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VALLÉE POUSSIN THEOREM AND REMEZ ALGORITHM IN THE CASE OF GENERALISED DEGREE POLYNOMIAL SPLINE APPROXIMATION

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Abstract: The classical Remez algorithm was developed for constructing the best polynomial approximations for continuous and discrete functions in an interval. In this paper the classical Remez algorithm is generalised to the problem of polynomial spline (piece-wise polynomial) approximation with the spline defect equal to the spline degree. Also, the values of the splines in the end points of the approximation interval may be fixed. Polynomial splines combine simplicity of polynomials and flexibility, which allows one to significantly decrease the degree of the corresponding polynomials and oscillations of deviation functions. Therefore polynomial splines are a powerful tool for function and data approximation. The generalisation of the Remez algorithm developed in this research has been tested on several approximation problems. The results of the numerical experiments are presented.

Key words: *nonsmooth optimization, polynomial spline, Remez algorithm*

Mathematics Subject Classification: *90C26, 90C30, 41A50*

The goal of this paper is to generalise the classical Remez algorithm (see [9]), developed for constructing the best polynomial approximation to continuous and discrete functions, and the famous Vallée Poussin theorem (see [3, 8]) to the case of polynomials spline approximation.

The original Remez algorithm requires the construction of a polynomial of degree m, such that the sign of the deviation function is alternating at basis points (a predefined set of $m+2$ points) and the absolute deviation is the same at these points. This polynomial is always unique (see [3, 9]). Then the basis should be updated according to a certain rule, such that the new polynomial, constructed over the new basis, has higher deviation at the new basis points and the deviation signs are alternating. This rule is based on the results of the Vallée Poussin theorem. In this paper the results of this theorem are generalised to the case of polynomial spline approximation.

Since its first appearing in 1950s the original Remez algorithm has been generalised to some particular cases. Among them:

- approximation by more than one polynomial (see [1, 2]);
- approximation by polynomial splines of defect one with fixed knots [7] and free knots [5].

The main advantage of the original Remez method is that at each iteration it requires the solving of one-dimensional subproblems. Therefore, there is minimal dependence on the dimension of the original problem. Of course, in the case of a higher dimension of the original optimisation problem, the subproblems are more complex; however, the dimension of these problems remains one. All the above generalisations of the Remez algorithm preserve this important characteristic to a certain extend.

The paper is organised as follows. In section 1 we introduce necessary definitions and formulate the optimisation problem. In section 2 we present a basis update rule for polynomial spline approximation. In section 3 we prove a theorem which is a generalisation of the Vallée-Poussin theorem to the case of polynomial spline approximation. In section 4 we propose an algorithm which is a generalisation of the Remaz algorithm to the case of polynomial splines. In section 5 we present the results of numerical experiments, with the proposed generalisation of the Remez algorithm. In section 6 we give conclusions to the conducted research and identify future research directions.

1 Preliminaries

1.1 Definitions

Definition 1.1. A function $S(t)$, determined in $[a, b]$, is called a polynomial spline of degree m with internal knots θ_i $(i = 1, 2, \dots, n-1;\ \ a = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{n-1} < \theta_n = b)$, if in each segment $[\theta_{i-1}; \theta_i], i = 1, \dots, n$ the function $S(t)$ is a polynomial of a degree not exceeding m, and at each point $\theta_i, 1 = 1, 2, \dots, n-1$ the derivative $S^{(\nu)}$ $(1 \le \nu \le m_i, i = 1, \dots, n)$ may be discontinuous.

Definition 1.2. The difference between the degree of the spline and the order of the highest continuous derivative is called the defect of the spline.

Most researchers work with smooth polynomial splines. In this paper the research is concentrated on highest defect splines, namely, the splines of defect equals the degree of the spline. Such splines are continuous functions which may be nonsmooth at their knots.

