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ON THE SOLUTION CONTINUITY OF PARAMETRIC GENERALIZED SYSTEMS*

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Abstract: In this paper, we give some results on the continuity of the efficient solution set, the weak efficient solution set and various proper efficient solution sets to a parametric generalized system in real locally convex Hausdorff topological vector spaces. The results obtained improve several ones in the literature.

Key words: *continuity, lower semicontinuity, solution set mappings, parametric generalized systems, scalarization*

Mathematics Subject Classification: *49K40, 90C31, 49J53*

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1 Introduction

In the last two decades, many results on the existence of solutions to various kinds of vector variational inequalities and vector equilibrium problems have been established, see [6, 9] and the references therein. Recently, the semicontinuity, especially, the lower semicontinuity of the solution sets to parametric vector variational inequalities and parametric vector equilibrium problems has been studied intensively in the literature, such as [1, 2, 4, 5, 7, 11, 13, 14, 15, 17, 18, 19, 20].

Lower semicontinuity and upper semicontinuity are both required in the continuity of a solution set mapping. Generally speaking, the lower semicontinuity of the solution set mapping for a parametric vector variational inequality or a parametric vector equilibrium problem is much stronger than upper semicontinuity, and consequently, it is much more difficult to derive conditions that guarantee lower semicontinuity because of the complexity of the problem structure. In the literature there are several approaches to study the lower semicontinuity and continuity of solution set mappings for parametric vector variational inequalities and parametric vector equilibrium problems. Cheng and Zhu [7] obtained a result on the lower semicontinuity of the solution set map to a parametric vector variational inequality in finite-dimensional spaces based on a scalarization method. Recently, by virtue of a density result and scalarization technique, Gong and Yao [13] have first discussed the lower semicontinuity of the efficient solutions to parametric vector equilibrium problems, which are called generalized systems in their paper. By using the ideas of Cheng and Zhu [7], Gong [11] has discussed the continuity of the solution set mapping for a class of parametric

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weak vector equilibrium problems in topological vector spaces. Anh and Khanh [2], Kimura and Yao [17] discussed the semicontinuity of solution mappings of parametric vector quasi-equilibrium problems by virtue of the closedness or openness assumptions for some certain sets. Huang et al.[14] used local existence results of the models considered and additional assumptions to establish the lower semicontinuity of solution mappings for parametric implicit vector equilibrium problems. Li and Chen [19] discussed the continuity and Hausdorff continuity of the solution set map for a parametric weak vector variational inequality by using a key assumption in virtue of the so-called parametric gap functions. Based on similar assumptions, both results on the upper semicontinuity and the lower semicontinuity of more general vector variational inequalities and a dual weak vector variational inequality were also obtained, see [4, 5, 20].

In this paper, we discuss and improve the results on lower semicontinuity and continuity of the efficient and weak efficient solution sets to parametric generalized systems given in the aforementioned papers [13] and [11], respectively. Based on a well-known conclusion with respect to the upper semicontinuity of a set-valued mapping (see Proposition 2.2), we show that the uniform compactness assumptions used in proving the lower semicontinuity of the efficient solution set in [13] and the weak efficient solution set in [11] are superfluous. Furthermore, we point out that under the assumptions of lower semicontinuity theorems, the solution set mappings are continuous actually. The upper semicontinuity of the solution set mappings are derived by scalarization methods and without using uniform compactness assumptions. In addition, we also give some continuity results of various proper efficient solution sets to parametric generalized systems.

The rest of the paper is organized as follows. In Section 2, we introduce the parametric generalized system (PGS), and recall some concepts and their properties. In Sections 3, 4 and 5, we discuss the continuity of the efficient solution set, the weak efficient solution set and various proper efficient solution sets for (PGS), respectively.

2 Preliminaries

Throughout this paper, let X be a real Hausdorff topological vector space, let Y be a real locally convex Hausdorff topological vector space and let Z be a metric space. Let Y^* be the topological dual space of Y. Let C be a pointed closed convex cone in Y with $\text{int}C \neq \emptyset$. Let $C^* := \{f \in Y^* \mid f(y) \geq 0, \forall y \in C\}$ be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e., $C^\sharp := \{f \in Y^* \mid f(y) > 0, \forall y \in C \backslash \{0\}\}$. Let D be a nonempty subset of Y. The cone hull of D is defined as $\text{cone}(D) := \{td \mid t \geq 0, d \in D\}$. Denote the closure of D by $\text{cl}(D)$ and the interior of D by $\text{int}D$. A nonempty convex subset M of the convex cone C is called a base of C if $C = \text{cone}(M)$ and $0 \notin \text{cl}(M)$. It is easy to see that $C^\sharp \neq \emptyset$ if and only if C has a base.

Let A be a nonempty subset of X and let $F : A \times A \to Y$ be a bifunction. We consider the following generalized system (GS): find $x \in A$ such that

$$F(x, y) \notin -K, \quad \forall y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y.

