n) weakly converges in $L^2_H([0, T]$ to $-\dot{u} - \gamma u - f$. By invoking Lemma 2.2, we conclude that

$$-\dot{u}(t) - \gamma u(t) - f(t) \in \partial\varphi(u(t)) \quad a.e.$$

In otherwords, u is a T-anti-periodic absolutely continuous solution of $(\mathcal{P}_1)$ satisfying $||\dot{u}||_{L^2_H([0,T])} \leq ||f||_{L^2_H([0,T])}$ by antiperiodicity. □

Remarks. Proposition 3.4 seems to be a corollary of the general theory in [3]. The above techniques led to a variational convergence result.

Theorem 3.5. *Let $\gamma > 0$, $f^n \in L^2_H([0, T])$, $\varphi_n, \varphi_\infty : H \to [0, +\infty]$ are proper, convex, l.s.c, even with $\varphi_n(0) = \varphi_\infty(0) = 0, \forall n \in \mathbf{N} \cup \{\infty\}$ satisfying:*

(i) *for every $n \in \mathbf{N}$, for every $r > 0$, $\sup_{x \in D(\varphi_n) \cap \overline{B}_H(0,r)} |\partial\varphi_n(x)|_0 < +\infty$,*

(ii) *for every $r > 0$, $\cup_n D(\varphi_n) \cap \overline{B}_H(0, r)$ is relatively compact in H, shortly $\cup_n D(\varphi_n)$ is ball-compact.*

Let u_n be a T-anti-periodic absolutely continuous of

$$\begin{cases} 0 \in \dot{u}^n(t) + \gamma u^n(t) + f^n(t) + \partial\varphi_n(u^n(t)), \quad a.e. \quad t \in [0, T], \\ u^n(T) = -u^n(0). \end{cases}$$

Assume that

(H_1): *(f^n) weakly converges to $f \in L^2_H([0, T])$.*

; (H_2): *(φ_n) epiconverges to φ_∞.*

Then, up to extracted subsequences, (u^n) converges uniformly to a T-anti-periodic absolutely continuous solution u of the inclusion

$$\begin{cases} 0 \in \dot{u} + \gamma u + f + \partial I_{\varphi_\infty}(u), \\ u(T) = -u(0). \end{cases}$$

with $\int_0^T \varphi_\infty(u(t))dt < +\infty$*, here* $\partial I_{\varphi_\infty}$ *denotes the subdifferential of the convex integral functional* I_{φ_∞} *defined on* $L^2_H([0,T])$ *by*

$$I_{\varphi_\infty}(u) = \begin{cases} \int_0^T \varphi_\infty(u(t))dt & \text{if} \quad \int_0^T \varphi_\infty(u(t))dt \quad \text{is finite} \\ +\infty \quad \text{otherwise.} \end{cases}$$

Proof. *Step 1* Thanks to the estimate $||\dot u^n||_{L^2_H} \le ||f^n||_{L^2_H}$ and Lemma 3.2 (a) we have

$$||u^n||_{\mathcal{C}_H([0,T])} \le \frac{\sqrt{T}}{2}||\dot u^n||_{L^2_H([0,T])} \le \frac{\sqrt{T}}{2}||f^n||_{L^2_H}$$

so that $\sup_{n\ge 1} ||u^n||_{\mathcal{C}_H([0,T])} < +\infty$. Furthermore, using the absolute continuity of $\varphi_n(u^n)$ and the chain rule theorem [6], yields

$$\langle -\dot u^n(t) - \gamma u^n(t) - f_n(t), \dot u^n(t)\rangle = \frac{d}{dt}\varphi_n(u^n(t))$$

for every $n \in \mathbf{N}$. Hence by integrating

$$+\infty > \sup_{n\ge 1}\int_0^T |\langle \dot u^n(t), \dot u^n(t) + \gamma u^n(t) + f^n(t)\rangle|dt = \sup_{n\ge 1}\int_0^T |\frac{d}{dt}\varphi_n(u^n(t))|dt.$$

Further apply the classical definition of the subdifferential to convex lsc funtion φ_n yields

$$0 = \varphi_n(0)) \ge \varphi_n(u^n(t)) + \langle u^n(t), \dot u^n(t) + \gamma u^n(t) + f^n(t)\rangle$$

or

$$0 \le \varphi_n(u^n(t)) \le \langle u^n(t), -\dot u^n(t) - \gamma u^n(t) - f^n(t)\rangle.$$

Hence $\sup_{n\ge 1} |\varphi_n(u^n)|_{L^1_{\mathbf{R}}([0,T])} < +\infty$. Now we assert that $|\varphi_n(u^n(t))| \le L$ for all $t \in [0,T]$ and all $n \in N$, here L is a positive constant. Indeed we have

$$\begin{aligned} \varphi_n(u^n(0)) &\le |\varphi_n(u^n(t)) - \varphi_n(u^n(0))| + \varphi_n(u^n(t)) \\ &\le \int_0^T |\frac{d}{dt}\varphi_n(u^n(t))|dt + \varphi_n(u^n(t)). \end{aligned}$$

Hence

$$\varphi_n(u^n(0)) \le \sup_{n\ge 1}\int_0^T |\frac{d}{dt}\varphi_n(u^n(t))|dt + \frac{1}{T}\sup_{n\ge 1}\int_0^T \varphi_n(u^n(t))dt < +\infty.$$

Whence we get the estimate

$$M := \sup_{n\ge 1}\sup_{t\in[0,T]} ||u^n(t)|| < +\infty, \quad L = \sup_{n\ge 1}\sup_{t\in[0,T]} \varphi_n(u^n(t)) < +\infty. \qquad (^*)$$

Using the ball-compactness assumption and Ascoli theorem we may assume that (u^n) converges uniformly to a T-anti-periodic absolutely continuous function u with $\dot u \in L^2_H([0,T]$, taking account into the above estimate. So, in view of (H_2) and $(^*)$ we have

$$\int_0^T \varphi_\infty(u(t))dt \le \liminf_n \int_0^T \varphi_n(u^n(t))dt \le LT < +\infty.$$