Consider an example of polynomial spline construction (see [12]).

$$S_m(A, t) = a_0 + \sum_{i=1}^{n} \sum_{j=m-d+1}^{m} a_{ij}(t - \theta_{i-1})_+^j, \tag{1.1}$$

where m is the spline degree, d is the spline defect, $\theta_i, i = 0, \dots, n$ are the spline knots, $A = (a_0, a_{11}, \dots, a_{nm}) \in \mathbb{R}^{mn+1}$ is a vector of spline parameters,

$$(\xi(x))_+ = \begin{cases} \xi(x), & \xi(x) > 0, \\ 0, & \xi(x) \le 0. \end{cases}$$

Remark 1.3. Notice also that according to Haar theorem (see [4] and references within) in the case of polynomial approximation the best polynomial approximation is unique, since $1, x, x^2, \dots, x^n$ form a Chebyshev system. In the case of polynomial spline approximation the best polynomial spline approximation is not necessarily unique, since

$$(t - \theta_{i-1})_+^j, \quad i = 1, \dots, n, \quad j = 1, \dots, m$$

does not form a Chebyshev system.

Definition 1.4. The borders of the approximation interval $[a, b]$ are called the external knots, the points $\theta_i, \quad i = 1, \dots, n-1$ are called the internal knots of the polynomial spline.

Definition 1.5. A function $S(t)$, determined in $[a, b]$, is called a polynomial spline of generalised (vector) degree $M = (m_1, m_2, \ldots, m_n)$ with internal knots θ_i $(i = 1, 2, \ldots, n-1;$ $a = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n = b)$, if in each interval $[\theta_{i-1}; \theta_i], i = 1, \ldots, n$ the function $S(t)$ is a polynomial of degree not exceeding m_i, and at each point $\theta_i, i = 1, 2, \ldots, n-1$ the derivative $S^{(\nu)}$ $(1 \le \nu \le m)$ may be discontinuous.

A spline of generalised degree $M = (m_1, \ldots, m_n)$ may be constructed as follows: in each segment T_i it is represented by a polynomial $P_i(t)$, such that

$$P_1(t) = \sum_{j=1}^{m_1} a_{1j}(t - \theta_0)^j + a_0, \quad P_i(t) = \sum_{j=1}^{m_i} a_{ij}(t - \theta_{i-1})^j + P_{i-1}(\theta_{i-1}), \quad i = 2, \ldots, n.$$

Definition 1.6. The vector $A = (a_0, a_{11}, a_{12}, \ldots, a_{1m_1}, a_{21}, \ldots a_{2m_2}, \ldots, a_{nm_n})$ is called a vector of spline parameters (VSP).

In some applications it is required that polynomial splines have certain conditions at one or both end point of the approximation interval $[a, b]$.

Definition 1.7. If the value of the spline is fixed at the left (right) end point then this spline is called a spline with fixed left (right) tail.

Definition 1.8. A function $g(t)$ alternates p times in an interval $[a, b]$ if there exist $p + 1$ points $t_i < t_{i+1} \in [a, b]$, such that

$$g(t_i) = -g(t_{i+1}) = \pm \max_{t \in [a,b]} |g(t)|.$$

Definition 1.9. Alternance points are the points where the absolute value of the deviation is maximal and the sign of the deviation at any two consequent points is opposite.

1.2 Optimisation Problem

Suppose that a continuous function $f(t)$ is approximated in the interval $[a, b]$, which has been divided into n sub-intervals, by a fixed knots polynomial spline of generalised degree $M = (m_1, \ldots, m_n)$. The polynomial spline $S(A, t)$, used in the approximation, is a polynomial spline with the knots $\Theta = (\theta_0, \ldots, \theta_n) \in \mathbb{R}^{n+1}$, such that $a = \theta_0$, and $b = \theta_n$ and the vector of spline parameters $A = (a_0, a_{11}, \ldots, a_{nm_n}) \in \mathbb{R}^{\gamma+1}$, where $\gamma = \sum_{k=1}^{n} m_k$. Then the optimisation problem for finding an optimal polynomial spline is as follows

$$\text{minimise} \max_{t \in [a,b]} |f(t) - S(A, t)| \quad \text{subject} \quad \text{to} \quad A \in \mathbb{R}^{\gamma+1}. \tag{1.2}$$

Problem (1.2) is a polynomial spline Chebyshev approximation problem. If $\Theta = (a, b)$ in (1.2) then problem (1.2) is a polynomial Chebyshev approximation problem.

