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GENERALIZED MIXED VARIATIONAL-LIKE INEQUALITIES WITH COMPOSITELY PSEUDOMONOTONE MULTIFUNCTIONS*

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Dedicated to Professor Guangya Chen On the occasion of his 70^{th} birthday.

Abstract: In this paper, we introduce and consider a class of generalized mixed variational-like inequalities with compositely pseudomonotone multifunctions in Banach spaces. By using the Brouwer's fixed point theorem and the Nadler's fixed point theorem, we prove the existence of solutions for generalized mixed variational-like inequalities with compositely relaxed $\eta - \alpha$ pseudomonotone multifunctions in reflexive Banach spaces. On the other hand, we also derive the solvability of generalized mixed variational-like inequalities with compositely relaxed $\eta - \alpha$ semi-pseudomonotone multifunctions in nonreflexive Banach spaces by virtue of the method of finite-dimensional successive approximation. The results presented in this paper extend and improve some earlier and recent results in the literature.

Key words: *generalized mixed variational-like inequalities, compositely pseudomonotone multifunctions, Hausdorff metric, $\widetilde{H}$-hemicontinuity, coercivity, finite-dimensional successive approximation*

Mathematics Subject Classification: *49J40, 47J20, 49J53, 47H05*

1 Introduction

Variational inequality theory has become an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including partial differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, and structure analysis. Because of its important applicability, variational inequality problems have been extensively studied and generalized in various directions by many authors. For more details, the reader is refereed to [1–7, 9–15] and references therein.

Generalized monotonicity concepts like quasimonotonicity, pseudomonotonicity, relaxed monotonicity, p-monotonicity, semimonotonicity, relaxed $\eta - \alpha$ monotonicity, and relaxed

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$\eta - \alpha$ semimonotonicity, have been introduced and studied in the context of variational inequalities and complementarity problems; See for instance [2–4, 6, 7, 12–14] and references therein. In 1997, Verma [12] studied a class of nonlinear variational inequalities with p-monotone and p-Lipschitz mappings in reflexive Banach spaces and gave some existence theorems of solutions. Subsequently, Chen [3] introduced a class of variational inequalities with semimonotone mappings in nonreflexive Banach spaces and obtained existence theorems of solutions by using the Kakutani-Fan-Glicksberg fixed-point theorem. Recently, Fang and Huang [4] introduced two concepts of relaxed $\eta - \alpha$ monotonicity and relaxed $\eta - \alpha$ semimonotonicity as well as two classes of variational-like inequalities with relaxed $\eta - \alpha$ monotone mappings and relaxed $\eta - \alpha$ semimonotone mappings. Using the KKM technique, they proved the existence of solutions for variational-like inequalities with relaxed $\eta - \alpha$ monotone mappings in reflexive Banach spaces. Moreover, they also derived the solvability of variational-like inequalities with relaxed $\eta - \alpha$ semimonotone mappings in arbitrary Banach spaces by means of the Kakutani-Fan-Glicksberg fixed-point theorem.

In this paper, we introduce and consider a class of generalized mixed variational-like inequalities with compositely pseudomonotone multifunctions in Banach spaces. Utilizing the Brouwer's fixed point theorem and the Nadler's result, we prove the existence of solutions for generalized mixed variational-like inequalities with compositely relaxed $\eta - \alpha$ pseudomonotone multifunctions in reflexive Banach spaces. On the other hand we also derive the solvability of generalized mixed variational-like inequalities with compositely relaxed $\eta - \alpha$ semi-pseudomonotone multifunctions in arbitrary Banach spaces by virtue of the method of finite-dimensional successive approximation. The results presented in this paper extend and improve some earlier and recent results in the literature including [2–5, 11–13].

Throughout this paper, we denote by "$\to$", "$\rightharpoonup$" and "$\rightharpoonup^*$" the strong convergence, weak convergence and weak* convergence, respectively. We also denote by 2^X the collection of all nonempty subsets of X.

2 Generalized Mixed Variational-like Inequalities in Reflexive Banach Spaces

Throughout the paper, unless otherwise specified, let X be a real Banach space with dual space X^* and K be a nonempty closed convex subset of X. Also, let $A : X^* \times X^* \to X^*$, $g : K \to X^*$, $f : K \to \mathbb{R} \cup \{+\infty\}$ and $\eta : K \times K \to X$ be mappings and let $V : K \to 2^X$ and $H : K \times K \to 2^{X^*}$ be vector multifunctions. We consider the following *generalized mixed variational-like inequality problem* (for short, GMVLIP):

$$\text{(GMVLIP)} \quad \begin{cases} \text{Find } \hat{x} \in K, \ \hat{z} \in V(\hat{x}) \text{ and } \hat{\xi} \in H(\hat{x}, \hat{z}) \text{ such that} \\ \langle A(g(\hat{x}), \hat{\xi}), \eta(y, \hat{x}) \rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K. \end{cases} \tag{2.1}$$

The GMVLIP is a generalization of many problems studied in [4,11,14–17] and references therein.

Let us recall the following definitions and results which will be used in the sequel.

Lemma 2.1 ([10]). *Let X, Y and Z be real topological vector spaces, K and C be nonempty subsets of X and Y, respectively. Let $H : K \times C \to 2^Z$ and $V : K \to 2^Y$ be multifunctions. If both H and V are upper semicontinuous with compact values, then the multifunction*

$T : K \to 2^Z$ defined as

$$T(x) = \bigcup_{z\in V(x)} H(x, z) = H(x, V(x))$$

is upper semicontinuous with compact values.