Step 2 u is solution of

$$\begin{cases} 0 \in \dot{u} + \gamma u + f + \partial I_{\varphi_\infty}(u), \\ u(T) = -u(0). \end{cases}$$

with $\int_0^T \varphi_\infty(u(t))dt \leq LT < +\infty$, $\partial I_{\varphi_\infty}$ being the subdifferential of the convex integral functional I_{φ_∞} defined on $L^2_H([0,T])$ by

$$I_{\varphi_\infty}(u) = \begin{cases} \int_0^T \varphi_\infty(u(t))dt & \text{if} \quad \int_0^T \varphi_\infty(u(t))dt \quad \text{is finite} \\ +\infty \quad \text{otherwise.} \end{cases}$$

For simplicity let $z^n := \dot{u}^n + \gamma u^n + f^n$ and $z := \dot{u} + \gamma u + f$. Then

$$-z^n(t) \in \partial\varphi_n(u^n(t)) \tag{**}$$

a.e. As $(\dot{u}^n)$ converges weakly to $\dot{u}$ in $L^2_H([0,T])$, (z_n) converges weakly in $L^2_H([0,T])$ to z. The proof will be achieved by using some facts developed in ([11], Lemma 3.4 and Lemma 3.7).

Fact 1 If h_n, h are measurable mappings $h_n, h : [0,T] \to H$ such that (h_n) pointwisely converges to h. Then

$$\liminf_{n\to\infty} \int_B \varphi_n(h^n(t))dt \geq \int_B \varphi_\infty(h(t))dt$$

for every measurable subset B of $[0,T]$, using (H_2).

Fact 2 Let $v \in L^\infty_H([0,T])$. Then there exists a bounded sequence (v_n) in $L^\infty_H([0,T])$ which pointwisely converges to v and such that

$$\limsup_{n\to\infty} \int_B \varphi_n(v^n(t))dt \leq \int_B \varphi_\infty(v(t)dt$$

for every measurable subset B of $[0,T]$, using (H_2) and the estimate $(^*)$. From Fact 1 and the result obtained in Step 1, we have

$$+\infty > LT \geq \liminf_{n\to\infty} \int_0^T \varphi_n(u^n(t))dt \geq \int_0^T \varphi_\infty(u(t))dt.$$

From $(^{**})$ we have

$$\varphi_n(v(t)) \geq \varphi_n(u^n(t)) + \langle v(t) - u^n(t), -z^n(t)\rangle \quad \text{a.e.} \quad t \in [0,T]$$

for every $v \in L^\infty_H([0,T])$. By integrating

$$\int_0^T \varphi_n(v(t))dt \geq \int_0^T \varphi_n(u^n(t))dt + \int_0^T \langle v(t) - u^n(t), -z^n(t)\rangle dt.$$

For every $v \in L^\infty_H([0,T])$, from Fact 2, there is a bounded sequence (v^n) in $L^\infty_H([0,T])$ which converges pointwisely to v and such that

$$\limsup_{n\to\infty} \int_0^T \varphi_n(v^n(t))dt \leq \int_0^T \varphi_\infty(v(t))dt.$$

Combining this with Fact 1 gives

$$\lim_{n\to\infty} \int_0^T \varphi_n(v^n(t))dt = \int_0^T \varphi_\infty(v(t))dt.$$

As

$$\lim_{n\to\infty}\int_0^T \langle v^n(t)-u^n(t), z^n(t)\rangle dt = \int_0^T \langle v(t)-u(t), z(t)\rangle dt$$

because the sequence $(v^n - u^n)$ is bounded in $L^\infty_H([0,T])$ and converges pointwisely to $u - v$ and the sequence (z^n) converges to z with respect to the weak topology of $L^2_H([0,T])$. Finally by combining these facts and by passing to the limit when $n \to \infty$ in the integral subdifferential inequality

$$\int_0^T \varphi_n(v^n(t))dt \geq \int_0^T \varphi_n(u^n(t))dt + \int_0^T \langle v^n(t)-u^n(t), -z^n(t)\rangle dt$$

we get

$$\int_0^T \varphi_\infty(v(t))dt \geq \int_0^T \varphi_\infty(u(t))dt + \int_0^T \langle v(t)-u(t), -z(t)\rangle dt.$$

Hence we conclude that $-z = -\dot{u} - \gamma u - f \in \partial I_{\varphi_\infty}(u)$ with $I_{\varphi_\infty}(u) \leq LT < +\infty$. □

4 A Class of Second Order Evolution Inclusion via a Variational Approach

This section is devoted to a generalization of some results developed by [3, 7] in second order evolution inclusions with T-anti-periodic boundary conditions. For this purpose we will use essentially an existence result obtained by [3, 7] and some variational techniques developed in [10, 12]. We recall below some notations and summarize some results which describe the limiting behaviour of a bounded sequence in $L^1_H([0,T])$. See ([10], Proposition 6.5.17).

Proposition 4.1. *Let H be a separable Hilbert space. Let (ζ_n) be a bounded sequence in $L^1_H([0,T])$. Then the following hold:*

1) *(ζ_n) (up to an extracted subsequence) stably converges to a Young measure ν that is, there exist a subsequence (ζ'_n) of (ζ_n) and a Young measure ν belonging to the space of Young measure $\mathcal{Y}([0,T];\mathcal{M}^1_+(H_\sigma))$ with $t \mapsto bar(\nu_t) \in L^1_H([0,T])$ (here $bar(\nu_t)$ denotes the barycenter of ν_t) such that*

$$\lim_{n\to\infty}\int_0^T h(t, \zeta'_n(t)))\,dt) = \int_0^T [\int_H h(t,x)\,\nu_t(dx)]\,dt$$

for all bounded Carathéodory integrands $h : [0,T] \times H_{\text{weak}} \to \mathbf{R}$,

2) *(ζ_n) (up to an extracted subsequence) weakly biting converges to an integrable function $f \in L^1_H([0,T])$, which means that, there is a subsequence (ζ'_m) of (ζ_n) and an increasing sequence of Lebesgue-measurable sets (A_p) with $\lim_p \lambda(A_p) = 1$ and $f \in L^1_H([0,T])$ such that, for each p,*

$$\lim_{m\to\infty}\int_{A_p} \langle h(t), \zeta'_m(t)\rangle\,dt = \int_{A_p} \langle h(t), f(t)\rangle\,dt$$

for all $h \in L^\infty_H([0,T])$,

3) *(ζ_n) (up to an extracted subsequence) Komlós converges to an integrable function $g \in L^1_H([0,T])$, which means that, there is a subsequence $(\zeta_{\beta(m)})$ and an integrable function $g \in L^1_H([0,T])$, such that*

$$\lim_{n\to\infty}\frac{1}{n}\Sigma_{j=1}^n \zeta_{\gamma(j)}(t) = g(t),\ a.e. \in [0,T],$$

for every subsequence $(f_{\gamma(n)})$ *of* $(f_{\beta(n)})$.