1.3 Optimality Conditions

Necessary and sufficient optimality conditions for polynomial spline approximation have been obtained in [6, 12]. The following theorem holds.

Theorem 1.10 (Tarashnin). *Necessary and sufficient optimality conditions for the spline $S(A^*, t)$ of degree m and defect m are as follows:*

(i) *in one of the sub-intervals* $[\theta_{i-1}, \theta_i]$ *there exist at least* $m+2$ *alternance points* $t_1, \dots, t_{m+2}$,

or

(ii) *there exist sub-intervals* $[\theta_{i-1}, \theta_i]$ *and* $(\theta_{j-1}, \theta_j]$, *such that* $1 \le i < j \le n$ *and on the chain*

$$[\theta_{i-1}, \theta_i], (\theta_i, \theta_{i+1}], \dots, (\theta_{j-1}, \theta_j]$$

there exist $(m(j-i+1)+2)$ *alternance points which are distributed as follows:*

- *there exist at least* $m+1$ *alternance points in the* $i-th$ *interval,*
- *there exist at least* $m+1$ *alternance points in the* $j-th$ *and*
- *there exist at least* m *alternance points in the* $k-th$ *interval* $(i < k < j)$.

Definition 1.11. A minimal length chain of sub-intervals, where the conditions of the Tarashnin theorem are satisfied is called a minimal chain (see [12]).

The length of the minimal chain is 1 if the first condition of the Tarashnin theorem is satisfied or $(j-i+1)$ if the second condition is satisfied.

Remark 1.12. In the case of generalised degree polynomial splines the necessary and sufficient optimality conditions are the same, but for each interval m should be replaced by the corresponding m_i (see [11]).

Remark 1.13. In the case of polynomial splines with fixed left (right) tails the necessary and sufficient optimality conditions are similar, but the number of alternance points is reduced by one if the first (last) interval is included in the minimal chain. If the left (right) tail is fixed and a (b) is a maximal deviation point then the obtained spline is optimal.

Consider now the splines only within their minimal chains. Then assume for simplicity that minimal chains contain n intervals.

Definition 1.14. A set of points $\{t_{ij}\}$, $\quad i = 1, \dots, n, \quad j = 1, \dots, k_i$, such that

$$\theta_0 \le t_{11} < t_{12} < \dots < t_{1m_1+1} < \theta_1 < t_{21} < \dots < t_{2m_2} < \theta_2 < \dots < t_{n1} < \dots < t_{nm_n+1} \le \theta_n, \tag{1.3}$$

is called a basis of a polynomials spline of generalised degree $M = (m_1, \dots, m_n)$.

If the left (right) tail is fixed then basis can be obtained from the free tails basis by removing one of the point from the first (last) interval.

For minimal chains which contain exactly n intervals we use the following notation. $S_1^0, S_2^0, \dots, S_n^0$ are the polynomials which correspond to the spline constructed on the initial basis. The parameters of the polynomials can be obtained as a solution of a linear nonhomogeneous system with a full rank matrix (see [10]). The spline is continuous, the degrees of the corresponding polynomials are less than or equal to $m_1, m_2, \dots, m_n$ respectively. $S_1^1, S_1^2, \dots, S_n^1$ are the polynomials which correspond to the spline constructed on the updated basis. The spline is continuous, the degrees of the corresponding polynomials are less than or equal to $m_1, m_2, \dots, m_n$ respectively. $t_{1i}^0, \dots, t_{k_i i}^0$ are the points of the initial basis located in the i-th interval, $i = 1, \dots, n$, $t_{1i}^1, \dots, t_{k_i i}^1$ are the points of the updated basis located in the i-th interval, $i = 1, \dots, n$, k_i is the number of the basis points located in the $i-$th interval (m_i or $m_i + 1$).

