Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 5, No. 3, pp. 465–475, September 2009

ON DERIVATIVES OF THE GENERALIZED PERTURBATION MAPS*

X.W. Xue and S.J. Li†

Dedicated to professor Guang-Ya Chen in honour of his 70th birthday.

Abstract: The aim of this paper is to study the derivatives of the generalized perturbation maps. Under some mild conditions, the contingent and adjacent derivatives of the generalized perturbation maps are obtained, respectively. Furthermore, the generalized perturbation maps are shown to be proto-differentiable under some stronger conditions.

Key words: *set-valued maps, generalized perturbation map, contingent derivative, adjacent derivative*

Mathematics Subject Classification: *47H04, 58C20, 90C31*

1 Introduction

Consider the generalized perturbation map G from $R^d \times R^m$ to R^n defined by

$$G(u,z) = \{x \in D \mid z \in F(u,x) + K(u,x)\}, \tag{1.1}$$

where F and K are set-valued maps from $R^d \times R^n$ to R^m and D is a nonempty subset of R^n.

Since many first-order conditions for optimization and variational inequality problems can be expressed in the form of generalized perturbation problems (1.1), there is much interest to discuss the sensitivity of (1.1) (see [3, 4, 6, 8–11]). Robinson [9] did an excellent work on the sensitivity of perturbation problem, in which the set-valued map Q is given by :

$$Q(u) = \{x \in D \mid -F(u,x) \in M(x)\}.$$

When M is the normal cone mapping associated with the polyhedral, convex set $D \subset R^n$, Robinson pointed out that under certain conditions of differentiability on F and regularity "linearized" conditions, Q is a single-valued, Lipschitz and "Bouligand" differentiable map. However, it is obvious that the perturbed map Q, in general, is a set-valued map, but not

*This research was partially supported by the National Natural Science Foundation of China (Grant number: 10871216).

†Corresponding author.

single-valued map. For this reason, by introducing a generalized derivative, called a "proto-derivative", Rockafellar [10] proved that the generalized perturbation map W from $R^d \times R^m$ to R^n given by

$$W(u, z) = \{x \in D \mid z - F(u, x) \in M(x)\},$$

is proto-differentiable under conditions that F is a Fréchet differentiable function, and M is the normal cone associated with the polyhedral, convex set $D \in R^n$. Recently, Lee and Huy [3] also proved that the generalized perturbation map W is proto-differentiable under conditions that F is semi-differentiable, M is proto-differentiable on D and an exactly regular condition is satisfied.

However, the conditions of these theorems seems to be strong. In this paper, we study the contingent and adjacent derivatives of the generalized perturbation map G. Under weaker conditions, in which F, K are generally set-valued maps and D is any subset of R^n, we give the exact expressions for the contingent, adjacent and proto-derivatives of the generalized perturbation map (1.1), respectively.

This paper is organized as follows. In Section 2, we recall some concepts of derivatives of set-valued maps and prove an important proposition. In Section 3, we investigate calculus rules for the contingent, adjacent and proto-derivatives of the generalized perturbation map. Finally, we give two special cases and discuss their relations to the preceding results.

2 Preliminary Results

Let X, Y, Z be real normed spaces and F be a set-valued map from X to Y. The effective domain and the graph of F are denoted respectively by

$$domF := \{x \in X \mid F(x) \neq \emptyset\},$$

and

$$graphF := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Let the inverse of F be the set-valued map F^{-1} defined from Y to X such that $x \in F^{-1}(y)$ if and only if $y \in F(x)$. Throughout this paper, let the set of positive real numbers be denoted by R_{++}, the set of nonnegative real numbers be denoted by R_+ and the origins of all real normed spaces be denoted by 0.

Definition 2.1 ([1, 2]). Let $\hat{x} \in X$ and B be the closed unit ball in Y. F is said to be upper locally Lipschitz (for short, u.l.l.) at $\hat{x}$ if and only if there exist a neighborhood $\mathcal{N}(\hat{x})$ of $\hat{x}$ and a positive constant L such that for any $x \in \mathcal{N}(\hat{x})$,

$$F(x) \subset F(\hat{x}) + L\|x - \hat{x}\|B.$$

Definition 2.2 ([1,2]). Let $\hat{x} \in X$. F is said to be compact-valued at $\hat{x}$ if $F(\hat{x})$ is a compact set. And F is said to be compact at $\hat{x}$ if for any sequence $\{(x_k, y_k)\} \subset graphF$ such that $x_k \to \hat{x}$, there exists a subsequence $\{(x_{k_i}, y_{k_i})\}$ such that $(x_{k_i}, y_{k_i}) \to (\hat{x}, \hat{y}) \in graphF$.