Lemma 2.2 (Nadler's fixed point theorem [9]). *Let $(Y, \|\cdot\|)$ be a normed vector space and $\widetilde{H}(\cdot, \cdot)$ be a Hausdorff metric on the collection $CB(Y)$ of all nonempty, closed and bounded subsets of Y, induced by a metric d in terms of $d(u, v) = \|u - v\|$, which is defined by*

$$\widetilde{H}(\Delta, \Lambda) = \max\left(\sup_{u\in\Delta}\inf_{v\in\Lambda}\|u - v\|,\ \sup_{v\in\Lambda}\inf_{u\in\Delta}\|u - v\|\right),$$

for all Δ and Λ in $CB(Y)$. If Δ and Λ are two nonempty, closed and bounded subsets in Y, then for each $\varepsilon > 0$ and each $u \in \Delta$, there exists $v \in \Lambda$ such that

$$\|u - v\| \leq (1 + \varepsilon)\widetilde{H}(\Delta, \Lambda).$$

In particular, if Δ and Λ are two compact subsets in Y, then for each $u \in \Delta$, there exists $v \in \Lambda$ such that

$$\|u - v\| \leq \widetilde{H}(\Delta, \Lambda).$$

Lemma 2.3 (Brouwer's fixed point theorem [1]). *Let D be a nonempty, compact and convex subset of a finite dimensional space and $h : D \to D$ be a continuous mapping. Then there exists $x \in D$ such that $h(x) = x$.*

Definition 2.4. (i) [15] Let $T : K \to X^*$ and $\eta : K \times K \to X$ be two mappings. T is said to be *η-hemicontinuous* if for any fixed $x, y \in K$, the mapping $f : [0, 1] \to \mathbb{R}$ defined by $f(t) = \langle T(x + t(y - x)), \eta(y, x)\rangle$ is continuous at 0^+;

(ii) [18] A nonempty compact-valued multifunction $T : K \to 2^{X^*}$ is called *$\widetilde{H}$-hemicontinuous* if for any fixed $x, y \in K$, the mapping $f : [0, 1] \to \mathbb{R}$ defined by $f(t) = \widetilde{H}(T(x+t(y-x)), T(x))$ is continuous at 0^+, where $\widetilde{H}$ is the Hausdorff metric defined on $CB(X^*)$.

Now we first present some concepts and results.

Definition 2.5. Let $A : X^* \times X^* \to X^*$, $g : K \to X^*$, $f : K \to \mathbb{R} \cup \{+\infty\}$ and $\eta : K \times K \to X$ be mappings, $V : K \to 2^X$ and $H : K \times K \to 2^{X^*}$ vector multifunctions and $\alpha : X \to \mathbb{R}$ a real valued function such that $\liminf_{t\to 0^+} \alpha(tx)/t = 0$, $\forall x \in X$. Then H and V are said to be *compositely relaxed $\eta - \alpha$ pseudomonotone with respect to* (A, f, g) if there exist $x_0 \in K$, $z_0 \in V(x_0)$, $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A(g(x_0), \xi_0), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0, \quad \forall y \in K,$$

implies

$$\langle A(g(x_0), \xi), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0), \quad \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

Definition 2.6. Let $A : X^* \times X^* \to X^*$, $g : K \to X^*$, $f : K \to \mathbb{R} \cup \{+\infty\}$ and $\eta : K \times K \to X$ be mappings and $V : K \to 2^X$ and $H : K \times K \to 2^{X^*}$ be multifunctions.

(i) Let Ω be a nonempty subset of K. The pair (H, V) is said to be *η-completely semi-continuous on Ω with respect to* (A, f, g) if for each $y \in \Omega$,

$$\{x \in \Omega : \langle A(g(x), \xi), \eta(y, x)\rangle + f(y) - f(x) < 0, \quad \forall z \in V(x), \xi \in H(x, z)\}$$

is open in Ω with respect to $\sigma(X, X^*)$;

(ii) H and V are said to be *of locally complete semicontinuity on* K if for any finite subset $\mathcal{B}$ of K, H and V are η-completely semicontinuous on co$\mathcal{B}$ with respect to (A, f, g), where co$\mathcal{B}$ denotes the convex hull of $\mathcal{B}$.

Theorem 2.7. *Let K be a nonempty, closed and convex subset of a real Banach space X. Let $g : K \to X^*$ be a mapping, $A(\zeta, \cdot) : X^* \to X^*$ a continuous map from the $\sigma(X^*, X)$-topology to itself for each fixed $\zeta \in X^*$, $f : K \to \mathbb{R} \cup \{+\infty\}$ a proper convex function and $\eta : K \times K \to X$ a mapping such that (a) $\langle A(g(x), \xi), \eta(\cdot, x)\rangle : K \to \mathbb{R}$ is convex for each fixed $(x, \xi) \in K \times X^*$ and (b) $\langle A(g(x), \xi), \eta(x, x)\rangle = 0$, $\forall (x, \xi) \in K \times X^*$. Let $V : K \to 2^X$ take weakly compact values in X and be upper semicontinuous from the $\sigma(X, X^*)$-topology to itself, and let $H : K \times K \to 2^{X^*}$ take weak* compact values in X^* and be upper semicontinuous from the product topology of $\sigma(X, X^*)$ and itself to the $\sigma(X^*, X)$-topology. If H and V are compositely relaxed $\eta - \alpha$ pseudomonotone with respect to (A, f, g), and the multifunction $T : K \to 2^{X^*}$ defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-hemicontinuous, then the following are equivalent:

(i) *there exist $x_0 \in K$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that*

$$\langle A(g(x_0), \xi_0), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0, \quad \forall y \in K; \tag{2.2}$$

(ii) *there exists $x_0 \in K$ such that*

$$\langle A(g(x_0), \xi), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0), \quad \forall y \in K,\ z \in V(y),\ \xi \in H(x, z). \tag{2.3}$$

Proof. Suppose that there exist $x_0 \in K$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A(g(x_0), \xi_0), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0, \quad \forall y \in K.$$

Since H and V are compositely relaxed $\eta - \alpha$ pseudomonotone with respect to (A, f, g), we have

$$\langle A(g(x_0), \xi), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0)$$

for all $y \in K$, $z \in V(y)$ and $\xi \in H(y, z)$.