2 A Generalisation of Basis Exchange Rules

Suppose that there exists an interval i such that the initial basis is represented by the points $T_i^0 = \{t_i^0, \ldots, t_{m_i}^0\}$ for $1 < i < n$ (internal intervals) or by the points $T_i^0 = \{t_1^0, \ldots, t_{m_i+1}^0\}$ for $i = 1$ or $i = n$ (end intervals).

Assume that in this interval there exists a point $t^* \notin T_i^0$ such that the absolute deviation in this point is higher than it is at basis points. Assume also that t^* is not an internal spline knot. If in this interval there exists a basis point $\bar{t} \in T_i^0$ (the nearest to t^* from the left or from the right) such that the sign of the deviation in this basis point coincides with the deviation sign in the point t^*, then t^* has to replace $\bar{t}$ in the new basis, the other basis points remain the same. Consider following examples.

Example 2.1. t^* is located between two basis of the same sub-interval of the minimal chain (e.g., t' and t''). Assume that the deviation signs at t^* and t' coincide. Then t^* should replace t'.

Example 2.2. t^* is located to the right (left) of all the other basis points. t' is the nearest basis point to t^* and the deviation signs at these points are the same. Then t^* should replace t'.

Example 2.3. t^* is located to the right (left) of all the other basis points. t' is the nearest basis point to t^* and the deviation signs at these points are not the same. Then t^* should not replace t'.

3 A Generalisation of the Vallée-Poussin Theorem and Remez Algorithm

First of all, we have to prove several auxiliary theorems.

Theorem 3.1. *The basis exchange rules established in the previous section preserve the deviation signs at the basis points.*

Proof. Assume that the deviation sign may change (proof by contradiction).

Consider the interval where the basis points have been replaced. First assume that it is one of the end intervals. Without loss of generality it is enough to present the proof for the case of the first interval.

The first interval contains $m_1 + 1$ basis points. Consider the polynomial $\bar{S}_1 = S_1^0 - S_1^1$. Recall that the polynomial S_1^0 is a polynomial of degree m_1 which represents the initial spline in the first interval, S_1^1 is a polynomial of degree m_1 which represents the new spline obtained after basis exchange, in the first interval. Also notice that the degree of the polynomial $\bar{S}_1$ does not exceed m_1.

Assume that the index $j \in \{1, \ldots, m_1 + 1\}$ represents the point which was removed from the basis, therefore $t_j^1 = \bar{t}$ and t^* replaced t_j^1. Also assume that $f(t_1^1) - S_1^0(t_1) > 0$. In the case of the opposite inequality sign the proof is similar. Then consider the following inequality system:

$$\begin{cases} (-1)^{i-1}(f(t_i^1) - S_1^0(t_i^1)) > 0, \quad i = 1, \ldots, m_1 + 1, \quad i \neq j, \\ (-1)^{j-1}(f(t^*) - S_1^0(t^*)) > 0. \end{cases} \tag{3.1}$$

Notice that $(-1)^{j-1}(f(t_j^1) - S_1^0(t_j^1)) > 0$. Suppose that at the next iteration the deviation signs are changing. Then consider the following system:

$$\begin{cases} (-1)^{i}(f(t_i^1) - S_1^1(t_i^1)) > 0, \quad i = 1, \ldots, m_1 + 1, \quad i \neq j \\ (-1)^{j}(f(t^*) - S_1^1(t^*)) > 0. \end{cases} \tag{3.2}$$

From (3.1) and (3.2) obtain

$$\begin{cases} (-1)^{i-1}(S_1^1(t_i^1) - S_1^0(t_i^1)) = (-1)^i\bar{S}_1(t_i^1) > 0, & i = 1, \dots, m_1+1, i \neq j \\ (-1)^{j-1}(S_1^1(t^*) - S_1^0(t^*)) = (-1)^i\bar{S}_1(t^*) > 0. \end{cases} \tag{3.3}$$