Definition 2.3 ([2]). Let A be a nonempty subset of X and $\bar{x} \in cl(A)$, the closure of A.

(i) The (Bouligand) contingent cone $T(A, \bar{x})$ to A at $\bar{x}$ is the set of all $y \in X$ such that there exist sequences $\{t_k\} \subset R_{++}$ and $\{y_k\} \subset X$ such that $y_k \to y, t_k \to 0$ and $\{\bar{x} + t_k y_k\} \subset A$;

(ii) The adjacent cone $T^b(A, \bar{x})$ to A at $\bar{x}$ is the set of all $y \in X$ such that for any $\{t_k\} \subset R_{++}$ with $t_k \to 0$ there exists a sequence $\{y_k\} \subset X$ such that $y_k \to y$ and $\{\bar{x} + t_k y_k\} \subset A$;

(iii) A is said to be derivable at $\bar{x}$ if and only if $T(A,\bar{x}) = T^b(A,\bar{x})$.

Definition 2.4 ([2, 10]). Let $(\bar{x},\bar{y}) \in graphF$. The set-valued maps $DF(\bar{x},\bar{y})$ and $D^bF(\bar{x},\bar{y})$ defined from X to Y by

$$graphDF(\bar{x},\bar{y}) = T(graphF,(\bar{x},\bar{y})),$$

and

$$graphD^bF(\bar{x},\bar{y}) = T^b(graphF,(\bar{x},\bar{y})),$$

are called the contingent derivative and the adjacent derivative of F at $(\bar{x},\bar{y})$, respectively. Moreover, F is said to be proto-differentiable at $(\bar{x},\bar{y})$ if and only if graph$F$ is derivable at $(\bar{x},\bar{y})$, in which case the proto-derivable of F at $\bar{x}$ relative to $\bar{y}$ is denoted by $F^{'}_{\bar{x},\bar{y}}$.

Alternatively, $y \in DF(\bar{x},\bar{y})(x)$ if and only if there exist sequences $\{t_k\} \subset R_{++}$ and $\{(x_k,y_k)\} \subset X\times Y$ such that $t_k \to 0$, $(x_k,y_k)\to(x,y)$ and for all k, $\bar{y}+t_ky_k \in F(\bar{x}+t_kx_k)$; $y \in D^bF(\bar{x},\bar{y})(x)$ if and only if for any $\{t_k\} \subset R_{++}$ with $t_k \to 0$, there exists a sequence $\{(x_k,y_k)\} \subset X\times Y$ such that $(x_k,y_k)\to(x,y)$ and for all k, $\bar{y}+t_ky_k \in F(\bar{x}+t_kx_k)$.

Definition 2.5 ([4]). For a set-valued map F from X to Y and a nonempty subset C of X, let the restricted set-valued map F_C be defined by

$$F_C(x) = \begin{cases} F(x), & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If F_C is proto-differentiable at $x \in C$ relative to $y \in F(x)$, then F is said to be proto-differentiable on C at x relative to y, and the proto-derivative is denoted by $(F_C)^{'}_{x,y}$.

Definition 2.6 ([11]). Let $(\hat{x},\hat{y}) \in graphF$. The set-valued map $D^pF(\hat{x},\hat{y})$ defined from X to Y is called the TP-derivative of F at $(\hat{x},\hat{y})$ if and only if there exist sequences $\{t_k\} \subset R_{++}$ and $\{(x_k,y_k)\} \subset X\times Y$ such that $t_kx_k \to 0$, $(x_k,y_k)\to(x,y)$ and for all k, $\hat{y}+t_ky_k \in F(\hat{x}+t_kx_k)$.

Definition 2.7 ([8, 10]). Let $(\bar{x},\bar{y}) \in graphF$. F is said semi-differentiable at $(\bar{x},\bar{y})$ if and only if for any $y \in DF(\bar{x},\bar{y})(x)$, any $\{t_k\} \subset R_{++}$ with $t_k \to 0$ and any $\{x_k\} \subset X$ such that $x_k \to x$, there exists a sequence $\{y_k\} \subset Y$ such that $y_k \to y$ and for all k, $\bar{y}+t_ky_k \in F(\bar{x}+t_kx_k)$.