(GS) includes as a special case a vector variational inequality (VVI) involving

$$F(x, y) = \langle T(x), y - x \rangle,$$

where T is a map from A to $L(X, Y)$, the space of all continuous linear operators from X to Y.

When the set A and the function F are perturbed by a parameter μ which varies over a set Λ of Z, we can consider the following parametric generalized system (PGS): find $x \in A(\mu)$ such that

$$F(x, y, \mu) \notin -K, \quad \forall y \in A(\mu),$$

where $A : \Lambda \to 2^X$ is a set-valued mapping, $F : B \times B \times \Lambda \to Y$ is a trifunction with $A(\mu) \subset B$ for all $\mu \in \Lambda$ and $K \cup \{0\}$ is a convex cone in Y.

In this paper, we will discuss the continuity of the efficient solution set, the weak efficient solution set and various proper efficient solution sets of (PGS).

Let $\varphi : A \times A \to Y$. The mapping φ is called C-monotone on $A \times A$ if $\varphi(x, y) + \varphi(y, x) \in -C$, for all $x, y \in A$. The mapping φ is called C-strictly monotone on $A \times A$ if φ is C-monotone and, if $x, y \in A, x \neq y$, then $\varphi(x, y) + \varphi(y, x) \in -\text{int}C$.

Let $\psi : A \to Y$. The mapping ψ is called C-convex if, for every $x_1, x_2 \in A$, $t \in [0, 1]$, $t\psi(x_1) + (1 - t)\psi(x_2) \in \psi(tx_1 + (1 - t)x_2) + C$.

We say that $D \subset Y$ is a C-convex set if $D + C$ is a convex set in Y.

Let Λ, Ω be Hausdorff topological spaces and let $G : \Lambda \to 2^\Omega$ be a set-valued mapping with nonempty values. In what follows, the symbol 0_Ω denotes the origin of the space Ω.

Definition 2.1. (i) G is called lower semicontinuous (l.s.c) at $\bar{\lambda} \in \Lambda$ if for any open set $Q \subset \Omega$ with $G(\bar{\lambda}) \cap Q \neq \emptyset$, there exists a neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$ such that $G(\lambda) \cap Q \neq \emptyset$, for all $\lambda \in N(\bar{\lambda})$. Remark that G is l.s.c at $\bar{\lambda}$ if and only if for any net $\{\lambda_\alpha\} \subset \Lambda$ with $\lambda_\alpha \to \bar{\lambda}$ and any $\bar{x} \in G(\bar{\lambda})$, there exists $x_\alpha \in G(\lambda_\alpha)$ such that $x_\alpha \to \bar{x}$.

(ii) G is called upper semicontinuous (u.s.c) at $\bar{\lambda}$ if for any open set $Q \subset \Omega$ with $G(\bar{\lambda}) \subset Q$, there exists a neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$ such that $G(\lambda) \subset Q$, for all $\lambda \in N(\bar{\lambda})$.

(iii) G is called Hausdorff lower semicontinuous (H-l.s.c) at $\bar{\lambda}$ if for each neighborhood B_0 of 0_Ω, there is a neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$ such that for every $\lambda \in N(\bar{\lambda})$, $G(\bar{\lambda}) \subset G(\lambda) + B_0$.

(iv) G is called Hausdorff upper semicontinuous (H-u.s.c) at $\bar{\lambda}$ if for each neighborhood B_0 of 0_Ω, there is a neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$ such that for every $\lambda \in N(\bar{\lambda})$, $G(\lambda) \subset G(\bar{\lambda}) + B_0$.

(v) G is called closed at $\bar{\lambda}$ if for each net $(\lambda_\alpha, x_\alpha) \in \text{graph}(G) := \{(\lambda, x) \mid \lambda \in \Lambda, x \in G(\lambda)\}$, $(\lambda_\alpha, x_\alpha) \to (\bar{\lambda}, \bar{x})$, it follows that $(\bar{\lambda}, \bar{x}) \in \text{graph}(G)$.

(vi) G is called uniformly compact near $\bar{\lambda}$, if there exists a neighborhood U of $\bar{\lambda}$ such that $\text{cl}(\bigcup_{\lambda \in U} G(\lambda))$ is compact.

We say G is l.s.c (resp. u.s.c, H-l.s.c, H-u.s.c, closed) on Λ, if it is l.s.c (resp. u.s.c, H-l.s.c, H-u.s.c, closed) at each $\lambda \in \Lambda$. G is said to be continuous (resp. H-continuous) on Λ if it is both l.s.c (resp. H-l.s.c) and u.s.c (resp. H-u.s.c) on Λ. Moreover, we say that G has compact (resp. closed, convex) values, if $G(\lambda)$ is a compact (resp. closed, convex) set for each $\lambda \in \Lambda$.