From (3.3) obtain that $\bar{S}_1$ changes its sign at least $m_1 + 1$ times in the interval $[a, \theta_1)$. The degree of the polynomial $\bar{S}_1$ does not exceed m_1, then

$$\mathrm{sign}\,(\bar{S}_1(\theta_1)) = \mathrm{sign}\,(\bar{S}_1(t^1_{m_1+1})), \quad \text{if} \quad j \neq m_1 + 1,$$

$$\mathrm{sign}\,(\bar{S}_1(\theta_1)) = \mathrm{sign}\,(\bar{S}_1(t^*)), \quad \text{if} \quad j = m_1 + 1.$$

Now consider the second interval. According to our assumption the deviation sign changed (the basis points remain the same). Then the following inequality system holds (similar to (3.1) and (3.2)):

$$(-1)^{m_1+i}(f(t_i^2) - S_2^0(t_i^2)) > 0, \quad i = 1, \dots, m_2, \tag{3.4}$$

$$(-1)^{m_1+i+1}(f(t_i^2) - S_2^1(t_i^2)) > 0, \quad i = 1, \dots, m_2. \tag{3.5}$$

Again, the degree of the polynomial $\bar{S}_2 = S_2^1 - S_2^0$ does not exceed m_2, therefore this polynomial changes its sign no more than m_2 times in the interval $[\theta_1, \theta_2]$. Similar to (3.3) obtain the following system:

$$(-1)^{m_1+i}(S_2^1(t_i^2) - S_2^0(t_i^2)) = (-1)^{m_1+i}\bar{S}_2(t_i^2) > 0, \quad i = 1, \dots, m_2, \tag{3.6}$$

Since the spline is continuous in its knots

$$S_2^1(\theta_1) = S_1^1(\theta_1) \Rightarrow S_2^1(\theta_1) - S_2^0(\theta_1) = S_1^1(\theta_1) - S_1^0(\theta_1) \Rightarrow \bar{S}_2(\theta_1) = \bar{S}_1(\theta_1) \tag{3.7}$$

Combine (3.6) and (3.7): from (3.6) obtain that $\bar{S}_2$ changes its sign at least m_2 times in the interval (θ_1, θ_2); from (3.7) obtain that $\mathrm{sign}\,(\bar{S}_2(\theta_1)) = -\mathrm{sign}\,(\bar{S}_2(t_1^2))$.

Therefore,

$$\mathrm{sign}\,(\bar{S}_2(\theta_2)) = \mathrm{sign}\,(\bar{S}_2(t^2_{m_2})). \tag{3.8}$$

Continue the process till the last interval. Obtain the following equations

$$\mathrm{sign}\,(\bar{S}_n(\theta_{n-1})) = (-1)^i\mathrm{sign}\,(\bar{S}_n(t_i^n)), \quad i = 1, \dots, m_n + 1. \tag{3.9}$$

Therefore, in the $n-$th interval there exist at least $m_n + 1$ roots of the polynomial $\bar{S}_n = \bar{S}_n^1 - \bar{S}_n^0$. However, the degree of this polynomial does not exceed m_n. The obtained contradiction proves our theorem for the case when basis exchange occurred in one of the end intervals.

Assume now that basis exchange occurred in one of the internal intervals. Suppose that it is the $k-$th interval ($1 < k < n$). Then the degree of the polynomial $\bar{S}_k = \bar{S}_k^1 - S_k^0$ does not exceed m_k, and this polynomial has at least $m_k - 1$ roots in the interval (θ_{k-1}, θ_k). Two situations are possible.

(i) The following conditions hold:

- $\mathrm{sign}\,(\bar{S}_k(\theta_{k-1})) \neq \mathrm{sign}\,(\bar{S}_k(t_1^k))$;
- $\mathrm{sign}\,(\bar{S}_k(\theta_{k-1})) \neq \mathrm{sign}\,(\bar{S}_k(t^k_{m_k}))$.

In this case $\bar{S}_k$, has at least $m_k + 1$ roots in the interval $[\theta_{k-1}, \theta_k]$, which contradicts to the fact that the degree of this polynomial does not exceed m_k.