Proposition 2.8. *Let F be a set-valued map from X to Y, C be a nonempty subset of X, $\hat{x} \in C$ and $\hat{y} \in F(\hat{x})$. If F is semi-differentiable at $(\hat{x},\hat{y})$, then,*

$$DF_C(\hat{x},\hat{y}) = DF(\hat{x},\hat{y})_{T(C,\hat{x})},$$

and

$$D^bF_C(\hat{x},\hat{y}) = D^bF(\hat{x},\hat{y})_{T^b(C,\hat{x})}.$$

That is, for any $x \in X$,

$$DF_C(\hat{x},\hat{y})(x) = \begin{cases} DF(\hat{x},\hat{y})(x), & \text{if } x \in T(C,\hat{x}), \\ \emptyset, & \text{if } x \notin T(C,\hat{x}), \end{cases} \tag{2.1}$$

and

$$D^bF_C(\hat{x},\hat{y})(x) = \begin{cases} D^bF(\hat{x},\hat{y})(x), & \text{if } x \in T^b(C,\hat{x}), \\ \emptyset, & \text{if } x \notin T^b(C,\hat{x}). \end{cases} \tag{2.2}$$

Furthermore, if C is derivable at $\hat{x}$, then F is proto-differentiable on C at $(\hat{x},\hat{y})$.

Proof. We only need to prove (2.1), since (2.2) can be proved similarly. Let $x \in domDF_C(\hat{x}, \hat{y})$ and $y \in DF_C(\hat{x}, \hat{y})(x)$. Then, by the definition of contingent derivative, there exist sequences $\{t_k\} \subset R_{++}$ with $t_k \to 0$ and $\{(x_k, y_k)\} \subset X \times Y$ with $(x_k, y_k) \to (x, y)$ such that for all k, $\hat{y} + t_k y_k \in F_C(\hat{x} + t_k x_k)$. That is, $\hat{y} + t_k y_k \in F(\hat{x} + t_k x_k)$ and $\hat{x} + t_k x_k \in C$. Then, it follows directly from the definitions of contingent derivative and contingent cone that $y \in DF(\hat{x}, \hat{y})(x)$ and $x \in T(C, \hat{x})$. Thus, $domDF_C(\hat{x}, \hat{y}) \subset T(C, \hat{x})$.

Conversely, let $x \in T(C, \hat{x})$ and $y \in DF(\hat{x}, \hat{y})(x)$. By the definition of contingent cone, there exist $\{t_k\} \subset R_{++}$ with $t_k \to 0$ and $\{x_k\} \subset X$ with $x_k \to x$ such that for all k, $\hat{x} + t_k x_k \in C$. Since F is semi-differentiable at $(\hat{x}, \hat{y})$, for the above sequences $\{t_k\}$ and $\{x_k\}$, there exists a sequence $\{y_k\}$ with $y_k \to y$ and for all k, $\hat{y} + t_k y_k \in F(\hat{x} + t_k x_k) = F_C(\hat{x} + t_k x_k)$. Thus, $y \in DF_C(\hat{x}, \hat{y})(x)$. This completes the proof. □

We demonstrate by the following example that if F is semi-differentiable at $(\hat{x}, \hat{y})$, formulas (2.1) and (2.2) hold, and that if C is not derivable at $\hat{x}$, F may be not proto-differentiable on C at $(\hat{x}, \hat{y})$ even if it is semi-differentiable at $(\hat{x}, \hat{y})$.

Example 2.9. Let $C = \{x \mid x = 1/n, n = 1, 2, \cdots\} \cup \{0\}$ and

$$F(x) = \begin{cases} x, & \text{if } \ x \geq 0, \\ -x, & \text{otherwise.} \end{cases}$$

Then,

$$F_C(x) = \begin{cases} x, & \text{if } \ x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly, F is semi-differentiable at $(\hat{x}, \hat{y}) = (0, 0)$, and C is not derivable at $\hat{x}$ since

$$T(C, \hat{x}) = R_+,$$

and

$$T^b(C, \hat{x}) = 0.$$

By directly calculating, we have

$$DF(\hat{x}, \hat{y})(x) = D^b F(\hat{x}, \hat{y})(x) = \begin{cases} x, & \text{if } \ x > 0, \\ -x, & \text{if } \ x \leq 0, \end{cases}$$

$$DF_C(\hat{x}, \hat{y})(x) = \begin{cases} x, & \text{if } \ x \geq 0, \\ \emptyset, & \text{if } \ x < 0, \end{cases}$$

$$D^b F_C(\hat{x}, \hat{y})(x) = \begin{cases} 0, & \text{if } \ x = 0, \\ \emptyset, & \text{if } \ x \neq 0. \end{cases}$$

Thus, $DF_C(\hat{x}, \hat{y}) = DF(\hat{x}, \hat{y})_{T(C,\hat{x})}$ and $D^b F_C(\hat{x}, \hat{y}) = D^b F(\hat{x}, \hat{y})_{T^b(C,\hat{x})}$. However, F is not proto-differentiable on C at $(\hat{x}, \hat{y})$.

In the following example, we show that formulas (2.1) and (2.2) hold even if F is not semi-differentiable at $(\hat{x}, \hat{y})$. Moreover, we show that F is not proto-differentiable on C at $(\hat{x}, \hat{y})$ when C is derivable at $\hat{x}$ but F is not semi-differentiable at $(\hat{x}, \hat{y})$.