The following proposition is a well-known fact in the literature, for instance, we can refer to [8, p.23], [22, Proposition 1] and [21, Lemma 2.1]. For its importance in this paper and for the convenience of the reader, we shall give its proof.

Proposition 2.2. *If G has compact values, then G is u.s.c at $\bar{\lambda}$ if and only if for any net $\{\lambda_\alpha\} \subset \Lambda$ with $\lambda_\alpha \to \bar{\lambda}$ and for any $x_\alpha \in G(\lambda_\alpha)$, there exist $\bar{x} \in G(\bar{\lambda})$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$, such that $x_\beta \to \bar{x}$.*

Proof. "$\Rightarrow$" Suppose that there is no subnet of the net $\{x_\alpha\}$ converges to a point in $G(\bar{\lambda})$. Then for each $x \in G(\bar{\lambda})$, there exist an open neighborhood $N(x)$ of x and some $\alpha(x)$ such that $x_\alpha \notin N(x)$, $\forall \alpha \succeq \alpha(x)$. Clearly, $G(\bar{\lambda}) \subset \bigcup_{x \in G(\bar{\lambda})} N(x)$. By the compactness of $G(\bar{\lambda})$, there exist $x_i \in G(\bar{\lambda})$, $i = 1, \cdots, n$ such that

$$G(\bar{\lambda}) \subset \bigcup_{i=1}^{n} N(x_i) =: U.$$

Moreover, $x_\alpha \notin U$, $\forall \alpha \geq \max\{\alpha(x_i) : i = 1, \cdots, n\}$.

On the other hand, since $G(\cdot)$ is u.s.c at $\bar{\lambda}$, there exists an open neighborhood V of $\bar{\lambda}$ such that $G(V) := \bigcup_{\lambda \in V} G(\lambda) \subset U$. It follows from $\lambda_\alpha \to \bar{\lambda}$ that $\lambda_\alpha \in V$ eventually, and hence $G(\lambda_\alpha) \subset U$. Consequently, we get $x_\alpha \in G(\lambda_\alpha) \subset U$ eventually, which leads to a contradiction.

"$\Leftarrow$" Suppose that $G(\cdot)$ is not u.s.c at $\bar{\lambda}$. Then there exist an open set V satisfying $G(\bar{\lambda}) \subset V$, and nets $\lambda_\alpha \to \bar{\lambda}$ and $x_\alpha \in G(\lambda_\alpha)$, such that $x_\alpha \notin V$, $\forall \alpha$. Whence, there exist $\bar{x} \in G(\bar{\lambda})$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x_\beta \to \bar{x}$. Since $\bar{x} \in V$, there exists β_1 such that $x_\beta \in V$ when $\beta \geq \beta_1$, a contradiction. □

Proposition 2.3. (i) *If G is u.s.c at $\bar{\lambda}$, then G is H-u.s.c at $\bar{\lambda}$. Conversely if G is H-u.s.c at $\bar{\lambda}$ and $G(\bar{\lambda})$ is compact, then G is u.s.c at $\bar{\lambda}$.*

(ii) *If G is H-l.s.c at $\bar{\lambda}$, then G is l.s.c at $\bar{\lambda}$. Conversely if G is l.s.c at $\bar{\lambda}$ and cl$(G(\bar{\lambda}))$ is compact, then G is H-l.s.c at $\bar{\lambda}$.*

(iii) *If $G(\bar{\lambda})$ is compact, then G is u.s.c at $\bar{\lambda}$ if and only if G is H-u.s.c at $\bar{\lambda}$, and G is l.s.c at $\bar{\lambda}$ if and only if G is H-l.s.c at $\bar{\lambda}$.*

Proof. (i) See Proposition 3.1(i) of [1].

(ii) The first implication is obvious from the definition.

For the inverse suppose to the contrary that G is not H-l.s.c at $\bar{\lambda}$. Then there exists a neighborhood B_0 of 0_Ω, nets $\{\lambda_\alpha\} \subset \Lambda$ with $\lambda_\alpha \to \bar{\lambda}$ and $\{x_\alpha\}$ such that $x_\alpha \in G(\bar{\lambda})$ but $x_\alpha \notin G(\lambda_\alpha) + B_0$, $\forall \alpha$.

Since $\mathrm{cl}(G(\bar{\lambda}))$ is compact, we may assume that there exists $x_0 \in \mathrm{cl}(G(\bar{\lambda}))$ such that $x_\alpha \to x_0$. Because $x_0 \in \mathrm{cl}(G(\bar{\lambda}))$, so for any neighborhood $V(x_0)$ of x_0, $V(x_0) \cap G(\bar{\lambda}) \neq \emptyset$. Hence, there exist $y_0 \in G(\bar{\lambda})$ and a neighborhood U of 0_Ω satisfying $U \subset B_0$ such that $x_0 - y_0 \in U$.