Conversely, suppose that there exists $x_0 \in K$ such that

$$\langle A(g(x_0), \xi), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0)$$

for all $y \in K$, $z \in V(y)$ and $\xi \in H(y, z)$. For any given $y \in K$, we know that $y_t = ty + (1 - t)x_0 \in K$, $\forall t \in (0, 1)$ since K is convex. Replacing y by y_t in the left-hand side of the above inequality, one deduces from assumptions (a) – (b) that for each $\xi_t \in T(y_t) = H(y_t, V(y_t))$

$$\begin{aligned}
\alpha(t(y - x_0)) &= \alpha(y_t - x_0) \\
&\leq \langle A(g(x_0), \xi_t), \eta(y_t, x_0)\rangle + f(y_t) - f(x_0) \\
&= \langle A(g(x_0), \xi_t), \eta(ty + (1-t)x_0, x_0)\rangle + f(ty + (1-t)x_0) - f(x_0) \\
&\leq t\langle A(g(x_0), \xi_t), \eta(y, x_0)\rangle + (1-t)\langle A(g(x_0), \xi_t), \eta(x_0, x_0)\rangle + tf(y) \\
&\quad +(1-t)f(x_0) - f(x_0) \\
&= t[\langle A(g(x_0), \xi_t), \eta(y, x_0)\rangle + f(y) - f(x_0)],
\end{aligned} \tag{2.4}$$

which hence implies that

$$\langle A(g(x_0), \xi_t), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(t(y - x_0))/t, \quad \forall \xi_t \in T(y_t),\ t \in (0, 1). \tag{2.5}$$

We remark that according to Lemma 2.1 the multifunction $T : K \to 2^{X^*}$ defined as

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

takes weak* compact values in X^* and is upper semicontinuous from $\sigma(X, X^*)$ to $\sigma(X^*, X)$. Thus, $T(y_t)$ and $T(x_0)$ are weak* compact and hence are nonempty, bounded and closed subsets in X^*. So from Lemma 2.2, it follows that for each fixed $\xi_t \in T(y_t)$ there exists an $\zeta_t \in T(x_0)$ such that

$$\|\xi_t - \zeta_t\| \leq (1 + t)\widetilde{H}(T(y_t), T(x_0)),$$

where $\widetilde{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(X^*)$. Since $T(x_0)$ is weak*ly compact, without loss of generality, we may assume that $\zeta_t \rightharpoonup^* \xi_0 \in T(x_0)$ as $t \to 0^+$. Since T is $\widetilde{H}$-hemicontinuous, $\widetilde{H}(T(y_t), T(x_0)) \to 0$ as $t \to 0^+$. Consequently one derives for each $u \in X$

$$\begin{aligned} |\langle \xi_t - \xi_0, u\rangle| &\leq |\langle \xi_t - \zeta_t, u\rangle| + |\langle \zeta_t - \xi_0, u\rangle| \\ &\leq \|\xi_t - \zeta_t\|\|u\| + |\langle \zeta_t - \xi_0, u\rangle| \\ &\leq \|u\| \cdot \widetilde{H}(T(y_t), T(x_0)) + |\langle \zeta_t - \xi_0, u\rangle| \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

Note that $A(g(x_0), \cdot) : X^* \to X^*$ is continuous from $\sigma(X^*, X)$ to itself. Then $A(g(x_0), \xi_t) \rightharpoonup^* A(g(x_0), \xi_0)$ as $t \to 0^+$. Thus letting $t \to 0^+$, we obtain

$$\langle A(g(x_0), \xi_t), \eta(y, x_0)\rangle \to \langle A(g(x_0), \xi_0), \eta(y, x_0)\rangle.$$

Consequently, from (2.5) we deduce that for any given $y \in K$,

$$\langle A(g(x_0), \xi_0), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0.$$

Since $\xi_0 \in T(x_0) = \bigcup_{z \in V(x_0)} H(x_0, z) = H(x_0, V(x_0))$, there exists $z_0 \in V(x_0)$ such that $\xi_0 \in H(x_0, z_0)$. This completes the proof. □

Remark 2.8. Theorem 2.7 generalizes Theorem 2.1 of Fang and Huang [4], Theorem 2.1 of Verma [12] and Theorem 2.1 of Verma [13].

Utilizing the Brouwer's fixed point theorem and Nadler's fixed point theorem, we now establish the existence of solutions of GMVLIP with compositely relaxed $\eta - \alpha$ pseudomonotone multifunctions in reflexive Banach spaces.

Theorem 2.9. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X, and let X^* be the dual space of X. Assume that the following conditions are satisfied:*

(i) $\langle A(g(x), \xi), \eta(x, x)\rangle = 0,\ \forall (x, \xi) \in K \times X^*$;
(ii) $\langle A(g(x), \xi), \eta(\cdot, x)\rangle : K \to \mathbb{R}$ *is convex for each fixed* $(x, \xi) \in K \times X^*$;
(iii) $\eta(x, \cdot) : K \to X$ *is continuous from* $\sigma(X, X^*)$ *to* $\sigma(X, X^*)$ *for each fixed* $x \in K$;
(iv) $f : K \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex lower semicontinuous function;*
(v) $A(\zeta, \cdot) : X^* \to X^*$ *is continuous from* $\sigma(X^*, X)$ *to itself for each fixed* $\zeta \in X^*$;
(vi) $A(g(\cdot), \xi) : K \to X^*$ *is continuous from* $\sigma(X, X^*)$ *to the norm topology of* X^* *for each fixed* $\xi \in X^*$;

(vii) *H and V are of locally complete semicontinuity on K.*

Suppose additionally that $V : K \to 2^X$ takes weakly compact values in X and is upper semicontinuous from $\sigma(X, X^*)$ to itself, and that $H : K \times K \to 2^{X^*}$ takes weak* compact values in X^* and is upper semicontinuous from the product topology of $\sigma(X, X^*)$ and itself to $\sigma(X^*, X)$. If H and V are compositely relaxed $\eta - \alpha$ pseudomonotone with respect to (A, f, g), and the multivalued map $T : K \to 2^{X^*}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-hemicontinuous, then GMVILP has a solution.

