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SOME w-P PROPERTIES FOR LINEAR TRANSFORMATIONS ON EUCLIDEAN JORDAN ALGEBRAS*

Jiyuan Tao

Abstract: In this paper, we introduce the concepts of w-P and w-uniqueness properties for a linear transformation defined on a Euclidean Jordan algebra V and study some interconnections between these concepts. We also specialize them to the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices with the semidefinite cone $\mathcal{S}^n_+$ and to the space R^n with the Lorentz cone $\mathcal{L}^n_+$.

Key words: *Euclidean Jordan algebra, P-property, w-P property, column sufficient property, column competent property, complementarity problem, Z-property, w-uniqueness*

Mathematics Subject Classification: *90C33, 17C20, 17C55*

1 Introduction

For a given matrix $M \in R^{n\times n}$ and a vector $q \in R^n$, the linear complementarity problem [2], LCP(M, q), is to find a vector $z \in R^n$ such that

$$z \geq 0, \quad w := Mz + q \geq 0, \quad \textit{and} \quad \langle z, w\rangle = 0. \tag{1.1}$$

This problem has been well studied in the literature [2], [3].

A pair of vectors (w, z) satisfying (1.1) is called a *solution* of the LCP(M, q). A vector z is called a z-solution if there exists a vector w such that (w, z) is solution of the LCP(M, q). Similarly, A vector w is called a w-solution if there exists a vector z such that (w, z) is solution of the LCP(M, q).

In [5], Fiedler and Pták introduced the notion of P-matrix: A real square matrix M is a P-matrix if all principal minors of M are positive. It is well known (see [2]) that the P-matrix property can be equivalently described by the following condition:

$$x \in R^n, \quad x * Mx \leq 0 \Rightarrow x = 0,$$

where the asterisk denotes the componentwise product. The equivalence of the P-matrix property and the existence of a unique z-solution for all linear complementarity problems LCP(M, q) was established in Murty [14]. Motivated by a study of dynamical systems

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subject to smooth unilateral constraints, Ingleton [12] studied the uniqueness of w-solutions to LCP problems, and showed that the LCP(M, q) has the uniqueness of w-solutions for all $q \in R^n$ if and only if the following condition holds:

$$z * Mz \leq 0 \Rightarrow Mz = 0. \tag{1.2}$$

We may call a matrix M satisfying (1.2) as w-P matrix. We note that P-matrix and w-P matrix concepts coincide when M is invertible.

Generalizing the P-property of a matrix, in [10], Gowda, Sznajder and Tao introduced and studied P and globally uniquely solvable (GUS) properties for linear transformations on Euclidean Jordan algebras. Motivated by these results, as a counterpart of P-matrix and z-uniqueness, we generalize the w-P property of a matrix and study w-P and w-uniqueness properties for linear transformations on Euclidean Jordan algebras in this paper.

Here is an outline of the paper. In Section 2, we cover the basic material dealing with the complementarity properties and Euclidean Jordan algebras. In Section 3, we introduce the Order w-P property, the Jordan w-P property, and the w-P property, and study some interconnections between them. In Section 4, we describe the w-P and the w-uniqueness properties. In Section 5, we specialize the w-P and the w-uniqueness properties for Lyapunov transformations L_A defined by $L_A(X) := AX + XA^T$ for a real $n \times n$ matrix A on the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices. In Section 6, we describe the w-P and the w-uniqueness properties for Lyapunov-like transformations. In Section 7, we specialize the w-P property for Stein transformations defined by $S_A(X) := X - AXA^T$ on $\mathcal{S}^n$. In Section 8, we specialize our results to symmetric linear transformations, to monotone transformations, to polyhedral cones and finally to algebra automorphisms on the Lorentz cone $\mathcal{L}^n_+$. In Section 9, we study the column competence property of linear transformations defined on V, and give some interconnections between the column competence property and finiteness of w-solutions of the LCP(L, q) for $q \in V$.

2 Preliminaries

2.1 Euclidean Jordan Algebras

In this subsection, we recall some concepts, properties, and results from Euclidean Jordan algebras. Most of these can be found in Refs. [4], [10], [16].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$ where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions for all x and y: $x \circ y = y \circ x$, $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 := x \circ x$, and $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$. In addition, an element $e \in V$ is called the *unit* element if $x \circ e = x$ for all $x \in V$. Henceforth, V denotes a Euclidean Jordan algebra.

In V, the set of squares

$$K := \{x^2 : x \in V\}$$

is a *symmetric cone* ([4], page 46). This means that K is a self-dual closed convex cone and for any two elements $x, y \in K^o$(=interior (K)), there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$. We defined

$$z^+ := \Pi_K(z) \qquad \text{and} \qquad z^- := z^+ - z,$$

where $\Pi_K(z)$ denotes the (orthogonal) projection of z onto K. Finally, for any two elements $x, y \in V$, we let

$$x \sqcap y := x - (x - y)^+ \quad \text{and} \quad x \sqcup y := y + (x - y)^+.$$

For an element $z \in V$, we write

$$z \geq 0 \quad (z > 0) \quad \text{if and only if} \quad z \in K \quad (z \in K^o),$$

and $z \leq 0$ $(z < 0)$ when $-z \geq 0$ $(z > 0)$.

An element $c \in V$ such that $c^2 = c$ is called an *idempotent* in V; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j, \text{ and } \sum_1^m e_i = e.$$

Given $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{2.1}$$

The numbers λ_i are called the eigenvalues of x, and the representation (2.1) is called the *spectral decomposition (or the spectral expansion)* of x.

Given (2.1), we have

$$x = \sum_1^r {\lambda_i}^+ e_i - \sum_1^r {\lambda_i}^- e_i \qquad \text{and} \qquad \langle \sum_1^r {\lambda_i}^+ e_i, \sum_1^r {\lambda_i}^- e_i \rangle = 0,$$

where for a real number α, $\alpha^+ := \max\{0, \alpha\}$ and $\alpha^- := (\alpha)^+ - \alpha$.

From this we easily verify that

$$x^+ = \sum_1^r {\lambda_i}^+ e_i \qquad \text{and} \qquad x^- = \sum_1^r {\lambda_i}^- e_i,$$

and so

$$x = x^+ - x^- \qquad \text{with} \qquad \langle x^+, x^- \rangle = 0.$$

For an $x \in V$, a linear transformation $L_x : V \to V$ is defined by $L_x(z) = x \circ z$, for all $z \in V$. We say that two elements x and y *operator commute* if $L_x L_y = L_y L_x$.

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Lemma X.2.2, Faraut and Korányi [4]).

Here are some standard examples.

Example 2.1. R^n is a Euclidean Jordan algebra with inner product and Jordan product defined respectively by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad x \circ y = x * y.$$

Here R^n_+ is the corresponding symmetric cone.

Example 2.2. $\mathcal{S}^n$, the set of all $n \times n$ real symmetric matrices, is a Euclidean Jordan algebra with the inner and Jordan product given by

$$\langle X, Y\rangle := \text{trace}(XY) \quad \text{and} \quad X \circ Y := \frac{1}{2}(XY + YX).$$

In this setting, the symmetric cone $\mathcal{S}^n_+$ is the set of all positive semidefinite matrices in $\mathcal{S}^n$. Also, X and Y operator commute if and only if $XY = YX$.