4) *There is a filter* $\mathcal{U}$ *finer than the Fréchet filter such that* $\mathcal{U} - \lim_n \zeta_n = l \in (L^\infty_H)'_{weak}$ *where* $(L^\infty_H)'_{weak}$ *is the second dual of* $L^1_H([0,T])$.

Let $w_{l_a} \in L^1_H([0,T])$ *be the density of the absolutely continuous part* l_a *of* l *in the decomposition* $l = l_a + l_s$ *in absolutely continuous part* l_a *and singular part* l_s. *If we have considered the same extracted subsequence in 1), 2), 3), 4), then one has*

$$f(t) = g(t) = bar(\nu_t) = w_{l_a}(t) \ a.e. \ t \in [0,T]$$

For more information on Young measures, see [10] and the references therein. Now comes our second epigraphical convergence.

Theorem 4.2. *Let* $H = \mathbf{R}^d$, $\gamma \in \mathbf{R}^+$. *Assume that* $\psi : \mathbf{R}^d \to \mathbf{R}$, $\varphi_n : \mathbf{R}^d \to [0,+\infty[$ *are* $\mathcal{C}^1$, *even, convex, Lipschitzean with* $\varphi_n(0) = 0$, $\forall n \geq 1$ *and,* $\varphi_\infty : \mathbf{R}^d \to [0,+\infty[$ *is even proper convex lower semicontinuous. Let* (f^n) *be sequence in* $L^2_H([0,T])$ *weakly converging to* $f^\infty \in L^2_H([0,T])$. *Let* u^n *be a* $W^{2,2}_{\mathbf{R}^d}([0,T])$ *solution of the problem*

$$\begin{cases} \ddot{u}^n(t) + \gamma \dot{u}^n(t) - \nabla\psi(u^n(t)) - f^n(t) + \nabla\varphi_n(u^n(t)) = 0 \quad t \in [0,T], \\ u_n(T) = -u_n(0), \dot{u}_n(T) = -\dot{u}_n(0) \end{cases}$$

Assume that

(i) φ_n *epi-converges to* φ_∞.

(ii) *There exist* $r_0 > 0$ *and* $x_0 \in \mathbf{R}^d$ *such that*

$$\sup_{n\in\mathbf{N}} \sup_{v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}} \int_0^T \varphi_n(x_0 + r_0 v(t))) < +\infty$$

here $\overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$ *is the closed unit ball in* $L^\infty_{\mathbf{R}^d}([0,T])$.

(a) *Then up to extracted subsequences,* (u^n) *converges uniformly to an absolutely continuous function* u^∞ *with* $u^\infty(T) = -u^\infty(0)$, $(\dot{u}^n)$ *pointwisely converges to a BV function* y^∞ *with* $y^\infty = \dot{u}^\infty$ *and* $\dot{u}^\infty(T) = -\dot{u}^\infty(0)$, *and* $(\ddot{u}^n)$ *weakly biting converges to a function* $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,T])$ *which satisfy the variational inclusion*

$$(\mathcal{Q}_\infty) \qquad 0 \in \zeta^\infty + \gamma\dot{u}^\infty - f^\infty - \nabla\psi(u^\infty) + \partial I_{\varphi_\infty}(u^\infty)$$

here $\partial I_{\varphi_\infty}$ *denotes the subdifferential of the convex lower semicontinuous integral functional* I_{φ_∞} *defined on* $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt, \ \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$

Furthermore $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt$.

(b) *There are a filter* $\mathcal{U}$ *finer than the Fréchet filter,* $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ *such that*

$$\mathcal{U} - \lim_n [-\ddot{u}^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n)] = l \in L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$$

where $L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$ *is the second dual of* $L^1_{\mathbf{R}^d}([0,T])$ *endowed with the topology* $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$ *and* $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ *such that*

$$\forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]), \lim_n \int_0^T \langle h, -\ddot{u}^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n)\rangle dt = \langle h, m\rangle$$

here $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ *denotes the space* $\mathcal{C}_{\mathbf{R}^d}([0,T])'$ *endowed with the weak topology* $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,T])', \mathcal{C}_{\mathbf{R}^d}([0,T]))$. *Let* l_a *be the density of the absolutely continuous part* l_a *of* l *in the decomposition* $l = l_a + l_s$ *in absolutely continuous part* l_a *and singular part* l_s. *Then*

$$l_a(h) = \int_0^T \langle h(t), -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty(t) + \nabla\psi(u^\infty(t))\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ *so that*

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(-\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty) + \delta^*(l_s, dom I_{\varphi_\infty})$$

here φ^*_∞ *is the conjugate of* φ_∞, $I_{\varphi^*_\infty}$ *the integral functional defined on* $L^1_{\mathbf{R}^d}([0,T])$ *associated with* φ^*_∞, $I^*_{\varphi_\infty}$ *the conjugate of the integral functional* I_{φ_∞}, $dom I_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}$ *and*

$$\langle m, h\rangle = \int_0^T \langle -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty(t)), h(t)\rangle dt + \langle m_s, h\rangle$$

$\forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$ *with* $\langle m_s, h\rangle = l_s(h)$, $\forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$. *Further* m *belongs to the subdifferential* $\partial J_{\varphi_\infty}(u^\infty)$ *of the convex lower semicontinuous integral functional* J_{φ_∞} *defined on* $\mathcal{C}_{\mathbf{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\, dt,\ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

(c) *Consequently the density* $-\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)$ *of the absolutely continuous part* m_a

$$m_a(h) := \int_0^T \langle -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty(t)), h(t)\rangle dt$$

for all $h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$, *satisfies the inclusion*

$$-\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty(t) + \nabla\psi(u^\infty(t)) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e..}$$

and for any nonnegative measure θ *on* $[0,T]$ *with respect to which* m_s *is absolutely continuous*

$$\int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi^*_\infty}$ *denotes the recession function of* φ^*_∞.