Proof. First we claim that for every finite subset E of K, there exist $\bar{x} \in \text{co}E$, $\bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A(g(\bar{x}), \bar{\xi}), \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in \text{co}E.$$

Indeed, suppose to the contrary that the assertion is not valid. Then for every $x \in \text{co}E$, there exists some $y_0 \in \text{co}E$ such that

$$\langle A(g(x), \xi), \eta(y_0, x)\rangle + f(y_0) - f(x) < 0 \tag{2.6}$$

for all $z \in V(x)$ and $\xi \in H(x, z)$. For every $y \in \text{co}E$, define the set N_y as follows:

$$N_y = \{x \in \text{co}E : \langle A(g(x), \xi), \eta(y, x)\rangle + f(y) - f(x) < 0, \quad \forall z \in V(x), \xi \in H(x, z)\}. \tag{2.7}$$

Since H and V are of locally complete semicontinuity on K by assumption (vii), the set N_y is open in co$E$ with respect to $\sigma(X, X^*)$ for every $y \in \text{co}E$.

Now we assert that $\{N_y : y \in \text{co}E\}$ is an open cover of co$E$ with respect to $\sigma(X, X^*)$. Indeed, first it is easy to see that

$$\bigcup_{y \in coE} N_y \subseteq \text{co}E.$$

Second, for each $x \in \text{co}E$, by (2.6) there exists $y_0 \in \text{co}E$ such that $x \in N_{y_0}$. Hence $x \in \bigcup_{y \in \text{co}E} N_y$. This shows that $\text{co}E \subseteq \bigcup_{y \in \text{co}E} N_y$. Consequently,

$$\text{co}E = \bigcup_{y \in \text{co}E} N_y.$$

So the assertion is valid.

The weak compactness of co$E$ implies that there exists a finite set $\{v_1, v_2, ..., v_m\} \subseteq \text{co}E$ such that

$$\text{co}E = \bigcup_{i=1}^{m} N_{v_i}.$$

Hence there exists a continuous (with respect to $\sigma(X, X^*)$) partition of unity $\{\beta_1, \beta_2, ..., \beta_m\}$ subordinated to $\{N_{v_1}, N_{v_2}, ..., N_{v_m}\}$ such that $\beta_j(x) \geq 0$, $\forall x \in \text{co}E$, $j = 1, 2, ..., m$,

$$\sum_{j=1}^{m} \beta_j(x) = 1, \quad \forall x \in \text{co}E,$$

and

$$\beta_j(x)\begin{cases} = 0, & \text{whwnever } x \notin N_{v_j}, \\ > 0, & \text{whenever } x \in N_{v_j}. \end{cases}$$

Let $h : \text{co}E \to X$ be defined as follows:

$$h(x) = \sum_{j=1}^{m} \beta_j(x)v_j, \quad \forall x \in \text{co}E. \tag{2.8}$$

Since β_j is continuous with respect to $\sigma(X, X^*)$ for each j, h is continuous with respect to $\sigma(X, X^*)$. Let $S = \text{co}\{v_1, v_2, ..., v_m\}$ be the convex hull of $\{v_1, v_2, ..., v_m\}$ in co$E$. Then S is a simplex of a finite dimensional space and h maps S into S. By Brouwer's fixed point theorem (Lemma 2.3), there exists some $x_0 \in S$ such that $h(x_0) = x_0$. Now for any given $x \in \text{co}E$, let

$$k(x) = \{j : x \in N_{v_j}\} = \{j : \beta_j(x) > 0\}.$$

Obviously, $k(x) \neq \emptyset$.

Utilizing assumption (i), for all $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ we have

$$\langle A(g(x_0), \xi_0), \eta(x_0, x_0)\rangle = 0.$$

Since x_0 is a fixed point of h, we have $x_0 = h(x_0) = \sum_{j=1}^{m} \beta_j(x_0)v_j$, and hence from the definition of N_y and assumptions (ii), (iv) we derive

$$\begin{aligned} 0 &= -\langle A(g(x_0), \xi_0), \eta(x_0, x_0)\rangle + f(x_0) - f(x_0) \\ &= -\langle A(g(x_0), \xi_0), \eta(h(x_0), x_0)\rangle + f(x_0) - f(h(x_0)) \\ &= -\langle A(g(x_0), \xi_0), \eta(\sum_{j=1}^{m} \beta_j(x_0)v_j, x_0)\rangle + f(x_0) - f(\sum_{j=1}^{m} \beta_j(x_0)v_j) \\ &\geq -\sum_{j=1}^{m} \beta_j(x_0)\langle A(g(x_0), \xi_0), \eta(v_j, x_0)\rangle + f(x_0) - \sum_{j=1}^{m} \beta_j(x_0)f(v_j) \\ &= -\sum_{j=1}^{m} \beta_j(x_0)[\langle A(g(x_0), \xi_0), \eta(v_j, x_0)\rangle + f(v_j) - f(x_0)] \\ &= -\sum_{j\in k(x_0)} \beta_j(x_0)[\langle A(g(x_0), \xi_0), \eta(v_j, x_0)\rangle + f(v_j) - f(x_0)] > 0, \end{aligned}$$

which leads to a contradiction. Therefore, there exist $\bar{x} \in \text{co}E$, $\bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A(g(\bar{x}), \bar{\xi}), \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in \text{co}E.$$