Example 2.3. Consider R^n $(n > 1)$ where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix}$$

with $x_0 \in R$ and $\overline{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \overline{y} \end{bmatrix} := \begin{bmatrix} \langle x, y\rangle \\ x_0\overline{y} + y_0\overline{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle\cdot,\cdot\rangle)$ by $\mathcal{L}^n$. In this algebra, the cone of squares, denoted by $\mathcal{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathcal{L}^n_+ = \{x : ||\overline{x}|| \leq x_0\}.$$

The unit element in $\mathcal{L}^n$ is $e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We note the spectral decomposition of any x with $\overline{x} \neq 0$:

$$x = \lambda_1 e_1 + \lambda_2 e_2$$

where

$$\lambda_1 := x_0 + ||\overline{x}||, \ \lambda_2 := x_0 - ||\overline{x}||$$

and

$$e_1 := \frac{1}{2}\begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix}, \text{ and } e_2 := \frac{1}{2}\begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix}.$$

In this setting, x and y operator commute if and only if either $\overline{y}$ is a multiple of $\overline{x}$ or $\overline{x}$ is a multiple of $\overline{y}$.

We recall the following propositions from Gowda, Sznajder and Tao (see [10]):

Proposition 2.1. *For x, $y \in V$, the following conditions are equivalent:*

(i) $x \geq 0$, $y \geq 0$, *and* $\langle x, y\rangle = 0$.

(ii) $x \geq 0$, $y \geq 0$, *and* $x \circ y = 0$.

In each case, the elements x and y operator commute.

Proposition 2.2. *For x, $y \in V$, consider the following statements:*

(i) *x and y operator commute and $x \circ y \leq 0$.*

(ii) *$x \circ y \leq 0$.*

(iii) *$x \sqcap y \leq 0 \leq x \sqcup y$.*

(iv) *$\langle x, y \rangle \leq 0$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Peirce Decomposition Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R\, e_i \text{ (where R is the set of all real numbers)}$$

and when $i \neq j$,

$$V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}.$$

Then, we have the following theorem

Theorem 2.3 (see [4], Theorem IV.2.1)). *The space V is the orthogonal direct sum of the spaces V_{ij} $(i \leq j)$. Furthermore,*

$$\begin{array}{l} V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj} \\ V_{ij} \circ V_{jk} \subset V_{ik} \textit{ if } i \neq k \\ V_{ij} \circ V_{kl} = \{0\} \textit{ if } \{i, j\} \cap \{k, l\} = \emptyset. \end{array}$$

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two Euclidean Jordan algebras. The classification theorem (See Faraut and Korányi, Chapter V, [4]) says that every simple Euclidean Jordan algebra is isomorphic to one of the algebras below:

(i) the algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices;

(ii) The algebra $\mathcal{L}^n$;

(iii) The algebra $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$;

(iv) The algebra $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$;

(v) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

Theorem 2.4 (See Faraut and Korányi, Prop. III.4.4, Prop. III.4.5, Thm. V.3.7, [4]). *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

2.2 Linear Complementarity Concepts

Throughout this paper, we assume that V is an Euclidean Jordan algebra with the corresponding symmetric cone K and $L : V \to V$ is a linear transformation. Given L on V and $q \in V$, the linear complementarity problem, LCP(L, q), is to find an $z \in V$ such that

$$z \in K,\ w := L(z) + q \in K,\ \text{ and } \langle z, w\rangle = 0. \tag{2.2}$$

A pair of elements (w, z) satisfying (2.2) is called a *solution* of the LCP(L, q). A vector z is called a z-solution if there exists a vector w such that (w, z) is solution of the LCP(L, q). Similarly, A vector w is called a w-solution if there exists a vector z such that (w, z) is solution of the LCP(L, q).

This problem is a particular case of a variational inequality problem [3]. Given L on V, we say that L has/is:

(a) monotone (strictly = strongly) if $\langle L(x), x\rangle \geq 0$ (respectively, > 0) for any $0 \neq x \in V$;

(b) the GUS (globally uniquely solvable) property on V if LCP(L, q) has a unique z-solution for all $q \in V$;

(c) has the Order P-property if

$$x \sqcap L(x) \leq 0 \leq x \sqcup L(x) \Rightarrow x = 0,$$

where $x \sqcap L(x) := x - (x - L(x))^+$ and $x \sqcup L(x) := L(x) + (x - L(x))^+$;

(d) the Jordan P-property if

$$x \circ L(x) \leq 0 \Rightarrow x = 0;$$

(e) the P-property if

$$\left.\begin{array}{r} x \quad \textit{and} \quad L(x) \quad \textit{operator} \quad \textit{commute} \\ x \circ L(x) \leq 0 \end{array}\right\} \Rightarrow x = 0;$$

(f) the P_0-property if $L + \epsilon I$ has the P-property for every $\epsilon > 0$;

(g) nondegenerate if

$$\left.\begin{array}{r} x \quad \textit{and} \quad L(x) \quad \textit{operator} \quad \textit{commute} \\ x \circ L(x) = 0 \end{array}\right\} \Rightarrow x = 0;$$

(h) the Q-property if LCP(L, q) has a solution for all $q \in V$.

These properties have been well studied, see e.g., [7], [10]. In particular, we always have the implications $(a)(strictly) \Rightarrow (b) \Rightarrow (e) \Rightarrow (f)$ and $(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (g)$, $(e) \Rightarrow (h)$.

Definition 2.5 (see Definition 13, [10]). L is said to have the *Cross Commutative* property if for any $q \in V$ and for any two solutions x_1 and x_2 of LCP(L, q), x_1 operator commutes with y_2 and x_2 operator commutes with y_1, where $y_i = L(x_i) + q\,(i = 1, 2)$.

Now we introduce the following definition.

Definition 2.6. Given L on V, let $K(L)$ denote the set of all $q \in V$ for which SOL$(L, q) \neq \emptyset$, where SOL(L, q) denotes the z-solution set of LCP(L, q). $K(L)$ is closed in standard LCP problems, but not necessarily closed in general setting (see Example 2.5.14, [3]). We say that L:

(i) has the w-uniqueness property if for any $q \in K(L)$, $L(x_1) = L(x_2)$ whenever x_1 and x_2 are two z-solutions of LCP(L, q);

(ii) is column sufficient if

$$\left.\begin{array}{r} x \quad \textit{and} \quad L(x) \quad \textit{operator} \quad \textit{commute} \\ x \circ L(x) \leq 0 \end{array}\right\} \Rightarrow x \circ L(x) = 0;$$

(iii) is column competent if

$$\left.\begin{array}{r} x \quad \textit{and} \quad L(x) \quad \textit{operator} \quad \textit{commute} \\ x \circ L(x) = 0 \end{array}\right\} \Rightarrow L(x) = 0.$$

We note that the *column sufficient* property was introduced for standard LCP problems in [2], it is equivalent to the convexity of SOL(L, q). Gowda and Song [7] defined the *column sufficient* property as the convexity of SOL(L, q) when $V = \mathcal{S}^n$. Recently, Qin, Kong and Han [15] extended the *column sufficient* property from R^n to the setting of Euclidean Jordan algebras, and they showed that the *column sufficient* property with the *Cross Commutative* property is equivalent to the convexity of SOL(L, q); *column competent* was introduced for LCP in [19].