Example 2.10. Let $X = Y = R$, $C = R_+$ and

$$F(x) = \begin{cases} x, & \text{if } \ x = 1/n, n = 1, 2, \cdots, \\ -x, & \text{if } \ x \leq 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then,

$$F_C(x) = \begin{cases} x, & \text{if } \ x = 1/n, n = 1, 2, \cdots, \\ 0, & \text{if } \ x = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $\hat{x} = \hat{y} = 0$. Then, by directly calculating, we have

$$DF(\hat{x}, \hat{y})(x) = \begin{cases} x, & \text{if } \ x > 0, \\ -x, & \text{if } \ x \leq 0, \end{cases}$$

$$D^b F(\hat{x}, \hat{y})(x) = \begin{cases} \emptyset, & \text{if } \ x > 0, \\ -x, & \text{if } \ x \leq 0, \end{cases}$$

$$DF_C(\hat{x}), \hat{y})(x) = \begin{cases} x, & \text{if } \ x \geq 0, \\ \emptyset, & \text{if } \ x < 0, \end{cases}$$

$$D^b F_C(\hat{x}, \hat{y})(x) = \begin{cases} 0, & \text{if } \ x = 0, \\ \emptyset, & \text{if } \ x \neq 0, \end{cases}$$

and

$$T(C, \hat{x}) = T^b(C, \hat{x}) = R_+.$$

Thus, F is not semi-differentiable at $(\hat{x}, \hat{y})$ and F is not proto-differentiable on C at $(\hat{x}, \hat{y})$. However, we still have $DF_C(\hat{x}, \hat{y}) = DF(\hat{x}, \hat{y})_{T(C,\hat{x})}$ and $D^b F_C(\hat{x}, \hat{y}) = D^b F(\hat{x}, \hat{y})_{T^b(C,\hat{x})}$.

$\boxed{3}$ Main Results

In this section, we shall study the contingent and adjacent derivatives of the generalized perturbation map G introduced in (1.1). Let $H : R^d \times R^n \to 2^{R^m}$ be defined by $H(u, x) = F(u, x) + K_{\tilde{D}}(u, x)$, where $\tilde{D} = R^d \times D$ and $K_{\tilde{D}}$ is the restricted set-valued map of K on $\tilde{D}$. Then, G can be rewritten as

$$G(u, z) = \{x \in R^n \mid z \in H(u, x)\}.$$

Clearly, $\hat{x} \in G(\hat{u}, \hat{z})$ if and only if $\hat{z} \in H(\hat{u}, \hat{x})$. Thus, $x \in DG((\hat{u}, \hat{z}), \hat{x})(u, z)$ if and only if $z \in DH((\hat{u}, \hat{x}), \hat{z})(u, x)$, and similarly $x \in D^b G((\hat{u}, \hat{z}), \hat{x})(u, z)$ if and only if $z \in D^b H((\hat{u}, \hat{x}), \hat{z})(u, x)$. That is, in order to study the differential properties of G, we can focus on the differential properties of H. Noting that H is the sum of F and $K_{\tilde{D}}$, we can then apply the calculus rules [5] for the sum of two set-valued maps to obtain some formulas for the contingent and adjacent derivatives of H, and hence for the contingent and adjacent derivatives of G. Following directly from Theorem 28 and Corollary 29 of [5], we have the following theorem.

Theorem 3.1. *Let $\hat{x} \in G(\hat{u}, \hat{z})$ and $T^+ : R^d \times R^n \times R^m \to 2^{R^m}$ be defined by*

$$T^+(u, x, z) = F(u, x) \cap (z - K_{\tilde{D}}(u, x)).$$

Suppose that T^+ is compact at $(\hat{u}, \hat{x}, \hat{z})$, and that for any $\bar{z} \in T^+(\hat{u}, \hat{x}, \hat{z})$,

$$DT^+((\hat{u}, \hat{x}, \hat{z}), \bar{z})(0, 0, 0) = \{0\}. \tag{3.1}$$

If for any $\bar{z} \in T^+(\hat{u}, \hat{x}, \hat{z})$ and $(u, x, z) \in R^d \times R^n \times R^m$,

$$DT^+((\hat{u}, \hat{x}, \hat{z}), \bar{z})(u, x, z) = DF((\hat{u}, \hat{x}), \bar{z})(u, x) \cap (z - DK_{\tilde{D}}((\hat{u}, \hat{x}), \hat{z} - \bar{z})(u, x)), \tag{3.2}$$

then for any $(u,z) \in R^d \times R^m$,

$$DG((\hat{u},\hat{z}),\hat{x})(u,z) = \{x \in R^n \mid z \in DH((\hat{u},\hat{x}),\hat{z})(u,x)\}, \tag{3.3}$$

where

$$DH((\hat{u},\hat{x}),\hat{z})(u,x) = \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [DF((\hat{u},\hat{x}),\bar{z})(u,x) + DK_{\tilde{D}}((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)].$$