Now, by Theorem 2.7 we conclude that for every finite subset E of K, there exists $\bar{x} \in \text{co}E$ such that

$$\langle A(g(\bar{x}), \xi), \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in \text{co}E,\ z \in V(y),\ \xi \in H(y, z).$$

Second, we claim that there exists $\hat{x} \in K$ such that

$$\langle A(g(\hat{x}), \xi), \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

Indeed, since X is reflexive and K is a nonempty, bounded closed and convex subset of X, so K is compact in $\sigma(X, X^*)$. Let $\Im$ be the family of all nonempty finite subsets of K. For each $E \in \Im$, consider the following set:

$$M_E = \{x \in K : \langle A(g(x), \xi), \eta(y, x)\rangle + f(y) - f(x) \geq 0, \quad \forall y \in \text{co}E,\ z \in V(y),\ \xi \in H(y, z)\}.$$

Then one has $M_E \neq \emptyset$ for each $E \in \Im$. We shall prove that $\bigcap_{E\in\Im} \overline{M}_E^w \neq \emptyset$, where $\overline{M}_E^w$ denotes the closure of E in $\sigma(X, X^*)$. For this, it suffices to show that the family $\{\overline{M}_E^w\}_{E\in\Im}$ has the finite intersection property. Let $E, F \in \Im$ and set $G = E \cup F \in \Im$. Then $M_G \subseteq M_E \cap M_F$ and it follows that $\overline{M}_E^w \cap \overline{M}_F^w \neq \emptyset$. This shows that the family $\{\overline{M}_E^w\}_{E\in\Im}$ has the finite intersection property. Since K is compact in $\sigma(X, X^*)$, it follows that $\bigcap_{E\in\Im} \overline{M}_E^w \neq \emptyset$. Let $\hat{x} \in \bigcap_{E\in\Im} \overline{M}_E^w$ and for an arbitrary $y \in K$ fixed, consider $F = \{y, \hat{x}\}$. Since $\hat{x} \in \overline{M}_F^w$, there exists $\{x_n\} \subseteq \overline{M}_F^w$ such that $\{x_n\} \subseteq K,\ x_n \rightharpoonup \hat{x}$ and for each n

$$\langle A(g(x_n), \xi), \eta(v, x_n)\rangle + f(v) - f(x_n) \geq 0, \quad \forall v \in \mathrm{co}F,\ z \in V(v),\ \xi \in H(v, z).$$

In particular, whenever $v = y$, one derives for each n

$$\langle A(g(x_n), \xi), \eta(y, x_n)\rangle + f(y) - f(x_n) \geq 0, \quad \forall z \in V(y),\ \xi \in H(y, z).$$

Since $f : K \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function, f is weakly lower semicontinuous. Note that $\eta(x, \cdot) : K \to X$ is continuous from $\sigma(X, X^*)$ to $\sigma(X, X^*)$ for each $x \in K$ fixed. Thus it follows from assumption (vi) that for each $y \in K$, $z \in V(y)$ and $\xi \in H(y, z)$ fixed,

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}[\langle A(g(x_n), \xi), \eta(y, x_n)\rangle + f(y) - f(x_n)] \\
&= \limsup_{n\to\infty}[\langle A(g(x_n), \xi), \eta(y, x_n)\rangle - \langle A(g(\hat{x}), \xi), \eta(y, x_n)\rangle \\
&\quad + \langle A(g(\hat{x}), \xi), \eta(y, x_n)\rangle + f(y) - f(x_n)] \\
&\leq \limsup_{n\to\infty}\langle A(g(x_n), \xi) - A(g(\hat{x}), \xi), \eta(y, x_n)\rangle \\
&\quad + \limsup_{n\to\infty}[\langle A(g(\hat{x}), \xi), \eta(y, x_n)\rangle + f(y) - f(x_n)] \\
&\leq \limsup_{n\to\infty} \|A(g(x_n), \xi) - A(g(\hat{x}), \xi)\|\|\eta(y, x_n)\| \\
&\quad + \limsup_{n\to\infty}\langle A(g(\hat{x}), \xi), \eta(y, x_n)\rangle + \limsup_{n\to\infty}(f(y) - f(x_n)) \\
&\leq \langle A(g(\hat{x}), \xi), \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}),
\end{aligned}$$

that is,

$$\langle A(g(\hat{x}), \xi), \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

Thus, the assertion is proved.

Now by Theorem 2.7 we infer that there exist $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(g(\hat{x}), \hat{\xi}), \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K.$$

This completes the proof. □

If K is unbounded, then we have the following result under a coercivity condition.