3 Order w-P, Jordan w-P, and w-P Properties

Motivated by implications (see [10]) of

$$\text{Order}\, P \Rightarrow \text{Jordan}\, P \Rightarrow P,$$

in this section, we ask if analogous implications hold for w-P properties.

Definition 3.1. Given L on V, we say that L has:

(i) the Order w-P property if

$$x \sqcap L(x) \leq 0 \leq x \sqcup L(x) \Rightarrow L(x) = 0;$$

(ii) the Jordan w-P property if

$$x \circ L(x) \leq 0 \Rightarrow L(x) = 0;$$

(iii) the w-P property if

$$\left.\begin{array}{r} x \quad and \quad L(x) \quad operator \quad commute \\ x \circ L(x) \leq 0 \end{array}\right\} \Rightarrow L(x) = 0.$$

Theorem 3.2. *Given L on V, we have Order w-P $\Rightarrow$ Jordan w-P $\Rightarrow$ w-P. Moreover, if L has the w-P property, then every real eigenvalue of L is nonnegative and the determinant of L is nonnegative.*

Proof. The implications follow immediately from Proposition 2.2. Now suppose that L has the w-P property. If λ is a real, negative eigenvalue of L, then there exists a nonzero $u \in V$ such that $L(u) = \lambda u$. It follows that u and $L(u)$ operator commute and $u \circ L(u) = \lambda u^2 \leq 0$. Then we have $L(u) = 0$ by the w-P property of L, hence $\lambda u = 0 \Rightarrow u = 0$. This is a contradiction. Therefore all real eigenvalues of L are nonnegative. It follows that the determinant of L (being the products of all eigenvalues) is also nonnegative. □

Remark 3.3. (1) Zero transformation always satisfies the above properties.

(2) When L is invertible, the above properties reduce to the Order P, the Jordan P and the P properties.

In standard LCP theory, the Jordan w-P property is the same as the w-P property. However, as the following example shows, this result is not necessarily true on a Euclidean Jordan algebra.

Example 3.1. Let $L = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$.

Then L has the w-P property but not the Jordan w-P property. The justification is given in Appendix.

$\boxed{4}$ w-Uniqueness and w-P Properties

In this section, we study the w-uniqueness, the w-P properties and some other properties related to these two properties.

Lemma 4.1. *If L is monotone, then L has the Cross Commutative property.*

Proof. For a given $q \in K(L)$, let x_1 and x_2 be two solutions of LCP(L, q). Letting $x := x_1 - x_2$ and $y_i = L(x_i) + q\,(i = 1, 2)$, we see that $x \circ L(x) = -[x_1 \circ y_2 + x_2 \circ y_1]$. Since L is monotone, thus we have

$$\begin{aligned} 0 &\leq \langle x, L(x)\rangle = \langle x \circ L(x), e\rangle \\ &= \langle -[x_1 \circ y_2 + x_2 \circ y_1], e\rangle \\ &= -[\langle x_1, y_2\rangle + \langle x_2, y_1\rangle] \leq 0. \end{aligned}$$

It follows that $\langle x_1, y_2\rangle = \langle x_2, y_1\rangle = 0$. By Proposition 2.1, x_1 (x_2) operator commutes with y_2 (respectively, y_1). □

Theorem 4.2. *Given L on V, w-uniqueness = w-P + Cross Commutative.*

Proof. Suppose that L has the w-uniqueness property. Let x and $L(x)$ operator commute, and $x \circ L(x) \leq 0$. Then there exists a Jordan frame $\{e_1, \dots, e_r\}$ such that $x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$. From $x \circ L(x) \leq 0$, we have $\sum x_i y_i e_i \leq 0$. It follows that $x_i y_i \leq 0$ for all i. This implies that ${x_i}^+ {y_i}^+ = {x_i}^- {y_i}^- = 0$ for all i. Thus we have $x^+ \circ [L(x)]^+ = x^- \circ [L(x)]^- = 0$. Now define $q := [L(x)]^+ - L(x^+)$. We see that $q = [L(x)]^- - L(x^-)$. Obviously x^+ and x^- are two solutions of LCP(L, q). Thus $L(x^+) = L(x^-) \Rightarrow L(x^+ - x^-) = 0 \Rightarrow L(x) = 0$. This proves the w-P property. By the w-uniqueness of solution, the cross commutative property is obvious.

Now for the converse. Suppose L has the w-P and the cross commutative properties. For any $q \in K(L)$, let x_1 and x_2 be two solutions of LCP(L, q) and $y_i = L(x_i) + q$ $(i = 1, 2)$. Since x_1 operator commutes with y_2, it follows that $x_1 \circ y_2 \geq 0$. Similarly, $x_2 \circ y_1 \geq 0$. Now $x_1 - x_2$ operator commutes with $L(x_1 - x_2) = y_1 - y_2$ and $(x_1 - x_2) \circ L(x_1 - x_2) = -[x_1 \circ y_2 + x_2 \circ y_1] \leq 0$. By the w-P property, $L(x_1) = L(x_2)$. This argument shows that L has the w-uniqueness property. $\square$

In standard LCP problems, the w-uniqueness and w-P properties coincide (see Theorem 3.4.4, [2]). However, the following example shows that the w-P property does not imply the w-uniqueness property.

Example 4.1. When $V = \mathcal{S}^n$, for a real $n \times n$ matrix A, the *Lyapunov transformation* is defined by

$$L_A(X) := AX + XA^T.$$

Now consider the matrices $A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$. Then A is positive stable (every eigenvalue of A has positive real part) and Q is positive definite. Thus L_A has the P-property (see Theorem 5, [7]). Hence L_A has the w-P property. It can be easily verified that zero matrix and $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are two solutions of LCP(L_A, Q). However $L_A(X) = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix} \neq L_A(0)$.

As we see from Theorem 4.2, the w-uniqueness property is tied to the w-P property and the cross commutative property. However, we do not know how to describe, apart from the definition, the cross commutative property. In what follows, we give a necessary condition for the w-uniqueness property.

It has been observed in Theorem 4.1 of [18] that if L has the GUS-property on V, then $\langle L(c), c \rangle \geq 0$ for any primitive idempotent $c \in V$. By modifying the proof, we get the following:

Theorem 4.3. *If L has the w-uniqueness property , then for any primitive idempotent $c \in V$, $\langle L(c), c \rangle \geq 0$.*

As an illustration of Theorem 4.3, we provide the following examples.

Example 4.2. When $V = \mathcal{S}^n$, for a real $n \times n$ matrix A, L_A is defined in Example 4.1. It can be easily verified that $\langle L_A(c), c \rangle \geq 0$ for all primitive idempotents c on $\mathcal{S}^n_+$ if and only if A is positive semidefinite.

Example 4.3. When $V = \mathcal{S}^n$, for a real $n \times n$ matrix A, the *Stein transformation* is defined by

$$S_A(X) = X - AXA^T.$$

It can be easily verified that $\langle S_A(c), c\rangle \geq 0$ for all primitive idempotents c on $\mathcal{S}^n_+$ if and only if $I \pm A$ are positive semidefinite, where I is the identity matrix.

In what follows, we give some interconnections between the concepts introduced in Section 2.2.

Lemma 4.4. *Given L on V, consider the following statements:*

(a) *L has the w-uniqueness property.*

(b) *L has the w-P property.*

(c) *L has the column sufficiency property.*

(d) *L has the P_0-property.*

(e) *L has the column competence property.*

(f) *$L + \epsilon I$ has the P-property for all $\epsilon > 0$.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (b) $\Rightarrow$ (e), (b) $\Rightarrow$ (f).