(ii) At least one of the above conditions does not hold.

Without loss of generality suppose that the second condition of the situation (i) does not hold. In this case we are in conditions similar to (10) (when the basis was updated in one of the end intervals). In this case, following the same scheme, moving to the $n-$th interval, obtain the same contradiction (a polynomial of degree less than or equal to m_n has at least $m_n + 1$ root in the interval $(\theta_{n-1}, b])$.

□

Theorem 3.2. *The absolute deviation in the points of each successive basis does not decrease.*

Proof of this theorem is also based on the analysis of the behaviour of the polynomials which compose the spline. The techniques for the proof of this theorem are similar to those used in Theorem 3.1, therefore we omit the proof of the theorem.

Corollary 3.3. *Since none of the basis points coincides with an internal knots (see definition 1.14), then the absolute deviation value in the points of each successive basis increases.*

Proof. It was already proven that the absolute deviation value in the points of each successive basis does not decrease. Therefore, we only have to prove that the deviation cannot remain the same.

First assume that the basis exchange occurred in one of the end intervals. Suppose that it is the last interval (in the case of the first interval the reasonings are similar). Then in the first interval the new and the old polynomials coincide in m_1 or more points, therefore, these polynomials are identical. Then in the second interval the new and the old polynomials coincide in $m_2 + 1$ points (m_2 basis points and the knot θ_1, which does not coincide with any of the basis points). Continue this process, in the last interval the old and the new polynomials coincide in $m_n + 2$ points (the knot θ_{n-1} and m_n basis points, which were not affected by basis exchange). Therefore, these polynomials are identical, however, they cannot coincide in the basis point which replaced the old basis point. The obtained contradiction proves that the deviation is increasing.

Assume now that the basis exchange occurred in one of the internal intervals. Then using the same reasonings as they are in the case of end intervals, moving from both sides to this internal interval, obtain the same contradiction.

Therefore, in both cases the absolute deviation value in the points of each successive basis increases. □

The next step is to generalise the well know Vallée-Poussin theorem ([8]), which plays a very important role in the theory of Chebyshev approximation (best polynomial approximation).

Theorem 3.4 (Vallée-Poussin). *Let $f \in C_{[a,b]}$. If for a polynomial $P_m^*(t)$ of degree m there exists a set of $m+2$ points $t_1, \dots, t_{m+2}$, such that $a \leq t_1 < t_2 < \cdots < t_{m+2} \leq b$, and the sign of the deviation $\Delta(t) = f(t) - P_m^*(t)$ alternates at these points, then*

$$\inf_{P_m} \|f - P_m\|_{C_{[a,b]}} \geq \min_{1 \leq k \leq m+2} |\Delta(t_k)|.$$

Now we generalise this theorem to the case of polynomial spline approximation.

Theorem 3.5 (Generalisation of Vallée-Poussin theorem). *Assume that a minimal chain is known. Assume also that it consists of n intervals and the spline within the minimal chain has a generalised degree $M = (m_1, \dots, m_n)$. If for a polynomial spline $\bar{S}_M(t)$ there exists a basis*

$$T = \{t_{11}, \dots, t_{1m_1+1}, t_{21}, \dots, t_{2m_2}, \dots, t_{n1}, \dots, t_{m_n+1}\}, \tag{3.10}$$

such that the sign of the deviation $\Delta(t) = f(t) - \bar{S}_M(t)$ is alternating, then

$$\inf_{S_M} \|f - S_M\|_{C_{[a,b]}} \geq \bar{\Delta}(T), \tag{3.11}$$

where $\bar{\Delta}(T) = \min\{|\Delta(t_{1m_1+1})|, |\Delta(t_{nm_n+1})|, |\Delta(t_{i,j})|, \quad i = 1, \dots, n, \quad j = 1, \dots, m_i\}$