For $\lambda_\alpha \to \bar{\lambda}$ and $y_0 \in G(\bar{\lambda})$, by the lower semicontinuity of G at $\bar{\lambda}$, there exists $y_\alpha \in G(\lambda_\alpha)$ such that $y_\alpha \to y_0$. It follows from $x_\alpha - y_\alpha \to x_0 - y_0$ that there exists some α_1 such that $x_\alpha - y_\alpha \in U$ whenever $\alpha \geq \alpha_1$. Consequently, we get that $x_\alpha \in y_\alpha + U \subset G(\lambda_\alpha) + B_0$, which leads to a contradiction.

(iii) It follows from (i) and (ii) readily. □

We remark that the second implication of Proposition 2.3(ii) improves Proposition 2.1(b) of [16], where the compactness of $G(\bar{\lambda})$ but not $\mathrm{cl}(G(\bar{\lambda}))$ is required.

3 Continuity of the Efficient Solution Set

For each $\mu \in \Lambda$, let $V(A, F, \mu)$ denote the efficient solution set of (PGS), i.e.,

$$V(A, F, \mu) = \{x \in A(\mu) \mid F(x, y, \mu) \notin -C\backslash\{0\},\ \forall y \in A(\mu)\}.$$

In this section, we discuss the continuity of $V(A, F, \cdot)$ as a set-valued mapping from the set Λ into X.

For each $f \in C^*\backslash\{0\}$ and for each $\mu \in \Lambda$, let $V_f(A, F, \mu)$ denote the set of f-efficient solutions to (PGS), i.e.,

$$V_f(A, F, \mu) = \{x \in A(\mu) \mid f(F(x, y, \mu)) \geq 0, \ \forall y \in A(\mu)\}.$$

The following lemma is an improvement of Lemma 2.2 in [13] (also Lemma 4.2 in [11]), which plays an important role in proving the lower semicontinuity of $V(A, F, \cdot)$, because the uniform compactness of the mapping A used in Lemma 2.2 of [13] (also Lemma 4.2 in [11]) is not required here.

Lemma 3.1. *Let B be a nonempty set such that $A(\mu) \subset B$ for all $\mu \in \Lambda$. Let $\psi : B \times \Lambda \to Y$ and $\varphi : B \times B \times \Lambda \to Y$ be mappings. Suppose that the following conditions are satisfied:*

(i) *A is continuous with nonempty compact convex values on Λ;*

(ii) *ψ is continuous on $B \times \Lambda$ and φ is continuous on $B \times B \times \Lambda$;*

(iii) *For any given $\mu \in \Lambda$, $\varphi(x, x, \mu) \in C$ for all $x \in A(\mu)$ and $\varphi(\cdot, \cdot, \mu)$ is C-strictly monotone on $A(\mu) \times A(\mu)$;*

(iv) *For each $\mu \in \Lambda$ and for each $x \in A(\mu)$, $\psi(\cdot, \mu) + \varphi(x, \cdot, \mu)$ is C-convex on $A(\mu)$.*

Then, for each $f \in C^*\backslash\{0\}$, $V_f(A, F, \cdot)$ is a singleton and is continuous on Λ, where $F(x, y, \mu) = \psi(y, \mu) + \varphi(x, y, \mu) - \psi(x, \mu)$.

Proof. Since all conditions of Lemma 2.1 of [13] are satisfied, $V_f(A, F, \mu)$ is a singleton for each $\mu \in \Lambda$ and for each $f \in C^*\backslash\{0\}$.

Now we show that $\forall \mu \in \Lambda$, $V_f(A, F, \cdot)$ is continuous at μ. Given any net $\mu_\alpha \to \mu$. Let $\{x\} = V_f(A, F, \mu)$ since $V_f(A, F, \mu)$ is a singleton. Then, $x \in A(\mu)$ and

$$f(\psi(y, \mu)) + f(\varphi(x, y, \mu)) - f(\psi(x, \mu)) \geq 0, \quad \forall y \in A(\mu). \tag{3.1}$$

Since A is l.s.c at μ, there exists $x_\alpha \in A(\mu_\alpha)$ such that $x_\alpha \to x$. Let $\{z_\alpha\} = V_f(A, F, \mu_\alpha)$. Then $z_\alpha \in A(\mu_\alpha)$ and

$$f(\psi(y, \mu_\alpha)) + f(\varphi(z_\alpha, y, \mu_\alpha)) - f(\psi(z_\alpha, \mu_\alpha)) \geq 0, \quad \forall y \in A(\mu_\alpha). \tag{3.2}$$

It follows from (3.2) and $x_\alpha \in A(\mu_\alpha)$ that

$$f(\psi(x_\alpha, \mu_\alpha)) + f(\varphi(z_\alpha, x_\alpha, \mu_\alpha)) - f(\psi(z_\alpha, \mu_\alpha)) \geq 0. \tag{3.3}$$

Since A is u.s.c at μ with compact values, by Proposition 2.2, for the nets $\{\mu_\alpha\}$ and $\{z_\alpha\}$, there exist $z \in A(\mu)$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \to z$.