Proof. The implication (a) $\Rightarrow$ (b) follows from Theorem 4.2.
The implication (b) $\Rightarrow$ (c) is obvious.
(c) $\Rightarrow$ (d): Suppose that x operator commutes with $(L + \epsilon I)(x)$ for every $\epsilon > 0$ and $x \circ (L + \epsilon I)(x) \leq 0$. Then $x \circ (L + \epsilon I)(x) \leq 0 \Rightarrow x \circ L(x) \leq -\epsilon x^2 \leq 0$ and x and $L(x)$ operator commute. Thus $x \circ L(x) = 0$ by the column sufficiency property of L. Hence $x \circ (L + \epsilon I)(x) \leq 0 \Rightarrow \epsilon x^2 \leq 0 \Rightarrow x = 0$. Therefore L has the P_0-property.
The implication (b) $\Rightarrow$ (e) is obvious.
(b) $\Rightarrow$ (f): Suppose that x operator commutes with $(L + \epsilon I)(x)$ for every $\epsilon > 0$ and $x \circ (L + \epsilon I)(x) \leq 0$. Then $x \circ (L + \epsilon I)(x) \leq 0 \Rightarrow x \circ L(x) \leq -\epsilon x^2 \leq 0$ and x and $L(x)$ operator commute. Thus $L(x) = 0$ by the w-P property of L. Hence $x \circ (L + \epsilon I)(x) \leq 0 \Rightarrow \epsilon x^2 \leq 0 \Rightarrow x = 0$. Therefore $L + \epsilon I$ has the P-property. ☐

Remark 4.5. In general, the P_0-property does not imply the w-P property even in standard LCP problems: An example is $M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

In the next section, we specialize the w-uniqueness and the w-P properties to Lyapunov transformations defined on $\mathcal{S}^n$.

$\boxed{5}$ The w-Uniqueness and w-P Properties for Lyapunov Transformations

It is well known (see [7]) that L_A has the P-property if and only if A is positive stable and L_A has the GUS-property if and only if A is positive stable and semidefinite. As the counterpart of the P and the GUS properties, in this section, we give a characterization of the w-P property and the w-uniqueness property for L_A.

Theorem 5.1. *For $A \in R^{n\times n}$, consider L_A on $\mathcal{S}^n$. Then L_A has the w-P property if and only if A is semipositive stable, i.e., all eigenvalues of A lie in the closed right half-plane.*

Proof. The technique used here is similar to the proof of Theorem 5 in [7]. Since the w-P property implies the P_0-property by Lemma 4.4, and

$$(L_A + \epsilon I)(X) = AX + XA^T + \epsilon X = L_{A+\frac{\epsilon}{2}I}(X),$$

we see that L_A has the P_0- property if and only A is semipositive stable. Thus the w-P property of L_A implies that A is semipositive stable. Therefore we only need to show the "if" part. Let A is semipositive stable and suppose that L_A does not have the w-P property. Then there is a nonzero X which commutes with nonzero $L_A(X)$ and $X \circ L_A(X) \preceq 0$. Because of commutativity, we can write

$$X = U^T DU, \quad L_A(X) = U^T EU \quad \text{and} \quad B = UAU^T,$$

where U is some orthogonal matrix and D and E are diagonal matrices. Then we have $E = BD + DB^T$ and $DE \preceq 0$. Without loss of generality, we can write

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \; B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \text{ and } E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix},$$

where D_1 is invertible and the sizes of B_1 and E_1 agree with that of D_1. Then from $E = BD + DB^T$, we have

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} B_1D_1 + D_1{B_1}^T & D_1{B_3}^T \\ B_3D_1 & 0 \end{bmatrix}.$$

We see that $E_2 = 0$ and $B_3D_1 = 0 \Rightarrow B_3 = 0$. Then every eigenvalue of B_1 is an eigenvalue of B. Since B is semipositive stable, B_1 is also semipositive stable, thus $tr(B_1) \geq 0$. From $E_1 = B_1D_1 + D_1{B_1}^T$, we have the ith diagonal entry of B_1 as $\frac{1}{2}e_{ii}\frac{1}{d_{ii}}$ where e_{ii} and d_{ii} denote the ith diagonal entries of E_1 and D_1 respectively. Then from $DE \preceq 0$, we have $\frac{1}{2}e_{ii}\frac{1}{d_{ii}} \leq 0 \Rightarrow tr(B_1) \leq 0$. Therefore, $tr(B_1) = 0 \Rightarrow \frac{1}{2}e_{ii}\frac{1}{d_{ii}} = 0$ for all i; this implies that $e_{ii} = 0$ for all i, thus $E_1 = 0$. Hence $L_A(X) = 0$ contradicts our assumption. Therefore, the "if" part holds. $\square$

Lemma 4.4 and Theorem 5.1 immediately yield the following corollary.

Corollary 5.2. *For $A \in R^{n\times n}$, the following statements are equivalent:*

(a) *A is semipositive stable.*

(b) *L_A has the w-P property.*

(c) *L_A has the column sufficiency property.*

(d) *L_A has the P_0-property.*

Remark 5.3. Since L_A has the Q-property if and only if L_A has the P-property (see Theorem 5, [7]), the w-P property does not imply the Q-property.

Theorem 5.4. *For $A \in R^{n\times n}$, the following statements are equivalent:*

(a) *L_A has the w-uniqueness property.*

(b) *A is semipositive stable and positive semidefinite.*

Proof. (a) $\Rightarrow$ (b): If L_A has the w-uniqueness property, then L_A has the w-P property by Theorem 4.2. Hence A is semipositive stable by Theorem 5.1. A is positive semidefinite following from Theorem 4.3 and Example 4.2.
(b) $\Rightarrow$ (a): If A is positive semidefinite, then L_A is monotone. Thus L_A has the cross commutative property by Lemma 4.1. Therefore (a) holds by Corollary 5.2 and Theorem 4.2. $\square$

$\boxed{6}$ The w-Uniqueness and w-P Properties for Lyapunov-like Transformations

Motivated by the equivalence between the w-P property together with positive semidefiniteness and the w-uniqueness property for L_A on $\mathcal{S}^n$, one may ask if this equivalence holds on symmetric cones for transformations that are similar to a *Lyapunov transformation.* Below, we will provide an answer to this question in the positive.

We say that L has the Z-property if

$$x, y \in K, \quad \text{and } \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

Recently, Gowda and Tao ([11]) introduced and studied the properties of such transformations.

Remark 6.1. It can be easily verified that L_A and S_A have the Z-property on $\mathcal{S}^n_+$.