Proof. assume that the opposite condition holds:

$$\inf_{S_M^*} \|f - S_M\|_{C_{[a,b]}} < \bar{\Delta}(T), \tag{3.12}$$

where S_M^* is the best polynomial spline approximation. Then the following inequality holds at the basis points (3.10):

$$|f(t) - S_M^*(t)| < \bar{\Delta}(T), \quad \forall t \in T. \tag{3.13}$$

Then

$$|f(t) - S_M^*(t)| < |f(t) - S_M(t)|, \quad \forall t \in T. \tag{3.14}$$

Let $\bar{S}_{M_i}$ be a polynomial, which corresponds to the spline $\bar{S}_M$ in the $i-$th interval, $i = 1, \dots, n$. Assume that $f(t_{11}) - \bar{S}_M(t_{11}) > 0$, then

$$f(t_{11}) - \bar{S}_M^*(t_{11}) < f(t_{11}) - \bar{S}_M(t_{11}) \Rightarrow \bar{S}_M(t_{11}) - S_M^*(t_{11}) < 0. \tag{3.15}$$

The following inequality holds:

$$f(t_{12}) - \bar{S}_M(t_{12}) < 0,$$

then

$$\bar{S}_M^*(t_{12}) - f(t_{12}) < \bar{S}_M(t_{12}) - f(t_{12}) \Rightarrow \bar{S}_M(t_{12}) - S_M^*(t_{12}) > 0. \tag{3.16}$$

Continue the process, then obtain the following result:

$$\operatorname{sign}(\bar{S}_{M1}(t_{1m_1+1})) = \operatorname{sign}(\bar{S}_{M1}(\theta_1)) = -\operatorname{sign}(\bar{S}_{M1}(t_{21})) \tag{3.17}$$

Similar to the proof of the Theorem 3.1, obtain a contradiction, which proves that (3.12) does not hold. □

Remark 3.6. In the case of polynomial splines with fixed tails all the proofs are similar, the additional root appears in the point a (left tail) or b (right tail). If the maximal deviation point is exactly a (left tail) or b (right tail) then the obtained spline is already optimal.

4 Generalized Remez Algorithm

The classical Remez algorithm can be adopted to the case of generalised degree polynomial splines (fixed or free tails) in the following way:

Generalised Remez algorithm (GRA)

- **Step 0** Choose an initial basis, which satisfies 1.3.
- **Step 1** Construct the polynomial spline which deviates by the same absolute value from the function to be approximated in the basis points, the deviation signs are alternating. The coefficients of this spline can be found as a solution of a full rank nonhomogeneous linear system (see [10]).
- **Step 2** Update the basis using the basis exchange rules, presented in section 2.
  - (i) If the obtained spline satisfies necessary and sufficient optimality conditions then **Stop** (EXIT 1).
  - (ii) If the proposed basis exchange rules cannot be applied (Example 3) or an internal knot should be included in the basis then **Stop** (EXIT 2).
  - (iii) Otherwise, go to **Step 1**.

One can see that the proposed algorithm cannot have cycles (corollary 3.3). Also, if this algorithm terminates with EXIT 1, then the obtained spline is optimal.

If this algorithm terminates with EXIT 2 then the minimal chain was not chosen correctly or the initial basis was not chosen efficiently. These two obstacles limit the practical value of the current version of the algorithm, since the procedure has to be repeated many times before an optimal spline would be obtained. However, this procedure is interesting from the theoretical point of view (as a generalisation of the classical results, developed for the case of polynomial approximation) and also as a first step to the development of an algorithm, which would be also practically efficient.

At the current stage, this algorithm can be used in the following two-step procedure:

- Use one of convex optimisation methods to minimise the objective function in (1.2).
- Apply GRA to refine the obtained solution.

In the above procedure the choice of the minimal chain and the efficient initial basis is much easier than in most general cases.

5 Numerical Experiments

In our experiments we use GRA, described in section 4. If at least one of the spline tails is fixed then the procedure is the same. The only difference is that in the case of fixed tails bases are determined differently (see Definition 1.14). In this case the algorithm is called the Generalised Remez Algorithm for Fixed Tails (GRAFT).

In some cases the proposed exchange rule can not be used (see section 2 Example 3). In this case more accurate techniques are required. This problem is out of the scope of this paper. We will talk about it in the last section.