Proof. Existence of u^n for the problem

$$\begin{cases} \ddot{u}^n(t) + \gamma\dot{u}^n(t) - \nabla\psi(u^n(t)) - f^n(t) + \nabla\varphi_n(u^n(t)) = 0 \quad t \in [0,T], \\ u_n(T) = -u_n(0), \dot{u}_n(T) = -\dot{u}_n(0) \end{cases}$$

is ensured by ([3], Lemme 3.6) or ([7], Theorem 3.1).
Step 1 Estimation of $||\dot{u}^n(.)||_{L^2_H([0,T])}$. Multiply scalarly the equation

$$\ddot{u}^n(t) + \gamma\dot{u}^n(t) = \nabla\psi(u^n(t)) + f^n(t) - \nabla\varphi_n(u^n(t))$$

by $\dot{u}^n(t)$ and applying the chain rule formula [20] for the C^1, Lipschitzean function $\psi - \varphi_n$ gives

$$\gamma||\dot{u}^n(t)||^2 = \frac{d}{dt}[\psi(u^n(t)) - \varphi_n(u^n(t)) - \frac{1}{2}||\dot{u}^n(t)||^2] + \langle \dot{u}^n(t), f^n(t)\rangle.$$

Hence by antiperiodicity conditions we get the estimate

$$\gamma||\dot{u}^n||_{L^2_H([0,T])} \leq ||f^n||_{L^2_H([0,T])}. \tag{4.1}$$

From Lemma 3.2 (a)

$$||u^n||_{\mathcal{C}_H([0,T])} \leq \frac{\sqrt{T}}{2}||\dot{u}^n||_{L^2_H([0,T])}$$

and (4.1), it is immediate (u^n) is bounded in $\mathcal{C}_H([0,T])$ and $(\nabla\psi(u^n(.)))$ is uniformly bounded.

Step 2 Estimation of $||\ddot{u}^n(.)||$. As

$$z^n(t) := -\ddot{u}^n(t) - \gamma\dot{u}^n(t) + f^n(t) + \nabla\psi(u^n(t)) = \nabla\varphi_n(u^n(t))$$

by the subdifferential inequality for convex lower semi continuous functions we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. Now let $v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$, the closed unit ball of $L^\infty_{\mathbf{R}^d}[0,T])$. By taking $x = w(t) := x_0 + r_0 v(t)$ in the preceding inequality we get

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle w(t) - u^n(t), z^n(t)\rangle.$$

Integrating the preceding inequality gives

$$\begin{aligned}\int_0^T \langle x_0 + r_0 v(t) - u^n(t), z^n(t)\rangle dt &= \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt + r_0\int_0^T \langle v(t), z^n(t)\rangle dt \\ &\leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt - \int_0^T \varphi_n(u^n(t))dt.\end{aligned}$$

Whence follows

$$r_0\int_0^T \langle v(t), z^n(t)\rangle dt \leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt - \int_0^T \varphi_n(u^n(t))dt - \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt. \tag{4.2}$$

For simplicity, let us set $v^n(t) = u^n(t) - x_0$ for all $t \in [0,T]$. We compute the last integral in the preceding inequality.

$$\begin{aligned}-\int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt &= -\int_0^T \langle v^n(t), \ddot{v}^n(t) + \gamma\dot{v}^n(t) - f^n(t) - \nabla\psi(u^n(t))\rangle dt \\ &= -\int_0^T \langle v^n(t), \ddot{v}^n(t) + \gamma\dot{v}^n(t)\rangle dt \\ &\quad + \int_0^T \langle v^n(t), f^n(t) + \nabla\psi(u^n(t))\rangle dt.\end{aligned} \tag{4.3}$$

Then it is immediate that the last integral

$$\int_0^T \langle v^n(t), f^n(t) + \nabla\psi(u^n(t))\rangle dt$$

is bounded using the above estimates. By integration by parts and taking account into (4.2) we have

$$\begin{aligned}
-\int_0^T \langle v^n(t), \ddot{v}^n(t) + \gamma \dot{v}^n(t)\rangle dt &= -[\langle v^n(t), \dot{v}^n(t) + \gamma v^n(t)]_0^T \\
&\quad + \int_0^T \langle \dot{v}^n(t), \dot{v}^n(t) + \gamma v^n(t)\rangle dt \qquad (4.4)\\
&= -\langle v^n(T), \dot{v}^n(T)\rangle \\
&\quad + \langle v^n(0), \dot{v}^n(0)\rangle - \gamma\langle v^n(T), v^n(T)\rangle \\
&\quad + \gamma\langle v^n(0), v^n(0)\rangle \\
&\quad + \int_0^T ||\dot{v}^n(t)||^2 dt + \gamma \int_0^T \langle \dot{v}^n(t), v^n(t)\rangle dt \\
&= \int_0^T ||\dot{v}^n(t)||^2 dt \quad \text{(by antiperiodicity)}.
\end{aligned}$$

By (4.1)–(4.4), we get

$$r_0 \int_0^T \langle v(t), z^n(t)\rangle dt \le \int_0^T \varphi_n(x_0 + r_0 v(t))dt + \int_0^T ||\dot{u}^n(t)||^2 dt + C \qquad (4.5)$$

for all $v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$, where

$$C := \sup_{n\ge 1} \int_0^T |\langle v^n(t), f^n(t) + \nabla\psi(u^n(t))\rangle| dt < \infty.$$

By (ii), (4.1)–(4.5), we conclude that

$$(\ddot{u}^n + \gamma \dot{u}^n - f^n - \nabla\psi(u^n))$$

is bounded in $L^1_{\mathbf{R}^d}([0,T])$, and so is $(\ddot{u}^n)$. It turns out that the sequence $(\dot{u}^n)$ of absolutely continuous functions is bounded in variation and by Helly theorem, we may assume that $(\dot{u}^n)$ pointwisely converges to a BV function $v^\infty : [0,T] \to \mathbf{R}^d$ and the sequence (u^n) converges uniformly to an absolutely continuous function u^∞ with $\dot{u}^\infty = v^\infty$ a.e. At this point, it is clear that $(\dot{u}^n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to v^∞, using (4.1) and the dominated convergence theorem. Hence $(\gamma\dot{u}_n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to γv^∞.