Definition 6.2. see ([9]) Given L on V, it is said to be a Lyapunov-like transformation if both L and $-L$ have the Z-property, that is,

$$x, y \geq 0, \ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

Theorem 6.3. *Let L be a Lyapunov-like transformation. Then the following are equivalent:*

(a) *L has the w-P property.*

(b) *L has the column sufficiency property.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious.
(b) $\Rightarrow$ (a): Suppose x and $L(x)$ operator commute and $x \circ L(x) \leq 0$. Then $x \circ L(x) = 0$ by the column sufficiency property. Without loss of generality, we can write

$$x = \sum_{1}^{k} x_i e_i \qquad \text{and} \qquad L(x) = \sum_{k+1}^{r} y_i e_i,$$

where $\{e_1, \ldots, e_r\}$ is a Jordan frame. Then we have $L(x) = \sum_1^k x_i L(e_i)$. Thus $\langle L(x), L(x)\rangle = \langle \sum_1^k x_i L(e_i), \sum_{k+1}^r y_i e_i\rangle = \sum_{i<j} x_i y_j \langle L(e_i), e_j\rangle = 0$. The last equality holds because L is Lyapunov-like transformation. Thus $L(x) = 0$. Therefore, L has the w-P property. $\square$

Theorem 6.4. *Let L be a Lyapunov-like transformation. Then the following are equivalent:*

(a) *L has the w-uniqueness property.*

(b) *$\langle L(c), c\rangle \geq 0$ for any primitive idempotent $c \in V$.*

Proof. The implication (a)$\Rightarrow$ (b) follows from Theorem 4.3.
(b)$\Rightarrow$ (a): Fix $q \in K(L)$ and suppose that x_1 and x_2 are two solutions of LCP(L, q) so that

$$x_i \geq 0,\ y_i = L(x_i) + q \geq 0,\ \text{and } \langle x_i, y_i\rangle = 0\,(i = 1, 2).$$

Now if (b) holds, then L is monotone by Theorem 7.1 in [9]. Thus the solution set of LCP(L, q) is convex by Theorem 2.3.5, [3]. Therefore for any $t \in [0, 1]$, $tx_1 + (1 - t)x_2$ is also a solution of LCP(L, q). Writing out the complementarity conditions, we have

$$\langle x_1, y_2\rangle = 0 = \langle x_2, y_1\rangle.$$

Hence we have that x_1 and x_2 operator commute with both y_1 and y_2 by Proposition 2.1, and $x_1 \circ y_2 = 0 = x_2 \circ y_1$; hence $z := x_1 - x_2$ operator commutes with $y_1 - y_2 = L(x_1 - x_2) = L(z)$, and $z \circ L(z) = 0$. Without loss of generality, we may assume that there exists a Jordan $\{e_1, \ldots, e_r\}$ such that

$$z = \sum_{1}^{k} \lambda_i e_i \text{ and } L(z) = \sum_{k+1}^{r} \mu_i e_i.$$

Thus $L(z) = \sum_1^k \lambda_i L(e_i) = \sum_{k+1}^r \mu_i e_i$. Hence $\langle L(z), L(z)\rangle = \langle \sum_1^k \lambda_i L(e_i), \sum_{k+1}^r \mu_i e_i\rangle = \sum_{i<j} \lambda_i \mu_j \langle L(e_i), e_j\rangle = 0$. The last equality holds because L is Lyapunov-like transformation. This implies that $L(z) = 0$. Therefore L has the w-uniqueness property. $\square$

In the next section, we specialize the w-P property to Stein transformations defined on $\mathcal{S}^n$.

$\boxed{7}$ The w-P Property for Stein Transformations

It is well known (see [6]) that S_A has the P- property if and only if A is Schur stable (all eigenvalues of A lie in the open unit disk). As the counterpart of the P-property, in this section, we give a characterization of the w-P property for S_A.

Theorem 7.1. *For $A \in R^{n\times n}$ and the corresponding S_A. Then S_A has the w-P property if and only if $\rho(A) \leq 1$, i.e., all eigenvalues of A lie in the closed unit disk.*

Proof. The technique used here is similar to the proof of Theorem 11 in [6]. Since the w-P property implies P_0-property by Lemma 4.4, also

$$(S_A + \epsilon I)(X) = X - AXA^T + \epsilon X = (1 + \epsilon)S_{\frac{1}{\sqrt{1+\epsilon}}A}(X),$$

we see that S_A has the P_0-property if and only all eigenvalues of A lie in the closed unit disk. Thus the w-P property of S_A implies that $\rho(A) \leq 1$. Therefore we only need to show the "if" part. Let $\rho(A) \leq 1$ and suppose that S_A does not have the w-P property. Then there is a nonzero X which commutes with nonzero $S_A(X)$ and $X \circ S_A(X) \preceq 0$. Because of commutativity, we can write

$$X = U^T DU, \quad S_A(X) = U^T EU \quad \text{and} \quad B = UAU^T,$$

where U is some orthogonal matrix and D and E are diagonal matrices. Then we have $E = D - BDB^T$, $DE \preceq 0$, and $\rho(B) \leq 1$.
Case 1: D is invertible. Then

$$D^{-1}E = I - D^{-1}BDB^T \preceq 0. \tag{7.1}$$

Since $D^{-1}E$ is diagonal matrix, $D^{-1}BDB^T$ is symmetric. Now we claim that every eigenvalue of $D^{-1}BDB^T$ is equal to one. Let λ be an eigenvalue of $D^{-1}BDB^T$ and a nonzero vector u such that $(D^{-1}BDB^T)u = \lambda u$. From (7.1), we get $||u||^2 - \lambda||u||^2 \leq 0 \Rightarrow \lambda \geq 1$. Since $\det(D^{-1}BDB^T) \leq 1$, we have that every eigenvalue of $D^{-1}BDB^T$ is equal to 1. Again from (7.1), we get every eigenvalue of $D^{-1}E$ is zero. Since D is invertible, we have every eigenvalue of E is zero. Thus $E = 0$. Hence $S_A(X) = 0$.
Case 2: D is not invertible. Without loss of generality, we can write

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \; B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \text{ and } E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix},$$

where D_1 is invertible and $D_1E_1 \preceq 0$. Then from $E = D - BDB^T$, we have

$$\begin{aligned} E &= \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} D_1 - B_1D_1{B_1}^T & -B_1D_1{B_3}^T \\ -B_3D_1{B_1}^T & -B_3D_1{B_3}^T \end{bmatrix} \Rightarrow \\ E_1 &= D_1 - B_1D_1{B_1}^T,\, B_1D_1{B_3}^T = 0 \,\text{and}\, E_2 = -B_3D_1{B_3}^T. \end{aligned} \tag{7.2}$$

We claim that B_1 is invertible. If not, let ${B_1}^T u = 0$ for some nonzero u. Then $E_1u = D_1u$ and so $0 < \langle {D_1}^2u, u\rangle = \langle D_1E_1u, u\rangle \leq 0$ leading to a contradiction. Hence B_1 is invertible. Therefore $B_3 = 0$. It follows (from $\rho(B) \leq 1$) that $\rho(B_1) \leq 1$ and $E_2 = 0$. This, together with (7.2), as in the Case 1, leads to $E_1 = 0$. Hence $S_A(X) = 0$. Thus if $\rho(A) \leq 1$, then S_A has the w-P property. □

Lemma 4.4 and Theorem 7.1 immediately yield the following corollary.

Corollary 7.2. *For $A \in R^{n\times n}$ and the corresponding S_A. Then the following statements are equivalent:*

(a) $\rho(A) \leq 1$.

(b) *S_A has the w-P property.*

(c) *S_A has the column sufficiency property.*

(d) *S_A has the P_0-property.*

8 Some Special Cases

In this section, we study some special cases of the linear transformations, in particular, L is monotone and/or self-adjoint.