Consider $f(t) = sin(t)$ in the interval $[0, 6]$. The task is to approximate this function by a polynomial spline of generalised degree $M = (2, 2, 2)$. determined in the sub-intervals $[0, 2], (2, 4], (4, 6]$.

First assume that the minimal chain contains all the subintervals. Apply GRA, obtain:

- basis $T = (0.21, 1.30, 1.99, 2.45, 3.51, 4.01, 4.90, 6.00)$;

- absolute deviation at the basis points $\delta = 0.038$;
- maximal absolute deviation for the whole interval $[0, 6]$ $\delta_{max} = 0.040$ at the point $t = 2$;
- minimal absolute deviation for the whole interval $[0, 6]$ $\delta_{min} = -0.039$ at the point $t = 4$;
- VSP $A = (0.0253, 1.1212, -0.3296, -0.7480, -0.0624, -0.4961, 0.3678)$

The obtained spline is close to an optimal one (Tarashnin theorem). We can try to improve this result.

Construct the minimal chain differently, assume that the minimal chain consists of two intervals: $[2, 4], (4, 6]$. First construct an optimal spline in this chain, using GRA. Then construct a polynomial in the remain interval $[0, 2)$, such that the resulting spline is continuous at the point 2. If the maximal deviation of the resulting spline in the first interval does not exceed the maximal deviation in the intervals $[2, 4]$ and $(4, 6]$ then the resulting spline is optimal. Otherwise the minimal chain has to be rearranged.

Applying GRA to $[2, 4], (4, 6]$ obtain $SP_1(t)$:

- basis $T = (2.0, 2.52, 3.5, 4.001, 4.82, 5.96)$;
- absolute deviation at the basis points $\delta = 0.039$;
- maximal absolute deviation for the whole interval $\delta_{max} = 0.039$ at the point $t = 2$;
- minimal absolute deviation for the whole interval $\delta_{min} = -0.039$ at the point $t = 4$;
- VSP $A = (-0.9480, -0.7447, -0.0636, -0.4963, 0.3676)$;
- $SP_1(2) = 0.9480$.

It is easy to see that the resulting spline satisfies necessary and sufficient optimality conditions. Applying GRAFT to $[0, 2]$ obtain:

- basis $T = (0.00, 0.45, 1.31)$;
- absolute deviation at the basis points $\delta = 0.039$;
- maximal absolute deviation for the whole interval $\delta_{max} = 0.039$ at the point $t = 2$;
- minimal absolute deviation for the whole interval $\delta_{min} = -0.0196$ at the point $t = 0$;
- VSP $A = (-0.0196, 1.2193, -0.3677)$;
- $SP_2(2) = 0.9480$.

The obtained spline satisfies necessary and sufficient optimality conditions.

6 Conclusions

In this paper the well-known Vallée Poussin theorem and the classical Remez algorithm for constructing the best polynomial approximations have been generalised to the case of polynomial spline approximation (fixed and free tails). We started with the theoretical study of this problem, then proposed an algorithm and tested it on some practical problems. The results are presented in this paper.

In the future we plan to continue working in the area of polynomial spline approximation. The future research direction are included but not limited to the following:

- The problem of polynomial spline approximation with free knots. Polynomial splines with free knots allow the construction of more precise approximations, however the complexity of the corresponding optimisation problems is much higher. The corresponding problems are nonconvex. In [5] the Remez algorithm has been generalised to the case of free knots polynomial spline approximation of defect one. In the case of higher defect polynomial splines the problem is still open.
- The problem of minimal chain identification. In most practical problems minimal chains are not known in advance. Therefore, in the current versions of GRA and GRAFT for some practical problems we have to apply the algorithm several times. This approach is not very efficient if the number of subintervals in the original chain is large.
- The problem of obtaining a good initial basis for GRA and GRAFT. This problem is also very important since good initial basis allows:
  (i) reach an optimal spline faster (fewer number of basis exchanges);
  (ii) avoid situations similar to Example 3 section 2.

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