Step 3. Weak biting limit of $\ddot{u}_n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$, we may assume that $(\ddot{u}_n)$ weakly biting converges to a function $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,T])$, that is, there exists a decreasing sequence of Lebesgue-measurable sets (B_p) with $\lim_p \lambda(B_p) = 0$ such that the restriction of $(\ddot{u}_n)$ on each B_p^c converges weakly in $L^1_{\mathbf{R}^d}([0,T])$ to ζ^∞. Noting that $(\dot{u}_n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to v^∞. It follows that the restriction of $(z^n = -\ddot{u}_n - \gamma\dot{u}_n + f^n + \nabla\psi(u^n))$ to each B_p^c weakly converges in $L^1_{\mathbf{R}^d}([0,T])$ to $z^\infty := -\zeta^\infty - \gamma v^\infty + f^\infty + \nabla\psi(u^\infty)$, because

$$\lim_n \int_B \langle \ddot{u}_n + \gamma\dot{u}_n - f^n - \nabla\psi(u^n), h\rangle\, dt = \int_B \langle \zeta^\infty + \gamma v^\infty - f^\infty - \nabla\psi(u^\infty), h\rangle\, dt$$

for every $B \in B_p^c \cap \mathcal{L}([0,T])$ and for every function $h \in L^\infty_{\mathbf{R}^d}([0,T])$.

Step 4. $L = \sup_{n\ge 1} \sup_{t\in[0,T]} \varphi_n(u^n(t)) < +\infty$

From the chain rule theorem given in Step 1, recall that

$$-\langle \dot{u}^n(t), \ddot{u}^n(t) + \gamma\dot{u}_n(t) - f^n - \nabla\psi(u^n)\rangle = \frac{d}{dt}[\varphi_n(u^n(t))]$$

that is

$$\langle \dot u^n(t), z^n(t)\rangle = \frac{d}{dt}[\varphi_n(u^n(t))].$$

From the above estimate and the anti-periodicity of $\dot u^n$, it is immediate that $(\frac{d}{dt}[\varphi_n(u^n(t))])$ is bounded in $L^1_{\mathbf{R}}([0,T])$ so that $(\varphi_n(u^n(.))$ is bounded in variation. In fact, we get more here by arguing as in the proof of Theorem 3.5. Apply the classical definition of the subdifferential to convex lsc funtion φ_n yields

$$0 = \varphi_n(0) \geq \varphi_n(u^n(t)) + \langle -u^n(t), z^n(t)\rangle$$

or

$$0 \leq \varphi_n(u^n(t)) \leq \langle u^n(t), z^n(t)\rangle = \langle u^n(t), -\ddot u^n(t) - \gamma \dot u^n(t) + f^n(t) + \nabla\psi(u^n(t)\rangle.$$

Hence $\sup_{n\geq 1} |\varphi_n(u^n)|_{L^1_{\mathbf{R}}([0,T])} < +\infty$. Now we assert that $|\varphi_n(u^n(t))| \leq L$ for all $t\in[0,T]$ and all $n \in N$, here L is a positive constant. Indeed we have

$$\begin{aligned}\varphi_n(u^n(0)) &\leq |\varphi_n(u^n(t)) - \varphi_n(u^n(0))| + \varphi_n(u^n(t))\\ &\leq \int_0^T |\frac{d}{dt}\varphi_n(u^n(t))|dt + \varphi_n(u^n(t)).\end{aligned}$$

Hence

$$\varphi_n(u^n(0)) \leq \sup_{n\geq 1}\int_0^T |\frac{d}{dt}\varphi_n(u^n(t))|dt + \frac{1}{T}\sup_{n\geq 1}\int_0^T \varphi_n(u^n(t))dt < +\infty.$$

Whence we get the estimates (*)

$$\begin{aligned} M: &= \sup_{n\geq 1}\sup_{t\in[0,T]} ||u^n(t)|| < +\infty, (\text{by Step 1})\\ L &= \sup_{n\geq 1}\sup_{t\in[0,T]} \varphi_n(u^n(t)) < +\infty.\end{aligned}$$

Step 5. Localization of the limits:

$$z^\infty = -\zeta^\infty - \gamma\dot u^\infty + f^\infty + \nabla\psi(u^\infty) \in \partial I_{\varphi_\infty}(u^\infty).$$

We will adapt the techniques developed in ([11], Lemma 3.7, Proposition 4.2). As (φ_n) epiconverges to φ_∞, by Lemma 3.4 in [11] we have

$$\liminf_n \int_B \varphi_n(u^n(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt,$$

for every $B \in \mathcal{L}([0,T])$. Let $h \in L^\infty_{\mathbf{R}^d}([0,T])$. Using the estimates (*) and applying Lemma 3.7 in [11] provides a bounded sequence (h^n) in $L^\infty_H([0,T])$, such that (h^n) pointwisely converges to h and such that

$$\limsup_n \int_B \varphi_n(h^n(t))\,dt \leq \int_B \varphi_\infty(h(t))\,dt$$

for every $B \in \mathcal{L}([0,T])$. Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. By substituting x by $h^n(t)$ in this inequality and by integrating on each $B \in B_p^c \cap \mathcal{L}([0,T])$,

$$\int_B \varphi_n(h^n(t))\,dt \geq \int_B \varphi_n(u^n(t))\,dt + \int_B \langle h^n(t) - u^n(t), z^n(t)\rangle\,dt$$

and passing to the limit in the preceding inequality when n goes to $+\infty$, we get

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt + \int_B \langle h(t) - u^\infty(t), z^\infty(t)\rangle\,dt.$$

As this inequality is true on each $B \cap B_p^c$

$$\begin{aligned}\int_{B\cap B_p^c} \varphi_\infty(h(t))\,dt \quad &\geq \quad \int_{B\cap B_p^c} \varphi_\infty(u^\infty(t))\,dt \\ &\quad + \int_{B\cap B_p^c} \langle h(t) - u^\infty(t), z^\infty(t)\rangle\,dt\end{aligned}$$

and $B_p^c \uparrow [0,T]$, by passing to the limit when p goes to ∞ in the last inequality, we get

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt + \int_B \langle z^\infty(t), h(t) - u^\infty(t)\rangle\,dt$$

for all $B \in \mathcal{L}([0,T])$ and for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$. In other words,

$$z^\infty = -\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty) \in \partial I_{\varphi_\infty}(u^\infty).$$