Lemma 8.1. *Suppose L is self-adjoint and monotone. Then L has the w-uniqueness property.*

Proof. For any $q \in K(L)$, let $y_0 = L(x_0) + q$ and $y = L(x) + q$, where x_0 is a given solution and x an arbitrary solution. Then $y - y_0 = L(x - x_0)$. Since L is monotone, we have that L has the cross commutative property by Lemma 4.1. Thus $(x - x_0) \circ L(x - x_0) = -[x \circ y_0 + x_0 \circ y] \leq 0 \Rightarrow (x - x_0)^T L(x - x_0) \leq 0 \Rightarrow (x - x_0)^T L(x - x_0) = 0$. The last equality follows by monotonicity of L. Since L is self-adjoint and monotone, we have $L(x - x_0) = 0$. Hence $L(x) = L(x_0)$. □

Theorem 8.2. *Let L be self-adjoint. Then the following are equivalent:*

(a) *L is monotone.*

(b) *L has the Order w-P property.*

(c) *L has the Jordan w-P property.*

(d) *L has the w-P property.*

(e) *L has the w-uniqueness property.*

Proof. (a) $\Rightarrow$ (b): Suppose $x \sqcap L(x) \leq 0 \leq x \sqcup L(x)$. Then $\langle x, L(x)\rangle \leq 0$ by Proposition 2.2. Since L is monotone, we have $\langle x, L(x)\rangle = 0$. Since L is self-adjoint and monotone, we have $L(x) = 0$.
The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) follow from Theorem 3.2.
Since (a) $\Rightarrow$ (e) by Lemma 8.1 and (e) $\Rightarrow$ (d) by Theorem 4.2, we only need to show that (d) $\Rightarrow$ (a). Suppose (d) holds, since L is self-adjoint, all eigenvalues of L are real. We claim that all eigenvalues of L are nonnegative. Suppose not, then there exists a eigenvalue $\lambda < 0$ and corresponding eigenvector u, such that $L(u) = \lambda u$. Thus we have $u \circ L(u) = \lambda u^2 \leq 0$, this implies that $L(u) = 0$ by the w-P property of L, Thus $\lambda u = 0 \Rightarrow u = 0$, this is a contradiction. Hence the claim is true. Therefore L is monotone. □

Corollary 8.3. *Given any element a in V, the quadratic representation of a is defined by $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$. Then the following are equivalent:*

(a) *P_a is positive semidefinite on V.*

(b) *P_a has the w-uniqueness property.*

(c) *P_a has the w-P property.*

If, in addition, V is simple, then the above conditions are further equivalent to

(d) $\pm a \in K$.

Proof. Since P_a is self-adjoint, we only need to show (a) is equivalent to (d) when V is simple. For a given $a \in V$, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that

$$a = a_1 e_1 + a_2 e_2 + \ldots + a_r e_r.$$

For any $x \in V$, write the Peirce decomposition of x with respect to this Jordan frame as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

(with $x_i \in R$ and $x_{ij} \in V_{ij}$). Then it can be easily verified that

$$P_a(x) = \sum_{i=1}^{r} {a_i}^2 x_i e_i + \sum_{i<j} a_i a_j x_{ij}.$$

When V is simple, V_{ij} is nonzero for each $i \leq j$ (see Corollary IV.2.4 in [4]), so we have

$$\begin{aligned} 0 \leq \langle x, P_a(x)\rangle &= \sum_{i=1}^{r} {a_i}^2 {x_i}^2 ||e_i||^2 + \sum_{i<j} a_i a_j ||x_{ij}||^2 \ (\forall x \in V) \\ &\Leftrightarrow a_i a_j \geq 0 \ (i \leq j) \\ &\Leftrightarrow a_i \geq 0 \text{ or } a_i \leq 0, \forall i. \end{aligned}$$

Hence P_a is positive semidefinite on V if and only if $\pm a \in K$ when V is simple. □

Remark 8.4. When $V = \mathcal{S}^n$, for a real $n \times n$ matrix A, the *two sided multiplicative transformation* is defined by

$$M_A(X) := AXA^T.$$

If we specialize P_a on $\mathcal{S}^n$, then it can be easily verified that $P_A(X) = AXA$ for $A \in \mathcal{S}^n$. Thus for M_A, the following are equivalent when A is a real symmetric square matrix.

(a) A is either positive semidefinite or negative semidefinite.

(b) M_A has the w-uniqueness property.

(c) M_A has the w-P property.

The following two theorems are modifications of Theorem 22 and Theorem 23 in [10].

Theorem 8.5. *When L is monotone,*

$$Order\, w - P = Jordan\, w - P \qquad \text{and} \qquad w - P = w - uniqueness.$$

Theorem 8.6. *When K is polyhedral,*

$$\text{Order}\, w - P = \text{Jordan}\, w - P \ = w - P = w - \text{uniqueness}.$$

Now we consider $Aut(\mathcal{L}^n)$, the set of all invertible linear transformations L such that $L(x \circ y) = L(x) \circ L(y)$ for all $x, y \in \mathcal{L}^n$. Such linear transformations are called algebra automorphisms on $\mathcal{L}^n$. In [9], Gowda and Sznajder showed that the GUS and P properties coincide for all $L \in Aut(\mathcal{L}^n)$. The w-P and P properties coincide for an $L \in Aut(\mathcal{L}^n)$ coincide as any such L is invertible. Therefore it is easy to get the following result by using Theorem 5.1 in [9].

Theorem 8.7. *For $L \in Aut(\mathcal{L}^n)$, the following are equivalent:*

(a) *L has the P-property.*

(b) *L has the GUS-property on V.*

(c) *L has the w-uniqueness property.*

(d) *L has the w-P property.*

(e) *L has the R_0-property.*

(f) *L has the Q-property.*

(g) $-1 \notin \sigma(L)$(= *spectrum of* L).

$\boxed{9}$ The Column Competence Property and the Finiteness of w-Solutions of the LCP(L, q)

In the standard LCP theory, the nondegeneracy of a matrix M and finiteness of SOL(M, q) coincide (see [2]). However, this need not be true when $V = \mathcal{S}^n$ (see [8]). In [8], Gowda and Song introduced the concepts of nondegeneracy for a linear transformation defined on $\mathcal{S}^n$ and the locally-star-like property of a solution point of an SDLCP(L, Q) for $Q \in \mathcal{S}^n$ and showed that nondegeneracy together with the locally-star-like property is equivalent to finiteness of SDLCP solution sets. In [13], Malik extended this result to any Euclidean Jordan algebra. In [19], Xu introduced the concept of the column competence for the standard LCP problems and showed that the column competence of a matrix M and finiteness of w-solutions property of LCP(L, q) coincide. Motivated by these results, in the first part of this section, we study the column competence property of L defined on V, and then we give some interconnections between the column competence property and finiteness of w-solutions property of LCP(L, q) for all $q \in V$.

Recall that L has the column competence property if

$$\left.\begin{array}{r} x \quad \textit{and} \quad L(x) \quad \textit{operator} \quad \textit{commute} \\ x \circ L(x) = 0 \end{array}\right\} \Rightarrow L(x) = 0.$$

Theorem 9.1. *If L is a Lyapunov-like transformation, then L is column competent.*

Proof. Suppose x and $L(x)$ operator commute and $x \circ L(x) = 0$. Without loss of generality, we can write

$$x = \sum_{1}^{k} x_i e_i \qquad \text{and} \qquad L(x) = \sum_{k+1}^{r} y_i e_i,$$

where $\{e_1, \ldots, e_r\}$ is a Jordan frame. Then we have $L(x) = \sum_1^k x_i L(e_i)$. Thus $\langle L(x), L(x)\rangle = \langle \sum_1^k x_i L(e_i), \sum_{k+1}^r y_i e_i\rangle = \sum_{i<j} x_i y_j \langle L(e_i), e_j\rangle = 0$. The last equality holds because L is Lyapunov-like transformation. Thus $L(x) = 0$. $\square$

Example 9.1. L_A has the column competence property for any $A \in R^{n\times n}$.