It follows from (3.1) that

$$f(\psi(z, \mu)) + f(\varphi(x, z, \mu)) - f(\psi(x, \mu)) \geq 0. \tag{3.4}$$

By (3.3) and the continuity of f, ψ, φ, taking limit on both sides of (3.3), we get

$$f(\psi(x, \mu)) + f(\varphi(z, x, \mu)) - f(\psi(z, \mu)) \geq 0. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$f(\varphi(x, z, \mu) + \varphi(z, x, \mu)) \geq 0.$$

Assume that $z \neq x$. Since $\varphi(\cdot,\cdot,\mu)$ is C-strictly monotone, we have

$$\varphi(x,z,\mu) + \varphi(z,x,\mu) \in -\text{int}C.$$

Thus, it follows from $f \in C^*\backslash\{0\}$ that

$$f(\varphi(x,z,\mu) + \varphi(z,x,\mu)) < 0,$$

which leads to a contradiction. Therefore $z = x$ and consequently, $V_f(A,F,\mu_\beta) \to V_f(A,F,\mu)$. Hence, by Proposition 2.2 we see that, $V_f(A,F,\cdot)$ is continuous at μ since $V_f(A,F,\cdot)$ is single-valued. □

The following result on the lower semicontinuity of $V(A,F,\cdot)$ has been obtained by Gong and Yao [13] recently, see Theorem 2.1 in [13]. However, noting that the well-known fact of Proposition 2.2, we see that the uniform compactness of A in Theorem 2.1 of [13] can be removed actually.

Theorem 3.2. *Suppose that all conditions of Lemma 3.1 are satisfied. Moreover, assume that $\psi(A(\mu))$ and $D = \{\varphi(x,y,\mu) \mid x,y \in A(\mu)\}$ are bounded subsets of Y for each $\mu \in \Lambda$, $C^\sharp \neq \emptyset$ and int$C \neq \emptyset$. Then, $V(A,F,\cdot)$ is l.s.c on Λ.*

Furthermore, we point out that under the assumptions of Theorem 3.2, the solution mapping $V(A,F,\cdot)$ is continuous.

Theorem 3.3. *Suppose that all conditions of Theorem 3.2 are satisfied. Then, $V(A,F,\cdot)$ is continuous on Λ.*

Proof. We need to prove that for each $\mu \in \Lambda$, $V(A,F,\cdot)$ is u.s.c at μ. Suppose that there exists some $\mu_0 \in \Lambda$ such that $V(A,F,\cdot)$ is not u.s.c at μ_0. Then there exist an open set M satisfying $V(A,F,\mu_0) \subset M$, and nets $\mu_\alpha \to \mu_0$ and $x_\alpha \in V(A,F,\mu_\alpha)$, such that $x_\alpha \notin M$, $\forall\alpha$.

By Lemma 1.2 of [13] (or Theorem 2.1 of [12]), for each fixed $\mu \in \Lambda$, we have

$$\bigcup_{f\in C^\sharp} V_f(A,F,\mu) \subset V(A,F,\mu) \subset \text{cl}(\bigcup_{f\in C^\sharp} V_f(A,F,\mu)).$$

Since $x_\alpha \in V(A,F,\mu_\alpha) \subset \text{cl}(\bigcup_{f\in C^\sharp} V_f(A,F,\mu_\alpha))$, for any neighborhood $U(0)$ of 0_X we have

$$(x_\alpha + U(0)) \cap \bigcup_{f\in C^\sharp} V_f(A,F,\mu_\alpha) \neq \emptyset.$$

Thus, there exist a symmetric neighborhood $U_1(0)$ of 0_X (i.e., $U_1(0) = -U_1(0)$) such that $U_1(0) + U_1(0) \subset U(0)$, and $z_\alpha \in \bigcup_{f\in C^\sharp} V_f(A,F,\mu_\alpha)$ such that $z_\alpha - x_\alpha \in U_1(0)$. Then there exists $f' \in C^\sharp$ such that $\{z_\alpha\} = V_{f'}(A,F,\mu_\alpha)$. Let $\{x_0\} = V_{f'}(A,F,\mu_0)$. Since $V_{f'}(A,F,\cdot)$ is continuous at μ_0 by Lemma 3.1, it follows from the above $U_1(0)$ that there exists a neighborhood $U(\mu_0)$ of μ_0 such that for all $\mu \in U(\mu_0)$, $V_{f'}(A,F,\mu) \in V_{f'}(A,F,\mu_0) + U_1(0)$. Because $\mu_\alpha \to \mu_0$, there exists α_1 such that $\mu_\alpha \in U(\mu_0)$ when $\alpha \geq \alpha_1$. Whence,

$$V_{f'}(A,F,\mu_\alpha) \in V_{f'}(A,F,\mu_0) + U_1(0).$$

That is, $z_\alpha - x_0 \in U_1(0)$. Consequently, we get

$$x_\alpha - x_0 = x_\alpha - z_\alpha + z_\alpha - x_0 \in -U_1(0) + U_1(0) = U_1(0) + U_1(0) \subset U(0).$$