Theorem 2.10. *Let K be a nonempty, unbounded, closed and convex subset of a real reflexive Banach space X, and let X^* be the dual space of X. Assume that the following conditions are satisfied:*

(i) $\langle A(g(x), \xi), \eta(x, x)\rangle = 0,\ \forall (x, \xi) \in K \times X^*$;
(ii) $\langle A(g(x), \xi), \eta(\cdot, x)\rangle : K \to R$ *is convex for each fixed* $(x, \xi) \in K \times X^*$;
(iii) $\eta(x, \cdot) : K \to X$ *is continuous from* $\sigma(X, X^*)$ *to* $\sigma(X, X^*)$ *for each fixed* $x \in K$;
(iv) $f : K \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex lower semicontinuous function;*

(v) *$A(\zeta,\cdot): X^* \to X^*$ is continuous from $\sigma(X^*,X)$ to itself for each fixed $\zeta \in X^*$;*

(vi) *$A(g(\cdot),\xi): K \to X^*$ is continuous from $\sigma(X,X^*)$ to the norm topology of X^* for each fixed $\xi \in X^*$;*

(vii) *H and V are of locally complete semicontinuity on K;*

(viii) *$\alpha : X \to \mathbb{R}$ is weakly lower semicontinuous.*

Suppose additionally that $V : K \to 2^X$ takes weakly compact values in X and is upper semicontinuous from $\sigma(X,X^*)$ to itself, and that $H : K \times K \to 2^{X^*}$ takes weak* compact values in X^* and is upper semicontinuous from the product topology of $\sigma(X,X^*)$ and itself to $\sigma(X^*,X)$. If H and V are compositely relaxed $\eta - \alpha$ pseudomonotone with respect to (A,f,g), and the multivalued map $T : K \to 2^{X^*}$ defined by

$$T(x) = \bigcup_{z\in V(x)} H(x,z) = H(x,V(x))$$

is $\widetilde{H}$-hemicontinuous such that H and V are η-coercive with respect to (A,f,g); i.e., there exist $x_0 \in K$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0,z_0)$ such that

$$\inf_{\xi\in T(x)} \frac{\langle A(g(x_0),\xi_0) - A(g(x),\xi), \eta(x_0,x)\rangle + f(x) - f(x_0)}{\|\eta(x_0,x)\|} \to +\infty$$

as $\|x\| \to +\infty$, then GMVLIP has a solution.

Proof. Let

$$K_r = \{y \in K : \|y\| \le r\}.$$

Consider the problem of finding $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r,z_r)$ such that

$$\langle A(g(x_r),\xi_r), \eta(v,x_r)\rangle + f(v) - f(x_r) \ge 0, \quad \forall v \in K_r. \tag{2.9}$$

One can readily see that all conditions of Theorem 2.7 are fulfilled for nonempty, bounded, closed and convex subset $K_r = K \cap B_r$, where $B_r = \{x \in X : \|x\| \le r\}$. Thus according to Theorem 2.9 we know that problem (2.9) has one solution; that is, there exist $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r,z_r)$ such that inequality (2.9) holds. Choose $r > \|x_0\|$ with x_0 as in the coercivity condition. Then we have

$$\langle A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r) \ge 0.$$

Moreover,

$$\begin{aligned}
&\langle A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r)\\
&= -\langle A(g(x_0),\xi_0) - A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r) + \langle A(g(x_0),\xi_0), \eta(x_0,x_r)\rangle\\
&\le -\langle A(g(x_0),\xi_0) - A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r) + \|A(g(x_0),\xi_0)\|\|\eta(x_0,x_r)\|\\
&= \|\eta(x_0,x_r)\| \cdot [-\tfrac{\langle A(g(x_0),\xi_0) - A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_r) - f(x_0)}{\|\eta(x_0,x_r)\|} + \|A(g(x_0),\xi_0)\|].
\end{aligned}$$

Now, if $\|x_r\| = r$ for all r, we may choose r large enough such that the above inequality and the η-coercivity of H and V with respect to (A,f,g) imply that

$$\langle A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r) < 0,$$

which contradicts

$$\langle A(g(x_r),\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r) \ge 0.$$

Hence there exists r such that $\|x_r\| < r$. For any $y \in K$, we can choose $\epsilon > 0$ small enough such that

$$\epsilon < 1 \quad \text{and} \quad x_r + \epsilon(y - x_r) \in K_r.$$

It follows from (2.9) that

$$\begin{aligned}&(1-\epsilon)\langle A(g(x_r), \xi_r), \eta(x_r, x_r)\rangle + \epsilon\langle A(g(x_r), \xi_r), \eta(y, x_r)\rangle + (1-\epsilon)f(x_r) + \epsilon f(y) - f(x_r)\\ &\geq \langle A(g(x_r), \xi_r), \eta(x_r + \epsilon(y - x_r), x_r)\rangle + f(x_r + \epsilon(y - x_r)) - f(x_r)\\ &\geq 0.\end{aligned}$$

This implies that

$$\langle A(g(x_r), \xi_r), \eta(y, x_r)\rangle + f(y) - f(x_r) \geq 0$$

for all $y \in K$, and so problem (2.2) has one solution. This completes the proof. □

Remark 2.11. Theorems 2.9 and 2.10 generalize Theorems 2.2 and 2.3 of Fang and Huang [4], the known results of Hartman and Stampacchia [8] and the corresponding results of [5,11,13].

3 Generalized Mixed Variational-like Inequalities in Nonreflexive Banach Spaces

Throughout this section, unless otherwise specified, X is a nonreflexive Banach space with its dual space X^*, X^{**} denotes the dual space of X^* and K is a nonempty closed convex subset of X^{**}.

Let $A : X^* \times X^* \to X^*$, $g : K \to X^*$ and $\eta : K \times K \to X^{**}$ be three mappings, $f : K \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function, and $V : K \to 2^{X^{**}}$ and $H : K \times K \to 2^{X^*}$ be vector multifunctions. In this section, we consider the following *generalized mixed variational-like inequality problem* (for short, GMVLIP*) in the setting of nonreflexive Banach spaces and prove the existence of its solution:

$$(\text{GMVLIP}^*) \quad \begin{cases} \text{Find } \hat{x} \in K,\ \hat{z} \in V(\hat{x}) \text{ and } \hat{\xi} \in H(\hat{x}, \hat{z}) \text{ such that} \\ \langle A(g(\hat{x}), \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K. \end{cases} \tag{3.1}$$