Theorem 9.2. *S_A has the column competence property for any $A \in R^{n\times n}$.*

Proof. Suppose a nonzero X which commutes with nonzero $S_A(X)$ and $X \circ S_A(X) = 0$. Then because of commutativity, we can write

$$X = U^T D U, \quad S_A(X) = U^T E U \quad \text{and} \quad B = UAU^T,$$

where U is some orthogonal matrix and D and E are diagonal matrices. Then we have $E = D - BDB^T$ and $DE = 0$. Without loss of generality, we can write

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \; B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \text{ and } E = \begin{bmatrix} 0 & 0 \\ 0 & E_2 \end{bmatrix},$$

where D_1 is invertible. Then from $E = D - BDB^T$, we have

$$E = \begin{bmatrix} 0 & 0 \\ 0 & E_2 \end{bmatrix} = \begin{bmatrix} D_1 - B_1 D_1 {B_1}^T & -B_1 D_1 {B_3}^T \\ -B_3 D_1 {B_1}^T & -B_3 D_1 {B_3}^T \end{bmatrix} \Rightarrow$$
$$D_1 = B_1 D_1 {B_1}^T,\ B_1 D_1 {B_3}^T = 0 \,\text{and}\, E_2 = -B_3 D_1 {B_3}^T.$$

We claim that B_1 is invertible. If this is not the case, let ${B_1}^T u = 0$ for some nonzero u. Then $D_1 u = B_1 D_1 {B_1}^T u = 0 \Rightarrow u = 0$ leading to a contradiction because D_1 is invertible. Hence B_1 is invertible. From $B_1 D_1 {B_3}^T = 0$, we have $B_3 = 0$. Thus $E_2 = -B_3 D_1 {B_3}^T = 0$. Therefore $S_A(X) = 0$. □

Theorem 9.3. *Suppose L is monotone. Then the following are equivalent:*

(a) *L has the w-uniqueness property.*

(b) *L has the w-P property.*

(c) *L is column competent.*

Proof. The implication (a)$\Leftrightarrow$ (b) follows from Theorem 8.5.
The implication (b)$\Rightarrow$ (c) is obvious.
(c)$\Rightarrow$ (b): Suppose that x and $L(x)$ operator commute, and $x \circ L(x) \leq 0$. We have

$$0 \geq \langle x \circ L(x),\, e\rangle = \langle x,\, L(x)\rangle \geq 0.$$

The last inequality is from monotonicity of L. Thus we have $\langle x,\, L(x)\rangle = 0$. Since x and $L(x)$ operator commute, we may write $x = \sum_1^r \lambda_i e_i$ and $L(x) = \sum_1^r \mu_i e_i$, where $\{e_1, \ldots, e_r\}$ is a Jordan frame. Then $x \circ L(x) \leq 0$ yields $\sum_1^r \lambda_i \mu_i e_i \leq 0$, which further implies $\lambda_i \mu_i \leq 0$ for all i. Now $\langle x,\, L(x)\rangle = 0$ yields $\sum_1^r \lambda_i \mu_i ||e_i||^2 = 0$. It follows that $\lambda_i \mu_i = 0$ for all i. This implies $x \circ L(x) = 0$. Since x and $L(x)$ operator commute, we have $L(x) = 0$ from the column competence property of L. Thus L has the w-P property. □

Theorem 9.4. *Let L be copositive on K (i.e., $\langle L(x), x\rangle \geq 0$ for all $x \in K$). Then L is the column competent only if L has the w-uniqueness property for all $q \in K$.*

Proof. Fix a $q \in K$, suppose that there exists $x \geq 0$, such that $y = L(x) + q \geq 0$ and $\langle x, y\rangle = 0$. Then x and y operator commute by Proposition 2.1. Since L is copositive on K, we have $\langle x, q\rangle = 0$, this implies that x and q operator commute by Proposition 2.1. Thus x and $L(x) = y - q$ operator commute, and $x \circ L(x) = 0$. By the column competence property of L, we have $L(x) = 0$. □

In view of the LCP result for column competence of a matrix mentioned at the beginning of the section, we may ask whether or not the LCP w-solution sets corresponding to a column competent transformation are finite. The following example shows that the answer is negative.

Example 9.2. In $R^{2\times 2}$, let $A = -\frac{1}{2}I$ and $Q = I$, where I is the identity matrix. Then L_A is column competent by Example 9.1. It can be easily verified that the solution set of the LCP(L_A, Q), consisting of all matrices of the form

$$\begin{bmatrix} \frac{1+\sqrt{1-4\lambda^2}}{2} & \lambda \\ \lambda & \frac{1-\sqrt{1-4\lambda^2}}{2} \end{bmatrix}$$

with λ real and $4\lambda^2 \leq 1$, is infinite. Thus the w-solution set of the LCP(L_A, Q), consisting of all matrices of the form

$$\begin{bmatrix} \frac{1-\sqrt{1-4\lambda^2}}{2} & -\lambda \\ -\lambda & \frac{1+\sqrt{1-4\lambda^2}}{2} \end{bmatrix}$$

with λ real and $4\lambda^2 \leq 1$, is infinite.

Now, to address the finiteness issue, we introduce the following definitions.

Definition 9.5. A w-solution y_0 of the LCP(L, q) is said to be locally w-unique if there exists a neighborhood of y_0 within which y_0 is the only w-solution.

Definition 9.6. A w-solution y_0 of the LCP(L, q) is said to be locally-star-like if there exists a ball B(y_0, r) such that for all $y \in B(y_0, r) \bigcap$ w-SOL(L, q), $[y_0, y] \subseteq$ w-SOL(L, q), or, equivalently,

$$(tx_0 + (1-t)x) \circ (ty_0 + (1-t)y) = 0 \ \forall t \in [0, 1],$$

where w-SOL(L, q) denotes the w-solution set of LCP(L, q), $y = L(x) + q$ and $y_0 = L(x_0) + q$.

We note that if SOL(L, q) is convex, then every w-solution in w-SOL(L, q) has the locally-star-like property.

Theorem 9.7. *Given L on V, consider the following statements:*

(a) *For all $q \in V$, the LCP(L, q) has a finite number (possible zero) of w-solutions.*

(b) *For all $q \in V$, any w-solution of the LCP(L, q), if exists, must be locally w-unique.*

(c) *L is column competent, and for all $q \in V$, each w-solution of LCP(L, q) is locally-star-like.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. The implication (a)$\Rightarrow$ (b) is obvious.
(b)$\Rightarrow$ (c): To show the column competent part, let x and $L(x)$ operator commute, and $x \circ L(x) = 0$. Then

$$x^+ \circ (L(x))^+ = x^- \circ (L(x))^- = x^+ \circ (L(x))^- = x^- \circ (L(x))^+ = 0.$$

If $L(x) \neq 0$, then $(L(x))^+ \neq (L(x))^-$. Defining $q := (L(x))^+ - L(x^+) = (L(x))^- - L(x^-)$, we see that the LCP$(L, q)$ has two distinct w-solutions $(L(x))^+$ and $(L(x))^-$ with

$$(tx^+ + (1-t)x^-) \circ (t(L(x))^+ + (1-t)(L(x))^-) = 0, \quad \forall t \in [0, 1].$$

i.e., $[(L(x))^-, (L(x))^+] \subseteq$ w-SOL(L, q). This contradicts (b). Thus $L(x) = 0$. Hence L is column competent. Now take any $q \in V$. For a $y_0 \in$ w-SOL(L, q), the locally-star-like property is trivially satisfied since y_0 is locally w-unique. $\square$

Remark 9.8. When K is polyhedral, the reverse implications in above theorem hold.