By the arbitrariness of $U(0)$, we obtain $x_\alpha \to x_0$. Note that $x_0 \in \bigcup_{f\in C^\sharp} V_f(A,F,\mu_0) \subset V(A,F,\mu_0) \subset M$. It follows from $x_\alpha \notin M$ and the openness of M that $x_0 \notin M$, which leads to a contradiction. □

Corollary 3.4. *Let Y be a real metric space. Suppose that all conditions of Lemma 3.1 are satisfied, $C^\sharp \neq \emptyset$ and int$C \neq \emptyset$. Then, $V(A, F, \cdot)$ is continuous on Λ.*

Proof. It follows from the continuity of ψ, φ and the compactness of $A(\mu)$ for each $\mu \in \Lambda$ that $\psi(A(\mu))$ and $D = \{\varphi(x, y, \mu) \mid x, y \in A(\mu)\}$ are compact subsets of Y for each $\mu \in \Lambda$. Since Y is a metric space, $\psi(A(\mu))$ and D are bounded subsets of Y for each $\mu \in \Lambda$. Thus, all conditions are satisfied and hence $V(A, F, \cdot)$ is continuous on Λ. □

4 Continuity of the Weak Efficient Solution Set

For each $\mu \in \Lambda$, let $V_W(A, F, \mu)$ denote the weak efficient solution set of (PGS), i.e.,

$$V_W(A, F, \mu) = \{x \in A(\mu) \mid F(x, y, \mu) \notin -\text{int}C,\ \forall y \in A(\mu)\}.$$

In this section, we discuss the continuity and closedness of $V_W(A, F, \cdot)$ as a set-valued mapping from the set Λ into X.

Very recently, Gong [11] has obtained the following result on the lower semicontinuity of $V_W(A, F, \cdot)$, see Theorem 4.1 in [11]. Here, by virtue of Lemma 3.1, we point out that the uniform compactness of A in Theorem 4.1 of [11] is also superfluous.

Theorem 4.1. *Suppose that all conditions of Lemma 3.1 are satisfied and int$C \neq \emptyset$. Then, $V_W(A, F, \cdot)$ is l.s.c on Λ.*

Furthermore, we point out that under the assumptions of Theorem 4.1, the solution mapping $V_W(A, F, \cdot)$ is continuous and closed. We remark that the upper semicontinuity of $V_W(A, F, \cdot)$ is derived as follows by a scalarization method and without using uniform compactness assumption, which is totally different from the proof of Theorem 3.1 in [11] with respect to the upper semicontinuity of the solution mapping. Our result improves Theorem 4.2 of [11].

Theorem 4.2. *Suppose that all conditions of Theorem 4.1 are satisfied. Then, $V_W(A, F, \cdot)$ is continuous and closed on Λ.*

Proof. We shall first prove that for each $\mu \in \Lambda$, $V_W(A, F, \cdot)$ is u.s.c at μ. Suppose that there exists some $\mu_0 \in \Lambda$ such that $V_W(A, F, \cdot)$ is not u.s.c at μ_0. Then there exist an open set M satisfying $V_W(A, F, \mu_0) \subset M$, and nets $\mu_\alpha \to \mu_0$ and $x_\alpha \in V_W(A, F, \mu_\alpha)$, such that $x_\alpha \notin M$, $\forall \alpha$.

For each $\mu \in \Lambda$ and for each $x \in A(\mu)$, since $\psi(\cdot, \mu) + \varphi(x, \cdot, \mu)$ is C-convex on $A(\mu)$, $F(x, A(\mu), \mu) = \{F(x, y, \mu) \mid y \in A(\mu)\}$ is a C-convex set. Then by Theorem 2.1(iii) of [10] (or Theorem 2.1 of [11]), we have that $x_\alpha \in V_W(A, F, \mu_\alpha) = \bigcup_{f \in C^* \setminus \{0\}} V_f(A, F, \mu_\alpha)$, thus there exists $f' \in C^* \setminus \{0\}$ such that $\{x_\alpha\} = V_{f'}(A, F, \mu_\alpha)$. Let $\{x_0\} = V_{f'}(A, F, \mu_0)$. Since $V_{f'}(A, F, \cdot)$ is continuous at μ_0 by Lemma 3.1, we have $x_\alpha \to x_0$. Note that $x_0 \in \bigcup_{f \in C^* \setminus \{0\}} V_f(A, F, \mu_0) = V_W(A, F, \mu_0) \subset M$. It follows from $x_\alpha \notin M$ and the openness of M that $x_0 \notin M$, which yields a contradiction. Thus, we prove that $V_W(A, F, \cdot)$ is u.s.c at μ. By the arbitrariness of μ, we have that $V_W(A, F, \cdot)$ is u.s.c on Λ.