Step 6. $\lim_n \int_0^T \varphi_n(u_n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt$.
From the estimates in Step 4 and Helly theorem, we may assume that $(\varphi_n(u_n(.))$ pointwisely converges to a BV function β. By (*), $(\varphi_n(u_n(.))$ converges in $L^1_{\mathbf{R}}([0,T])$ to β. In particular, for every $k \in L^\infty_{\mathbf{R}^+}([0,T])$ we have

$$\lim_{n\to\infty} \int_0^T k(t)\varphi_n(u_n(t))dt = \int_0^T k(t)\beta(t)dt.$$

Using this fact and repeating the biting arguments via the epi-limit results given in Step 5, it is easy to see that

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \beta(t)\,dt + \int_B \langle z^\infty(t), h(t) - u^\infty(t)\rangle\,dt$$

for all $B \in \mathcal{L}([0,T])$ and for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$. In particular, we get the estimate

$$\int_B \varphi_\infty(u^\infty(t))\,dt \geq \int_B \beta(t)\,dt$$

for all $B \in \mathcal{L}([0,T])$. Again by the epi-lower convergence result in Step 5, we have

$$\begin{aligned}\int_B \beta(t)\,dt \quad &= \quad \lim_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt \\ &= \quad \liminf_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt\end{aligned}$$

for all $B \in \mathcal{L}([0,T])$. It turns out that $\varphi_\infty(u^\infty(t)) = \beta(t)$ a.e.

Step 7. Localization of further limits and final step.

As $(z^n = -\ddot{u}^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n))$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$ in view of Step 3, it is relatively compact in the second dual $L^\infty_{\mathbf{R}^d}([0,T])'$ of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the weak topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$. Furthermore, (z^n) can be viewed as a bounded sequence in $\mathcal{C}_{\mathbf{R}^d}([0,T])'$. Hence there are a filter $\mathcal{U}$ finer than the Fréchet filter, $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ and $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'$ such that

$$\mathcal{U} - \lim_n z^n = l \in L^\infty_{\mathbf{R}^d}([0,T])'_{weak} \tag{4.6}$$

and

$$\lim_n z^n = m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak} \tag{4.7}$$

where $L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$ is the second dual of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$ and $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ denotes the space $\mathcal{C}_{\mathbf{R}^d}([0,T])'$ endowed with the weak topology $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,T])', \mathcal{C}_{\mathbf{R}^d}([0,T]))$, because $\mathcal{C}_{\mathbf{R}^d}([0,T])$ is a separable Banach space for the norm sup, so that we may assume by extracting subsequence that (z^n) weakly converges to $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])'$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s, in the sense there is an decreasing sequence (A_n) of Lebesgue measurable sets in $[0,T]$ with $A_n \downarrow \emptyset$ such that $l_s(h) = l_s(1_{A_n}h)$ for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ and for all $n \geq 1$. As $(z^n = -\ddot{u}^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n))$ weakly biting converges to $z^\infty = -\zeta^\infty(t) - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)$ in Step 4, it is already seen (cf. Proposition 4.1) that

$$l_a(h) = \int_0^T \langle h(t), -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$, shortly $z^\infty = -\zeta^\infty(t) - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)$ coincides a.e. with the density of the absolutely continuous part l_a. By [13, 23] we have

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(-\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)) + \delta^*(l_s, dom I_{\varphi_\infty})$$

here φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ is the integral functional defined on $L^1_{\mathbf{R}^d}([0,T])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ is the conjugate of the integral functional I_{φ_∞} and

$$dom I_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}.$$

Using the inclusion

$$z^\infty = -\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty) \in \partial I_{\varphi_\infty}(u^\infty).$$

that is

$$I_{\varphi^*_\infty}(-\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)) = \langle -\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty), u^\infty\rangle - I_{\varphi_\infty}(u^\infty)$$

we see that

$$I^*_{\varphi_\infty}(l) = \langle -\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty), u^\infty\rangle - I_{\varphi_\infty}(u^\infty) + \delta^*(l_s, dom I_{\varphi_\infty}).$$

Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. By substituting x by $h(t)$ in this inequality, here $h \in L^\infty_{\mathbf{R}^d}([0,T])$, and by integrating

$$\int_0^T \varphi_n(h(t))\,dt \geq \int_0^T \varphi_n(u^n(t))\,dt + \int_0^T \langle h(t) - u^n(t), z^n(t)\rangle\,dt.$$

Arguing as in Step 5 by passing to the limit in the preceding inequality, involving the epilimsup property for integral functionals $\int_0^T \varphi_n(h(t))dt$ defined on $L^\infty_{\mathbf{R}^d}([0,T])$, it is easy to see that

$$\int_0^T \varphi_\infty(h(t))\,dt \geq \int_0^T \varphi_\infty(u^\infty(t))\,dt + \langle h - u^\infty, m\rangle.$$

Since this holds, in particular, when $h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$, we conclude that m belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt, \ \ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

As $(z^n = -\ddot{u}^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n))$ weakly biting converges to $z^\infty = -\zeta^\infty(t) - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)$ in Step 5, we see that

$$l_a(h) = \int_0^T \langle h(t), -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty(t) + \nabla\psi(u^\infty(t))\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ (see Proposition 4.1) so that

$$l(h) = \int_0^T \langle -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty), h(t)\rangle dt + l_s(h)$$

$\forall h \in L^\infty_{\mathbf{R}^d}([0,T])$. Now let $B : \mathcal{C}_{\mathbf{R}^d}([0,T]) \to L^\infty_{\mathbf{R}^d}([0,T])$ be the continuous injection and let $B^* : L^\infty_{\mathbf{R}^d}([0,T])' \to \mathcal{C}_{\mathbf{R}^d}([0,T])'$ be the adjoint of B given by

$$\langle B^*l, h\rangle = \langle l, Bh\rangle = \langle l, h\rangle, \quad \forall l \in L^\infty_{\mathbf{R}^d}([0,T])', \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