Remark 9.9. When L has the R_0-property (i.e., when LCP$(L, 0)$ has only zero solution), we claim that the reverse implications in above theorem hold.
(b)$\Rightarrow$ (a): Since L has the R_0-property, the solution set of LCP(L, q) is compact (possibly empty) for all $q \in V$ (see Observation 1.2.1 in [13]). Thus the w-solution set of LCP(L, q) is compact for all $q \in V$. Hence LCP(L, q) has a finite number of w-solutions for all $q \in V$.
(c)$\Rightarrow$ (b): Suppose for some $q \in V$, a w-solution y_0 of LCP(L, q) is not locally w-unique. Then there exists a sequence $\{y_k\} \subseteq$ w-SOL(L, q) which converges to y_0 with $y_k \neq y_0$ for all k. By the locally-star-like condition, $[y_0, y_k] \subseteq$ w-SOL(L, q) for all large k, i.e.,

$$(tx_0 + (1-t)x_k) \circ (ty_0 + (1-t)y_k) = 0 \ \forall t \in [0, 1]$$

for all large k, where $y_k = L(x_k) + q$ and $y_0 = L(x_0) + q$. This yields

$$\langle x_0, y_k \rangle = 0 = \langle x_k, y_0 \rangle.$$

Hence we have that x_0 (x_k) operator commutes with y_k (respectively, y_0), and $x_0 \circ y_k = 0 = x_k \circ y_0$ for all large k by Corollary 2.1; Therefore $x_k - x_0$ operator commutes with $y_k - y_0 = L(x_k - x_0)$, and $(x_k - x_0) \circ L(x_k - x_0) = 0$. But from the column competence property, this implies that $y_k = y_0$ for all large k, contradicting our assumption. Therefore, for all $q \in V$, any w-solution of LCP(L, q), if it exists, must be locally w-unique. This proves the claim.

10 Concluding Remarks

In this paper, we have introduced some generalizations of the w-P matrix concept mentioned in Introduction for a linear transformation defined on a Euclidean Jordan algebra and studied some interconnections between these generalized concepts.

Appendix Here we justify the assertion made in Example 3.1. First we present a result from [17] (see Proposition 7.3.1); a proof is given for completeness.

Proposition 10.1. *Consider*

$$L = \begin{bmatrix} a & b & c \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

If $a + |\lambda| > 0$, $\lambda \neq 0$, $a > \lambda c$, $a > 0$ *and* $(c + \lambda)^2 < 4a$, *then* L *has the P-property.*

Proof. Let $x = [x_0 \quad x_1 \quad x_2]^T$. Then $L(x) = \begin{bmatrix} ax_0 + bx_1 + cx_2 \\ x_1 \\ \lambda x_0 + x_2 \end{bmatrix}$.
Now $x \circ L(x) \leq 0$, and x and $L(x)$ operator commute if and only if (see Corollary 7 in [1])

(1) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$

or

(2) $\begin{bmatrix} x_1 \\ \lambda x_0 + x_2 \end{bmatrix} = 0,$

or

(3) $\begin{bmatrix} x_1 \\ \lambda x_0 + x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for some $\alpha \neq 0$.

Now we analyze the above three cases.

(1) If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, then we have

$x = x_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $L(x) = x_0 \begin{bmatrix} a \\ 0 \\ \lambda \end{bmatrix}$ and $x \circ L(x) = {x_0}^2 \begin{bmatrix} a \\ 0 \\ \lambda \end{bmatrix} \leq 0$.

So, if $a + |\lambda| > 0$, we have $x_0 = 0$.

(2) If $\begin{bmatrix} x_1 \\ \lambda x_0 + x_2 \end{bmatrix} = 0$, then $x_1 = 0$ and $x_2 = -\lambda x_0$ $(\lambda \neq 0)$. We have

$x = x_0 \begin{bmatrix} 1 \\ 0 \\ -\lambda \end{bmatrix}$, $L(x) = (a - \lambda c)x_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $x \circ L(x) = (a - \lambda c){x_0}^2 \begin{bmatrix} 1 \\ 0 \\ -\lambda \end{bmatrix} \leq 0$.

So, if $a - \lambda c > 0$, we have $x_0 = 0$.

(3) Let $\begin{bmatrix} x_1 \\ \lambda x_0 + x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(i) If $\alpha = 1$, then we have $x_0 = 0$ $(\lambda \neq 0)$,

$L(x) = \begin{bmatrix} bx_1 + cx_2 \\ x_1 \\ x_2 \end{bmatrix}$ and $x \circ L(x) = \begin{bmatrix} {x_1}^2 + {x_2}^2 \\ x_1(bx_1 + cx_2) \\ x_2(bx_1 + cx_2) \end{bmatrix} \leq 0$.

So, we $x_1 = x_2 = 0$.

(ii) If $\alpha \neq 1$, then we have $x_1 = 0$, $x_2 = \frac{\lambda}{\alpha - 1} x_0$,

$$L(x) = \frac{x_0}{\alpha - 1} \begin{bmatrix} a(\alpha - 1) + c\lambda \\ 0 \\ \alpha\lambda \end{bmatrix},$$

and

$$x \circ L(x) = \frac{{x_0}^2}{(\alpha - 1)^2} \begin{bmatrix} a(\alpha - 1)^2 + \lambda(c + \lambda)(\alpha - 1) + \lambda^2 \\ 0 \\ \lambda[a(\alpha - 1) + c\lambda + \alpha(\alpha - 1)] \end{bmatrix} \leq 0.$$

Thus, we have $x_0 = 0$ or

$$|\lambda||(\alpha - 1)^2 + (a + 1)(\alpha - 1) + c\lambda| + a(\alpha - 1)^2 + \lambda(c + \lambda)(\alpha - 1) + \lambda^2 \leq 0.$$

The above inequality is violated if $a > 0$ and $(c + \lambda)^2 < 4a$ in which case we have $x_0 = 0$, hence $x_2 = 0$. Therefore, L has the P-property. □

Now consider L given in Example 3.1:

$$L = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3.$$

It can be easily verified that L has the P-property by the above proposition. Hence it has the w-P property. Take $x = \begin{bmatrix} -1.32 \\ 1 \\ 0 \end{bmatrix}$. Then we have

$x \circ L(x)$=[-2.5376 1.36 -1.32^2]$^T < 0$, but $L(x)$=[2.68 1 1.32]$^T \neq 0$. Therefore L has the w-P property but not the Jordan w-P property.

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Jiyuan Tao
Department of Mathematical Sciences, Loyola University Maryland
Baltimore, MD 21210, USA
E-mail address: `JTao@loyola.edu`