Next, we prove that $V_W(A, F, \cdot)$ has closed values on Λ. Take arbitrary $\mu_0 \in \Lambda$ and $x_\alpha \in V_W(A, F, \mu_0)$ with $x_\alpha \to x_0$. It follows from $x_\alpha \in V_W(A, F, \mu_0)$ that $x_\alpha \in A(\mu_0)$ and $F(x_\alpha, y, \mu_0) \notin -\text{int}C$, $\forall y \in A(\mu_0)$. Since $A(\mu_0)$ is a compact set, $x_0 \in A(\mu_0)$. By the continuity of ψ and φ, we get for any fixed $y \in A(\mu_0)$, $F(x_\alpha, y, u_0) \to F(x_0, y, u_0)$. Thus, $F(x_0, y, u_0) \notin -\text{int}C$, $\forall y \in A(\mu_0)$. This shows that $x_0 \in V_W(A, F, \mu_0)$ and hence $V_W(A, F, \mu_0)$ is a closed set.

Since $V_W(A, F, \cdot)$ is u.s.c on Λ with closed values, $V_W(A, F, \cdot)$ is closed on Λ by virtue of Proposition 7 of [3, p.110]. □

Now we give a sufficient condition of Hausdorff continuity and closedness for the solution mapping $V_W(A, F, \cdot)$.

Corollary 4.3. *Suppose that the conditions of Theorem 4.2 are satisfied. Then* $V_W(A, F, \cdot)$ *is H-continuous and closed on* Λ.

Proof. From the proof of Theorem 4.2 we see, $V_W(A, F, \mu)$ is a closed set for each $\mu \in \Lambda$. It follows from $V_W(A, F, \mu) \subset A(\mu)$ and the compactness of $A(\mu)$ that $V_W(A, F, \mu)$ is a compact set for each $\mu \in \Lambda$.

In view of Theorem 4.2, $V_W(A, F, \cdot)$ is continuous and closed on Λ. Since $V_W(A, F, \cdot)$ has compact values on Λ, by virtue of Proposition 2.3, the continuity of $V_W(A, F, \cdot)$ is equivalent to the Hausdorff continuity of $V_W(A, F, \cdot)$. □

Remark 4.4. Let $X = R^n$, $Y = R^p$ and $C = R^p_+$. Let $g_i : B \times \Lambda \to R^n$, $i = 1, 2, \cdots, p$, be mappings. Let $\varphi(x, y, \mu) = (\langle g_1(x, \mu), y - x\rangle, \cdots, \langle g_p(x, \mu), y - x\rangle)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Euclidean space. Then $V_W(A, F, \cdot)$ reduces to the solutions set of the parameterized weak vector variational inequality $(WVVI)_\mu$ considered in [11]. Theorem 4.2 improves Corollary 5.1 of [11], because the uniform compactness is not required. Furthermore, let $\psi \equiv 0$. Theorem 4.2 also improves Theorem 3.1 of [7].

5 Continuity of Proper Efficient Solution Sets

Let M be a base of C. Set

$$C^{\triangle} = \{f \in C^{\sharp} \mid \exists\, t > 0 \text{ such that } f(b) \geq t, \forall\, b \in M\}.$$

By the separation theorem of convex sets, we know that $C^{\triangle} \neq \emptyset$. It is clear that $C^{\triangle} \subset C^{\sharp}$. Since M is a base of C, $0 \notin \text{cl}(M)$. By the separation theorem of convex sets, there exists $f \in Y^* \backslash \{0\}$ such that $r = \inf\{f(b) \mid b \in M\} > f(0) = 0$. Set

$$V_M = \{y \in Y \mid |f(y)| < r/2\}.$$

Then, V_M is an open convex circled neighborhood of 0_Y.

Now we define some concepts of proper efficient solutions to (PGS). Let $\mu \in \Lambda$ and $x \in A(\mu)$. Define $F(x, A(\mu), \mu) := \{F(x, y, \mu) \mid y \in A(\mu)\}$.

A vector $x \in A(\mu)$ is called a globally efficient solution to (PGS) if there exists a point convex cone $H \subset Y$, with $C\backslash\{0\} \subset \text{int}H$, such that

$$F(x, A(\mu), \mu) \cap ((-H)\backslash\{0\}) = \emptyset.$$

The set of globally efficient solutions to (PGS) is denoted by $V_G(A, F, \mu)$.