And if for any $\bar{z} \in T^+(\hat{u},\hat{x},\hat{z})$ *and* $(u,x,z) \in R^d \times R^n \times R^m$,

$$D^bT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(u,x,z) = D^bF((\hat{u},\hat{x}),\bar{z})(u,x) \cap (z - D^bK_{\tilde{D}}((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)), \tag{3.4}$$

then for any $(u,z) \in R^d \times R^m$,

$$D^bG((\hat{u},\hat{z}),\hat{x})(u,z) = \{x \in R^n \mid z \in D^bH((\hat{u},\hat{x}),\hat{z})(u,x)\}, \tag{3.5}$$

where

$$D^bH((\hat{u},\hat{x}),\hat{z})(u,x) = \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [D^bF((\hat{u},\hat{x}),\bar{z})(u,x) + D^bK_{\tilde{D}}((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)].$$

According to [5], we have the following remarks.

Remark 3.2. For T^+ being compact at $(\hat{u},\hat{x},\hat{z})$, we need only one of the following conditions:

(i) F is compact at $(\hat{u},\hat{x})$ and $K_{\tilde{D}}$ is closed at $(\hat{u},\hat{x})$;

(ii) F is closed at $(\hat{u},\hat{x})$ and $K_{\tilde{D}}$ is compact at $(\hat{u},\hat{x})$;

(iii) T^+ is upper locally lipschitz (u.l.l, for short) and compact-valued at $(\hat{u},\hat{x},\hat{z})$.

The condition (3.1) holds, if

$$DF((\hat{u},\hat{x}),\bar{z})(0,0) \cap -DK_{\tilde{D}}((\hat{u},\hat{x}),\hat{z}-\bar{z})(0,0) = \{0\}$$

or a stronger condition

$$DF((\hat{u},\hat{x}),\bar{z})(0,0) \cap -DK((\hat{u},\hat{x}),\hat{z}-\bar{z})(0,0) = \{0\}$$

holds. Both conditions (3.2) and (3.4) will hold if F is semi-differentiable at $((\hat{u},\hat{x}),\bar{z})$ or $K_{\tilde{D}}$ is semi-differentiable at $((\hat{u},\hat{x}),\hat{z}-\bar{z})$.

It follows directly from Theorem 3.1 and Proposition 2.8 that we have the following corollary.

Corollary 3.3. *Let* $\hat{x} \in G(\hat{u},\hat{z})$. *In addition to the conditions assumed in Theorem 3.1, suppose that* K *is semi-differentiable at* $(\hat{u},\hat{x})$. *Then, for any* $(u,z) \in R^d \times R^m$,

$$\begin{aligned} & DG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \; & \{x \in T(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [DF((\hat{u},\hat{x}),\bar{z})(u,x) + DK((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)]\}. \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & D^bG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T^b(D,\hat{x}) | z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [D^bF((\hat{u},\hat{x}),\bar{z})(u,x)+D^bK((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)]\}. \end{aligned} \tag{3.7}$$

Furthermore, if F is proto-differentiable at $(\hat{u},\hat{x})$ and D is derivable at $\hat{x}$, then G is proto-differentiable at $(\hat{u},\hat{z})$ and for any $(u,z) \in R^d \times R^m$,

$$G'_{(\hat{u},\hat{z}),\hat{x}}(u,z) = \{x \in T(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [F'_{(\hat{u},\hat{x}),\bar{z}}(u,x) + K'_{(\hat{u},\hat{x}),\hat{z}-\bar{z}}(u,x)]\}.$$

We illustrate Theorem 3.1 and Corollary 3.3 by the following example, and show that the union operations in Theorem 3.1 and Corollary 3.3 are necessary.