Then we have $B^*l = B^*l_a + B^*l_s$, $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ being the limit of $(z^n = -\zeta^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n))$ under the filter $\mathcal{U}$ given in section 4 and $l = l_a + l_s$ being the decomposition of l in absolutely continuous part l_a and singular part l_s. It follows that

$$\langle B^*l, h\rangle = \langle B^*l_a, h\rangle + \langle B^*l_s, h\rangle = \langle l_a, h\rangle + \langle l_s, h\rangle$$

for all $h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$. But it is already seen that

$$\langle l_a, h\rangle = \int_0^T \langle -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty), h(t)\rangle dt,$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ so that the measure B^*l_a is absolutely continuous

$$\langle B^*l_a, h\rangle = \int_0^T \langle -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

and its density $-\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty)$ satisfies the inclusion

$$-\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e.}$$

and the singular part B^*l_s satisfies the equation

$$\langle B^*l_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

As we have $B^*l = m$, using (4.6)-(4.7), it turns out that m is the sum of the absolutely continuous measure m_a with

$$\langle m_a, h\rangle = \int_0^T \langle -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

and the singular part m_s given by

$$\langle m_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

which satisfies the property: for any nonnegative measure θ on $[0,T]$ with respect to which m_s is absolutely continuous

$$\int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi^*_\infty}$ denotes the recession function of φ^*_∞. Indeed, as m belongs to $\partial J_{\varphi_\infty}(u^\infty)$ by applying Theorem 5 in [23] we have

$$J^*_{\varphi_\infty}(m) = I_{\varphi^*_\infty}(\frac{dm_a}{dt}) + \int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) \tag{4.8}$$

with

$$I_{\varphi^*_\infty}(v) := \int_0^T \varphi^*_\infty(v(t))dt, \forall v \in L^1_{\mathbf{R}^d}([0,T]).$$

Recall that

$$\frac{dm_a}{dt} = -\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty) \in \partial I_{\varphi_\infty}(u^\infty)$$

that is

$$I_{\varphi^*_\infty}(\frac{dm_a}{dt}) + I_{\varphi_\infty}(u^\infty) = \langle -\zeta^\infty - \gamma\dot{u}^\infty + f^\infty + \nabla\psi(u^\infty), u^\infty\rangle_{\langle L^1_{\mathbf{R}^d}([0,T]), L^\infty_{\mathbf{R}^d}([0,T])\rangle}. \tag{4.9}$$

From (4.9) we deduce

$$\begin{aligned}
J^*_{\varphi_\infty}(m) &= \langle u^\infty, m\rangle_{\langle \mathcal{C}_{\mathbf{R}^d}([0,T]), \mathcal{C}_{\mathbf{R}^d}([0,T])'\rangle} - J_{\varphi_\infty}(u^\infty) \\
&= \langle u^\infty, m\rangle_{\langle \mathcal{C}_{\mathbf{R}^d}([0,T]), \mathcal{C}_{\mathbf{R}^d}([0,T])'\rangle} - I_{\varphi_\infty}(u^\infty) \\
&= \int_0^T \langle u^\infty(t), -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty + \nabla\psi(u^\infty)\rangle dt \\
&\quad + \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)) - I_{\varphi_\infty}(u^\infty) \\
&= I_{\varphi^*_\infty}(\frac{dm_a}{dt}) + \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t).
\end{aligned}$$

Coming back to (4.8) we get the equality

$$\int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t).$$

□

Remarks. Combining biting argument with the characterization of the decomposition formula in the dual of $L^\infty_{\mathbf{R}^d}([0,T])$ allows to localize the limits under consideration and their relationships via Proposition 4.1 and the continuous injection $B : \mathcal{C}_{\mathbf{R}^d}([0,T]) \to L^\infty_{\mathbf{R}^d}([0,T])$, namely the absolute continuous part m_a of the measure limit m and its singular part m_s. At this point, it is easy to see that, up to extracted subsequence, (z_n) stably converges to a Young measure $\nu^\infty \in \mathcal{Y}([0,T], \mathcal{M}^1_+(\mathbf{R}^d))$ with

$$bar(\nu_t) = \int_{\mathbf{R}^d} x\,\nu_t(dx) = -\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + f^\infty(t) + \nabla\psi(u^\infty(t))$$

for a.e. $t \in [0,T]$.

Taking account into the above remark and the results given in Theorem 4.2 and its proofs, we obtain

Corollary 4.3. *Under the hypotheses and notations of Theorem 4.2, assume that φ^*_n is non negative for all $n \in \mathbf{N} \cup \{\infty\}$ and $(\varphi^*_n)_{n\geq 1}$ epilower converges to φ^*_∞, then the following hold:*

$$\liminf_n \int_0^T \varphi^*_n(-\ddot{u}^n(t) - \gamma\dot{u}^n(t) + f^n(t) + \nabla\psi(u^n(t)))\,dt \geq \int_0^T [\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu^\infty_t(dx)]\,dt. \quad (*)$$

Consequently the limits under consideration satisfy

$$\begin{aligned} 0 &\geq \int_0^T [\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu^\infty_t(dx)]\,dt - \int_0^T \langle bar(\nu^\infty_t), u^\infty(t)\rangle\,dt \\ &\quad + \int_0^T \varphi_\infty(u^\infty(t))\,dt - \int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) \\ &\geq \int_0^T \varphi^*_\infty(bar(\nu^\infty_t))dt - \int_0^T \langle bar(\nu^\infty_t), u^\infty(t)\rangle\,dt \\ &\quad + \int_0^T \varphi_\infty(u^\infty(t))\,dt - \int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t). \end{aligned} \quad (**)$$

Proof. As (φ^*_n) epilower converges to φ^*_∞ and $(z^n = -\ddot{u}^n - \gamma\dot{u}^n + f^n + \nabla\psi(u^n))$ stably converges to $\nu^\infty \in \mathcal{Y}([0,T], \mathcal{M}^1_+(\mathbf{R}^d))$, by virtue of Lemma 3.4 in [11], we have

$$\liminf_n \int_0^T \varphi^*_n(-\ddot{u}^n(t) - \gamma\dot{u}^n(t) + f^n(t) + \nabla\psi(u^n(t)))\,dt \geq \int_0^T [\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu^\infty_t(dx)]\,dt. \quad (*)$$