Definition 3.1. Let $A : X^* \times X^* \to X^*$, $g : K \to X^*$, $f : K \to \mathbb{R} \cup \{+\infty\}$ and $\eta : K \times K \to X^{**}$ be mappings, $V : K \to 2^{X^{**}}$ and $H : K \times K \to 2^{X^*}$ vector multifunctions and $\alpha : X^{**} \to \mathbb{R}$ a real valued function such that $\liminf_{t\to 0^+} \alpha(tx)/t = 0$, $\forall x \in X^{**}$. Then H and V are said to be *compositely relaxed $\eta - \alpha$ semi-pseudomonotone with respect to* (A, f, g) if the following conditions hold:

(a) for each fixed $\zeta \in X^*$ $A(\zeta, \cdot) : X^* \to X^*$ is continuous from $\sigma(X^*, X)$ to itself, and H and V are compositely relaxed $\eta - \alpha$ pseudomonotone with respect to (A, f, g); i.e., the existence of $x_0 \in K$, $z_0 \in V(x_0)$, $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A(g(x_0), \xi_0), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0, \quad \forall y \in K,$$

implies

$$\langle A(g(x_0), \xi), \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0), \quad \forall y \in K,\ z \in V(y),\ \xi \in H(y, z);$$

(b) for each fixed $\xi \in X^*$, $A(g(\cdot), \xi) : K \to X^*$ is completely continuous; i.e., $A(g(\cdot), \xi) : K \to X^*$ is continuous from $\sigma(X^{**}, X^*)$ to the norm topology of X^*.

Next we derive the solvability of generalized mixed variational-like inequalities with compositely relaxed $\eta - \alpha$ semi-pseudomonotone multifunctions in nonreflexive Banach spaces by virtue of the method of finite-dimensional successive approximation.

Theorem 3.2. *Let X be a real Banach space and $K \subseteq X^{**}$ be a nonempty, bounded closed and convex subset. Assume that the following conditions are satisfied:*

(i) $\langle A(g(x), \xi), \eta(x, x)\rangle = 0, \ \forall (x, \xi) \in K \times X^*$;
(ii) $\langle A(g(x), \xi), \eta(\cdot, x)\rangle : K \to R$ *is convex for each fixed* $(x, \xi) \in K \times X^*$;
(iii) $\eta(x, \cdot) : K \to X^{**}$ *is continuous from* $\sigma(X^{**}, X^*)$ *to* $\sigma(X^{**}, X^*)$ *for each fixed* $x \in K$;
(iv) $f : K \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex lower semicontinuous function;*
(v) H *and* V *are of locally complete semicontinuity on* K;
(vi) $\alpha : X^{**} \to \mathbb{R}$ *is convex and lower semicontinuous.*

Suppose additionally that $V : K \to 2^{X^{**}}$ takes weakly compact values in X and $H : K \times K \to 2^{X^*}$ takes weak compact values in X^* such that for any finite-dimensional subspace $L \subseteq X^{**}$, (a) $V : K_L \to 2^{X^{**}}$ is upper semicontinuous from $\sigma(X^{**}, X^*)$ to itself and (b) $H : K_L \times K_L \to 2^{X^*}$ is upper semicontinuous from the product topology of $\sigma(X^{**}, X^*)$ and itself to $\sigma(X^*, X)$, where $K_L = K \cap L$. If H and V are compositely relaxed $\eta - \alpha$ semi-pseudomonotone with respect to (A, f, g), and the multivalued map $T : K \to 2^{X^*}$ defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-hemicontinuous, then GMVLIP has a solution.*

Proof. Let $L \subseteq X^{**}$ be a finite-dimensional subspace with $K_L = K \cap L \neq \emptyset$. For each $y \in K$, consider the following problem: Find $x_0 \in K_L$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A(g(x_0), \xi_0), \eta(v, x_0)\rangle + f(v) - f(x_0) \geq 0, \quad \forall v \in K_L. \tag{3.2}$$

Observe that $K_L \subseteq L$ is bounded, closed and convex. Note that H and V are compositely relaxed $\eta - \alpha$ semi-pseudomonotone with respect to (A, f, g). Hence $A(\zeta, \cdot) : X^* \to X^*$ is continuous from $\sigma(X^*, X)$ to itself for each fixed $\zeta \in X^*$, $A(g(\cdot), \xi) : K \to X^*$ is continuous from $\sigma(X^{**}, X^*)$ to the norm topology of X^* for each fixed $\xi \in X^*$, and H and V are compositely relaxed $\eta - \alpha$ pseudomonotone with respect to (A, f, g). Since assumptions (i)-(v) guarantee that conditions (i)–(iv), (vii) in Theorem 2.9 are fulfilled, from Theorem 2.9 it follows that problem (3.2) has a solution; that is, there exist $x_0 \in K_L$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that inequality (3.2) holds.

Let

$$\mho = \{L \subseteq X^{**} : L \text{ is finite dimensional with } K \cap L \neq \emptyset\}$$

and let

$$W_L = \{x \in K : \langle A(g(x), \xi), \eta(v, x)\rangle + f(v) - f(x) \geq \alpha(v - x), \ \forall v \in K_L, z \in V(v), \xi \in H(v, z)\}$$

for all $L \in \mho$. By (3.2) and Theorem 2.7, we know that W_L is nonempty and bounded. Denote by $\overline{W}_L$ the $\sigma(X^{**}, X^*)$-closure of W_L in X^{**}. Then, $\overline{W}_L$ is $\sigma(X^{**}, X^*)$-compact in X^{**}. For any $L_i \in \mho, i = 1, 2, ..., N$, we know that $W_{\cap_i L_i} \subseteq \cap_i W_{L_i}$, so $\{\overline{W}_L : L \in \mho\}$ has the finite intersection property. Therefore, it follows that

$$\bigcap_{L \in \mho} \overline{W}_L \neq \emptyset.$$

Let $\hat{x} \in \bigcap_{L\in\mho} \overline{W}_L \neq \emptyset$. We claim that there exist $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(g(\hat{x}), \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K.$$