Example 3.4. Let $d = m = n = 1$, $D = R_+$, $F(u,x) = \{0,1\}$, $K(u,x) = [0,x]$ if $x \geq 0$, and $K(u,x) = [x,0]$ if $x < 0$. Then, $\tilde{D} = R \times R_+$,

$$H(u,x) = \begin{cases} \{z \in R \mid 0 \leq z \leq x+1\}, & \text{if } \ x \geq 1, \\ \{z \in R \mid 0 \leq z \leq x, \text{ or } \ 1 \leq z \leq x+1\}, & \text{if } \ 0 \leq x \leq 1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$G(u,z) = \begin{cases} \{x \in R \mid z-1 \leq x < \infty\}, & \text{if } \ z \geq 1, \\ \{x \in R \mid z \leq x < \infty\}, & \text{if } \ 0 \leq z < 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $\hat{u} = 0$ and $\hat{x} = \hat{z} = 1$. Clearly, $T^+(\hat{u},\hat{x},\hat{z}) = \{0,1\}$. Set $\bar{z}^1 = 0$ and $\bar{z}^2 = 1$. By directly calculating, we have $T(D,\hat{x}) = T^b(D,\hat{x}) = R$, and for any $u \in R$, $x \in R$ and $z \in R$,

$$DG((\hat{u},\hat{z}),\hat{x})(u,z) = D^bG((\hat{u},\hat{z}),\hat{x})(u,z) = \begin{cases} R, & \text{if } \ z \geq 0, \\ \{x \in R \mid x \geq z\}, & \text{if } \ z < 0, \end{cases} \tag{3.8}$$

$$\begin{aligned} & DF((\hat{u},\hat{x}),\bar{z}^1)(u,x) = D^bF((\hat{u},\hat{x}),\bar{z}^1)(u,x) \\ = \ & DF((\hat{u},\hat{x}),\bar{z}^2)(u,x) = D^bF((\hat{u},\hat{x}),\bar{z}^2)(u,x) = \{0\}, \end{aligned} \tag{3.9}$$

$$DK((\hat{u},\hat{x}),\hat{z}-\bar{z}^1)(u,x) = D^bK((\hat{u},\hat{x}),\hat{z}-\bar{z}^1)(u,x) = \{z \in R \mid z \leq x\}, \tag{3.10}$$

and

$$DK((\hat{u},\hat{x}),\hat{z}-\bar{z}^2)(u,x) = D^bK((\hat{u},\hat{x}),\hat{z}-\bar{z}^2)(u,x) = R^+. \tag{3.11}$$

It is easy to check that for any $(u,z) \in R^2$,

$$\begin{aligned} & DG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [DF((\hat{u},\hat{x}),\bar{z})(u,x) + DK((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)]\} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} & D^bG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T^b(D,\hat{x}) | z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} D^bF((\hat{u},\hat{x}),\bar{z})(u,x)+D^bK((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)\}. \end{aligned} \tag{3.13}$$

However, it follows from (3.8), (3.9), (3.10) and (3.11) that the union operations in (3.12) and (3.13) are necessary.

However, when a regular condition is imposed on the TP-derivative of T^+, the union operations appearing in Theorem 3.1 and Corollary 3.3 are unnecessary.

Theorem 3.5. *In addition to the conditions assumed in Theorem 3.1, suppose that*

$$D^pT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(0,0,0)=\{0\} \tag{3.14}$$

holds for some $\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})$. *Then, for any* $(u,z)\in R^d\times R^m$,

$$DG((\hat{u},\hat{z}),\hat{x})(u,z)=\{x\mid z\in DF((\hat{u},\hat{x}),\bar{z})(u,x)+DK_{\tilde{D}}((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)\},$$

and

$$D^bG((\hat{u},\hat{z}),\hat{x})(u,z)=\{x\mid z\in D^bF((\hat{u},\hat{x}),\bar{z})(u,x)+D^bK_{\tilde{D}}((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)\}.$$

If $K(x)$ *is semi-differentiable at* $((\hat{u},\hat{x}),\hat{z}-\bar{z})$, *then for any* $(u,z)\in R^d\times R^m$,

$$DG((\hat{u},\hat{z}),\hat{x})(u,z)=\{x\in T(D,\hat{x})\mid z\in DF((\hat{u},\hat{x}),\bar{z})(u,x)+DK((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)\},$$

and

$$D^bG((\hat{u},\hat{z}),\hat{x})(u,z)=\{x\in T^b(D,\hat{x})\mid z\in D^bF((\hat{u},\hat{x}),\bar{z})(u,x)+D^bK((\hat{u},\hat{x}),\hat{z}-\bar{z})(u,x)\}.$$

Furthermore, if in addition, F *is proto-differentiable at* $(\hat{u},\hat{x})$ *and* D *is derivable at* $\hat{x}$, *then* G *is proto-differentiable at* $(\hat{u},\hat{z})$ *and for any* $(u,z)\in R^d\times R^m$,

$$G'_{(\hat{u},\hat{z}),\hat{x}}(u,z)=\{x\in T(D,\hat{x})\mid z\in F'_{(\hat{u},\hat{x}),\bar{z}}(u,x)+K'_{(\hat{u},\hat{x}),\hat{z}-\bar{z}}(u,x)\}.$$

Proof. In view of Conclusion of [5], Theorem 3.1 and Corollary 3.3, the conclusions follow readily. □

Remark 3.6. A sufficient condition for (3.14) is that

$$D^pF((\hat{u},\hat{x}),\bar{z})(0)\cap -D^pK((\hat{u},\hat{x}),\hat{z}-\bar{z})(0)=\{0\},$$

which is the regular condition introduced in Theorem 3.1 of [3].