A vector $x \in A(\mu)$ is called a Henig efficient solution to (PGS) if there exists some neighborhood U of 0_Y with $U \subset V_M$ such that

$$\text{cone}(F(x, A(\mu), \mu)) \cap (-\text{int}\,\text{cone}(U + M)) = \emptyset.$$

The set of Henig efficient solutions to (PGS) is denoted by $V_H(A, F, \mu)$.

A vector $x \in A(\mu)$ is called a super efficient solution to (PGS) if, for each neighborhood V of 0_Y, there exists some neighborhood U of 0_Y such that

$$\text{cone}(F(x, A(\mu), \mu)) \cap (U - C) \subset V.$$

The set of super efficient solutions to (PGS) is denoted by $V_S(A, F, \mu)$.

A vector $x \in A(\mu)$ is called a cone-Benson efficient solution to (PGS) if

$$\mathrm{cl}(\mathrm{cone}(F(x, A(\mu), \mu) + C)) \cap (-C) = \{0\}.$$

The set of cone-Benson efficient solutions to (PGS) is denoted by $V_{c-B}(A, F, \mu)$.

When $A(\mu) \equiv A$, where A is a nonempty subset of X, and $F(x, A(\mu), \mu) = F(x, A)$, the above concepts of proper efficient solutions of (PGS) reduce to the corresponding concepts of proper efficient solutions of (GS) introduced in [10].

Theorem 5.1. *Suppose that all conditions of Lemma 3.1 are satisfied and C has a base. Then, $V_G(A, F, \cdot)$ and $V_H(A, F, \cdot)$ are continuous on Λ.*

Proof. For each $\mu \in \Lambda$ and for each $x \in A(\mu)$, since $\psi(\cdot, \mu) + \varphi(x, \cdot, \mu)$ is C-convex on $A(\mu)$, $F(x, A(\mu), \mu) = \{F(x, y, \mu) \mid y \in A(\mu)\}$ is a C-convex set. Thus, in view of Theorem 2.1(i)-(ii) of [10], for each fixed $\mu \in \Lambda$, we have that

$$V_G(A, F, \mu) = \bigcup_{f \in C^\sharp} V_f(A, F, \mu),$$

and

$$V_H(A, F, \mu) = \bigcup_{f \in C^\triangle} V_f(A, F, \mu).$$

For each fixed $\mu \in \Lambda$, take arbitrary $x \in V_G(A, F, \mu) = \bigcup_{f \in C^\sharp} V_f(A, F, \mu)$ and $\{\mu_\alpha\}$ with $\mu_\alpha \to \mu$. Then there exists $f' \in C^\sharp$ such that $\{x\} = V_{f'}(A, F, \mu)$. By Lemma 3.1, $V_{f'}(A, F, \cdot)$ is continuous at μ. Hence, there exists $\{x_\alpha\} = V_{f'}(A, F, \mu_\alpha)$ such that $x_\alpha \to x$. Since $x_\alpha = V_{f'}(A, F, \mu_\alpha) \in \bigcup_{f \in C^\sharp} V_f(A, F, \mu_\alpha) = V_G(A, F, \mu_\alpha)$, we obtain $V_G(A, F, \cdot)$ is l.s.c at μ. On the other hand, by the similar proof of Theorem 4.2, we can prove that $V_G(A, F, \cdot)$ is u.s.c at μ. By the arbitrariness of μ, we have $V_G(A, F, \cdot)$ is continuous on Λ.

The continuity of $V_H(A, F, \cdot)$ can be deduced by the similar proof of $V_G(A, F, \cdot)$. □

Theorem 5.2. *Suppose that all conditions of Lemma 3.1 are satisfied and C has a bounded base. Then, $V_S(A, F, \cdot)$ is continuous on Λ.*

Proof. Since $F(x, A(\mu), \mu) = \{F(x, y, \mu) \mid y \in A(\mu)\}$ is a C-convex set, in view of Theorem 2.1(iv) of [10], for each fixed $\mu \in \Lambda$, we have that

$$V_S(A, F, \mu) = \bigcup_{f \in intC^*} V_f(A, F, \mu),$$

where $\mathrm{int}C^*$ denotes the interior of C^* (with respect to the strong topology $\beta(Y^*, Y)$ for Y^*). Then, the continuity result can be deduced by the similar proof of Theorem 5.1. □

Theorem 5.3. *Suppose that all conditions of Lemma 3.1 are satisfied and C has a weakly compact base. Then, $V_{c-B}(A, F, \cdot)$ is continuous on Λ.*

Proof. By virtue of Theorem 2.3 of [10], we have that $V_{c-B}(A, F, \mu) = V_S(A, F, \mu)$ for each $\mu \in \Lambda$. Thus, it is clear that the conclusion holds because of Theorem 5.2. □

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