Indeed, for each $v \in K$, let $L \in \mho$ be such that $v \in K_L$ and $\hat{x} \in K_L$. Then, there exists a net $\{x_\beta\} \in W_L$ such that x_β converges to $\hat{x}$ in $\sigma(X^{**}, X^*)$, which implies by the definition of W_L that

$$\langle A(g(x_\beta), \xi), \eta(v, x_\beta)\rangle + f(v) - f(x_\beta) \geq \alpha(v - x_\beta), \quad \forall z \in V(v),\ \xi \in H(v, z).$$

It follows that

$$\langle A(g(\hat{x}), \xi), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq \alpha(v - \hat{x}), \quad \forall v \in K,\ z \in V(v),\ \xi \in H(v, z),$$

by the complete continuity of $A(g(\cdot), v)$ and the proper convex lower semicontinuity of f and α. Therefore according to Theorem 2.7 there exist $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(g(\hat{x}), \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K.$$

This completes the proof. □

Theorem 3.3. *Let X be a real Banach space and let $K \subseteq X^{**}$ be a nonempty, unbounded, closed and convex subset. Assume that the following conditions are satisfied:*

- (i) $\eta(x, y) + \eta(y, x) = 0,\ \forall (x, y) \in K \times K$;
- (ii) $\langle A(g(x), \xi), \eta(\cdot, x)\rangle : K \to R$ *is convex for each fixed* $(x, \xi) \in K \times X^*$;
- (iii) $\eta(x, \cdot) : K \to X^{**}$ *is continuous from* $\sigma(X^{**}, X^*)$ *to* $\sigma(X^{**}, X^*)$ *for each fixed* $x \in K$;
- (iv) $f : K \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex lower semicontinuous function;*
- (v) H *and* V *are of locally complete semicontinuity on* K;
- (vi) $\alpha : X^{**} \to \mathbb{R}$ *is convex and lower semicontinuous.*

Suppose additionally that $V : K \to 2^{X^{**}}$ takes weakly compact values in X and $H : K \times K \to 2^{X^*}$ takes weak compact values in X^* such that for any finite-dimensional subspace $L \subseteq X^{**}$, (a) $V : K_L \to 2^{X^{**}}$ is upper semicontinuous from $\sigma(X^{**}, X^*)$ to itself and (b) $H : K_L \times K_L \to 2^{X^*}$ is upper semicontinuous from the product topology of $\sigma(X^{**}, X^*)$ and itself to $\sigma(X^*, X)$, where $K_L = K \cap L$. If H and V are compositely relaxed $\eta - \alpha$ semi-pseudomonotone with respect to (A, f, g), and the multivalued map $T : K \to 2^{X^*}$ defined by*

$$T(x) = \bigcup_{z\in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-hemicontinuous such that

(vii) there exists a point $x_0 \in K$ such that

$$\liminf_{\|x\|\to\infty} \inf_{\xi\in T(x)} [\langle A(g(x), \xi), \eta(x, x_0)\rangle + f(x) - f(x_0)] > 0.$$

Then GMVLIP has a solution.*

Proof. Denote by B_r the closed ball with radius r and center at 0 in X^{**}. First consider the problem of finding $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A(g(x_r), \xi_r), \eta(v, x_r)\rangle + f(v) - f(x_r) \geq 0, \quad \forall v \in K_r, \tag{3.3}$$

where $K_r = \{x \in K : \|x\| \le r\} = K \cap B_r$. By Theorem 3.2 problem (3.3) has a solution; that is, there exist $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that inequality (3.3) holds.

Let r be large enough such that $x_0 \in B_r$. Therefore,

$$\langle A(g(x_r), \xi_r), \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) \ge 0. \tag{3.4}$$

From condition (vii) it follows that $\{x_r\}$ is bounded. Indeed, if this is false, we may assume without loss of generality that $\|x_r\| \to \infty$ as $r \to \infty$. Now from (3.4) and assumption (i) we derive

$$\begin{aligned}&\inf_{\xi \in T(x_r)} [\langle A(g(x_r), \xi), \eta(x_r, x_0)\rangle + f(x_r) - f(x_0)]\\ &\le \langle A(g(x_r), \xi_r), \eta(x_r, x_0)\rangle + f(x_r) - f(x_0)\\ &\le 0,\end{aligned}$$

which hence implies that

$$\liminf_{\|x_r\| \to \infty} \inf_{\xi \in T(x_r)} [\langle A(g(x_r), \xi), \eta(x_r, x_0)\rangle + f(x_r) - f(x_0)] \le 0.$$

This contradicts condition (vii). So, we may suppose that x_r converges to $\hat{x}$ in $\sigma(X^{**}, X^*)$ as $r \to \infty$. On the other hand, it follows from Theorem 2.7 that

$$\langle A(g(x_r), \xi), \eta(v, x_r)\rangle + f(v) - f(x_r) \ge \alpha(v - x_r), \quad \forall v \in K,\ z \in V(v),\ \xi \in H(v, z).$$

Letting $r \to \infty$, we have

$$\langle A(g(\hat{x}), \xi), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \ge \alpha(v - \hat{x}), \quad \forall v \in K,\ z \in V(v),\ \xi \in H(v, z).$$

Again from Theorem 2.7 we know that there exist $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(g(\hat{x}), \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \ge 0, \quad \forall v \in K.$$

This completes the proof. □

Remark 3.4. Theorems 3.2 and 3.3 improve and generalize Theorems 3.1 and 3.2 of Fang and Huang [4] and Theorems 2.1 – 2.6 of Chen [3].

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