Corollary 3.7. *Suppose that the generalized perturbation map* G *from* $R^d\times R^m$ *to* R^n *is defined by*

$$G(u,z)=\{x\in D\mid z\in F(u,x)+C\},$$

where C *is a closed and convex subset of* R^m *and* D *is a nonempty subset of* R^n. *Let* $\hat{x}\in G(\hat{u},\hat{z})$ *and* $T^+(u,x,z)=F(u,x)\cap(z-C)$. *Suppose* $F(u,x)$ *is u.l.l. and compact-valued at* $(\hat{u},\hat{x})$ *and for any* $\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})$, $\bar{z}$ *is an isolated point of* $F(\hat{u},\hat{x})$. *Furthermore, if*

$$DT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(u,x,z)=DF((\hat{u},\hat{x}),\bar{z})(u,x)\cap(z-T(C,\hat{z}-\bar{z})),$$

holds for any $\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})$, *then*

$$\begin{aligned}&DG((\hat{u},\hat{z}),\hat{x})(u,z)\\=\;&\{x\in T(D,\hat{x})\mid z\in\bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})}[DF((\hat{u},\hat{x}),\bar{z})(u,x)+T(C,\hat{z}-\bar{z})]\}.\end{aligned} \tag{3.15}$$

And if

$$D^bT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(u,x,z) = D^bF((\hat{u},\hat{x}),\bar{z})(u,x) \cap (z - T(C,\hat{z}-\bar{z})),$$

holds for any $\bar{z} \in T^+(\hat{u},\hat{x},\hat{z})$, *then*

$$\begin{aligned} & D^bG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T^b(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [D^bF((\hat{u},\hat{x}),\bar{z})(u,x) + T(C,\hat{z}-\bar{z})]\}. \end{aligned} \quad (3.16)$$

Proof. We only need to prove (3.15), since (3.16) can be proved similarly. Since $F(u,x)$ is u.l.l. and compact-valued at $(\hat{u},\hat{x})$, we have F is compact at $(\hat{u},\hat{x})$. For a closed set C, follows from Remark 3.2 we have T^+ is compact at $(\hat{u},\hat{x},\hat{z})$. Let $\bar{z} \in T^+(\hat{u},\hat{x},\hat{z})$ and $\xi \in DT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(0,0,0)$. By the definition of contingent derivative, there exist sequences $\{t_k\} \subset R_{++}$, $\{(u_k,x_k,z_k)\} \subset R^d \times R^n \times R^m$ and $\{\xi_k\} \subset R^m$ such that $t_k \to 0$, $(u_k,x_k,z_k) \to (0,0,0)$, $\xi_k \to \xi$ and for all k, $\bar{z} + t_k\xi_k \in T^+((\hat{u},\hat{x},\hat{z}) + t_k(u_k,x_k,z_k))$. That is $\bar{z}+t_k\xi_k \in F(\hat{u}+t_ku_k,\hat{x}+t_kx_k) \cap (\hat{z}+t_kz_k - C)$. Then we have $\xi \in DF((\hat{u},\hat{x}),\bar{z})(0,0)$. Since $F(u,x)$ is u.l.l. at $(\hat{u},\hat{x})$ and $\bar{z}$ is an isolated point of $F(\hat{u},\hat{x})$, by Lemma 9 in [5] we have $\xi = 0$. Thus $DT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(0,0,0) = \{0\}$. Let $K(u,x) = C$, from the convexity of the set C, we can deduce that K is semi-differentiable at $((\hat{u},\hat{x}),\hat{z}-\bar{z})$. By Theorem 3.1, we have

$$\begin{aligned} & DG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [DF((\hat{u},\hat{x}),\bar{z})(u,x) + T(C,\hat{z}-\bar{z})]\}. \end{aligned}$$