Using the results obtained in the proof of Theorem 4.2 and (*), it is not difficult to check that

$$\begin{aligned}
0 \quad &\geq \quad \liminf_n [\int_0^T \varphi_n^*(-\ddot{u}^n(t) - \gamma\dot{u}^n(t) + f^n(t) + \nabla\psi(u^n(t)))\,dt \\
&\quad + \int_0^T \langle \ddot{u}^n(t) + \gamma\dot{u}^n(t)) - f^n(t) - \nabla\psi(u^n(t)), u^n(t)\rangle dt + \int_0^T \varphi_n(u_n(t)\,dt] \\
&\geq \int_0^T [\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)]\,dt - \langle u^\infty, m\rangle + \int_0^T \varphi_\infty(u^\infty(t))\,dt \\
&= \int_0^T [\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)]\,dt \\
&\quad + \int_0^T \langle \zeta^\infty(t) + \gamma\dot{u}^\infty(t) - f^\infty(t) - \nabla\psi(u^\infty(t)), u^\infty(t)\rangle\,dt \\
&\quad - \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t) + \int_0^T \varphi_\infty(u^\infty(t))\,dt \\
&= \int_0^T [\int_{\mathbf{R}^d} \varphi^*_\infty(x)\nu_t^\infty(dx)]\,dt \\
&\quad + \int_0^T \langle \zeta^\infty(t) + \gamma\dot{u}^\infty(t) - f^\infty(t) - \nabla\psi(u^\infty(t)), u^\infty(t)\rangle\,dt \\
&\quad - \int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) + \int_0^T \varphi_\infty(u^\infty(t))\,dt
\end{aligned}$$

thus proving (**). □

Remarks. 1) Some comments are in order. It is worthy to mention that there is no relationship between the $\nabla\Psi(x)$ and the $\nabla\varphi_n(x)$ and $\partial\varphi(x)$. Without additional assumptions one cannot expect to have the convergence of approximated solutions (u^n)

$$\begin{cases} \ddot{u}^n(t) + \gamma\dot{u}^n(t) - \nabla\psi(u^n(t)) - f^n(t) + \nabla\varphi_n(u^n(t)) = 0 \quad t \in [0,T], \\ u_n(T) = -u_n(0), \dot{u}_n(T) = -\dot{u}_n(0) \end{cases}$$

towards a $W^{2,2}_{\mathbf{R}^d}([0,T])$ T-anti-periodic solution u^∞ of the problem

$$\begin{cases} -\ddot{u}^\infty(t) - \gamma\dot{u}^\infty(t) + \nabla\psi(u^\infty(t)) + f^\infty(t) \in \partial\varphi_\infty(u^\infty(t)) \quad t \in [0,T], \\ u^\infty(T) = -u^\infty(0), \dot{u}^\infty(T) = -\dot{u}^\infty(0) \end{cases}$$

because $(\ddot{u}^n)$ is bounded in $L^1_H([0,T])$. Nevertheless Theorem 4.2 shows that (u^n) converges pointwisely to the absolutely continuous T-anti-periodic mapping u^∞, $(\dot{u}^n)$ pointwisely converges to the T-anti-periodic mapping $\dot{u}^\infty$, $(-\ddot{u}^n(t) - \gamma\dot{u}^n(t) + \nabla\psi(u^n(t)) + f^n(t))$ weak*-converges in $\mathcal{C}_{R^d}([0,T])^*$ to a vector measure $m \in \mathcal{C}_{\mathbf{R}^d}([0,T])^*$ such that the density of its absolutely continuous part m_a satisfies the inclusion

$$-\zeta^\infty(t) - \gamma\dot{u}^\infty(t) + \nabla\psi(u^\infty(t)) + f^\infty(t) \in \partial\varphi_\infty(u^\infty(t))$$

and such that the singular measure m_s in the decomposition $m = m_a + m_s$ satisfies the equality

$$\int_0^T h_{\varphi^*_\infty}(\frac{dm_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{dm_s}{d\theta}(t)\rangle d\theta(t)$$

for any nonnegative measure θ on $[0,T]$ with respect to which m_s is absolutely continuous. On account of the proof of Theorem 4.2, it is easily seen that when $(\ddot{u}^n)$ is bounded in $L^2_H([0,T])$, the proof of Theorem 4.2 is rather simple, because here one can assume that (u^n) converges uniformly to the T-anti-periodic mapping u^∞ and $(\dot{u}^n)$ converges pointwisely to the T-anti-periodic absolutely continuous mapping $\dot{u}^\infty$ (see Corollary 3.3) and $(\ddot{u}^n)$ converges weakly in $L^2_H([0,T])$ to $\ddot{u}^\infty$ which satisfy the problem under consideration. In this particular situation the variational inequality (**) in Corollary 4.3 is reduced to

$$\begin{aligned} 0 \geq & \int_0^T \varphi^*_\infty(-\ddot{u}^\infty(t) - \gamma\dot{u}^\infty(t) + \nabla\psi(u^\infty(t)) + f^\infty(t))\,dt \\ & + \int_0^T \langle \ddot{u}^\infty(t) + \gamma\dot{u}^\infty(t) - \nabla\psi(u^\infty(t)) - f^\infty(t), u^\infty(t)\rangle\,dt \\ & + \int_0^T \varphi_\infty(u^\infty(t))\,dt \end{aligned} \qquad (^{**})$$

that is equivalent to

$$-\ddot{u}^\infty(t) - \gamma\dot{u}^\infty(t) + \nabla\psi(u^\infty(t)) + f^\infty(t) \in \partial\varphi_\infty(u^\infty(t)) \quad a.e.$$

2) The existence and uniqueness of $W^{2,2}_{\mathbf{R}^d}([0,T])$ T-anti-periodic solution for the inclusion of the form

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in f(t,u(t)) + \partial\varphi(u(t)), \quad a.e. \quad t \in [0,T], \\ u^\infty(T) = -u^\infty(0), \dot{u}^\infty(T) = -\dot{u}^\infty(0) \end{cases}$$

where φ is lsc even function, $f : \mathbf{R} \times H \to H$ is a Carathéodory mapping satisfying: $||f(t,x) - f(t,y)|| \leq L||x-y||$ for all $(t,x) \in \mathbf{R} \times H$, for some positive constant $L > 0$ and: there is a $L^2_{\mathbf{R}}$ integrable function $r : \mathbf{R} \to \mathbf{R}^+$ such that $||f(t,x)|| \leq r(t)$ for all $(t,x) \in \mathbf{R}\times H$, and $0 < T < \frac{\pi}{\sqrt{L}}$, is availablable ([7], Theorem 3.2) using the specific inequalities given in Lemma 3.2.

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