This completes the proof. □

Corollary 3.8. *Suppose that the generalized perturbation map G from $R^d \times R^n$ to R^n is defined by*

$$G(u,z) = \{x \in D \mid z \in F(u,x) + N_D(x)\},$$

where F is a set-valued map and $D \subset R^n$ is a polyhedral convex set. Let $\hat{z} \in H(\hat{u},\hat{x})$ and $K(u,x) = N_D(x)$. Suppose $F(u,x)$ is u.l.l. and compact-valued at $(\hat{u},\hat{x})$ and, for any $\bar{z} \in T^+(\hat{u},\hat{x},\hat{z})$, $\bar{z}$ is an isolated point of $F(\hat{u},\hat{x})$. Furthermore, if for any $\bar{z} \in T^+(\hat{u},\hat{x},\hat{z})$,

$$DT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(u,x,z) = DF((\hat{u},\hat{x}),\hat{z})(u,x) \cap (z - N_{D'(\hat{x},\hat{z}-\bar{z})}(x))$$

holds, then

$$\begin{aligned} & DG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [DF((\hat{u},\hat{x}),\bar{z})(u,x) + N_{D'(\hat{x},\hat{z}-\bar{z})}(x)]\}, \end{aligned} \quad (3.17)$$

and if for any $\bar{z} \in T^+(\hat{u},\hat{x},\hat{z})$,

$$D^bT^+((\hat{u},\hat{x},\hat{z}),\bar{z})(u,x,z) = D^bF((\hat{u},\hat{x}),\hat{z})(u,x) \cap (z - N_{D'(\hat{x},\hat{z}-\bar{z})}(x))$$

holds, then

$$\begin{aligned} & D^bG((\hat{u},\hat{z}),\hat{x})(u,z) \\ = \ & \{x \in T^b(D,\hat{x}) \mid z \in \bigcup_{\bar{z}\in T^+(\hat{u},\hat{x},\hat{z})} [D^bF((\hat{u},\hat{x}),\bar{z})(u,x) + N_{D'(\hat{x},\hat{z}-\bar{z})}(x)]\}, \end{aligned} \quad (3.18)$$

where $D'(\hat{x},\hat{z}-\bar{z}) = \{x \in T(D,\hat{x}) \mid x \cdot (\hat{z}-\bar{z}) = 0\}$.

Proof. It directly follows from the proof of Theorem 5.6 in [10] and Corollary 3.7. □

Remark 3.9. If the set-valued map F is single-valued, continuous and Fréchet differentiable, then the conditions in Corollary 3.7 and Corollary 3.8 are hold naturally and the $\bar{z}$ is unique. Furthermore, since F is semi-differentiable, then G is proto-differentiable and its proto-derivative are respectively given by

$$G'_{(\hat{u},\hat{z}),\hat{x}}(u,z) = \{x \in T(D,\hat{x}) \mid z \in F'_{(\hat{u},\hat{x}),\bar{z}}(u,x) + T(C,\hat{z}-\bar{z})\},$$

and

$$G'_{(\hat{u},\hat{z}),\hat{x}}(u,z) = \{x \in T(D,\hat{x}) \mid z \in F'_{(\hat{u},\hat{x}),\bar{z}}(u,x) + N_{D'(\hat{x},\hat{z}-\bar{z})}(x)\},$$

where $D'(\hat{x},\hat{z}-\bar{z}) = \{x \in T(D,x) \mid x \cdot (\hat{z}-\bar{z}) = 0\}$. So, Corollary 3.7 generalizes Corollary 3.1 in [3] and Corollary 3.8 generalizes Corollary 3.2 in [3] and Theorem 5.6 in [10].

Acknowledgements

The authors wish to express their sincere appreciation to the anonymous referees whose careful reading improved this paper.

References

[1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley, New York, 1984.

[2] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[3] G.M. Lee and N.Q. Huy, On proto-differentiability of generalized perturbation maps, *J. Math. Anal. Appl.* 324 (2006) 1297–1309.

[4] A.B. Levy and R.T. Rockafellar, Sensitivity analysis of solutions to generalized equations, *Trans. Amer. Math. Soc.* 345 (1994) 661–671.

[5] S.J. Li, K.W. Meng and J.P. Penot, Calculus rules for derivatives of multimaps, *Set-valued Anal.* 17 (2009) 21–39.

[6] S.J. Li, and K.W. Meng, Contingent derivatives of set-valued maps with applications to vector optimization (submitted).

[7] D.T. Luc, Contigent derivatives of set-valued maps and applications to vector optimization, *Math. Program.* 50 (1991) 99–111.

[8] J.P. Penot, Differentiability of relations and stability of perturbed optimization problems, *SIAM J. Control Optim.* 22 (1984) 529–551.

[9] S.M. Robinson, Implicit B-differentiability in generalized equations, Univ. of Wisconsin-Madison Math. Res. Center, Technical Summary Rep. 2854, 1985.

[10] R.T. Rockafellar, Proto-differentiability of set-valued mappings and its applications in optimization, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 6 (1989) 449–482.

[11] D.S. Shi, Contigent derivative of the perturbation map in multiobjective optimization *J. Optim. Theory Appl.* 70 (1991) 385–396.

[12] T. Tanino, Sensitivity analysis in multiobjective optimization, *J. Optim. Theory Appl.* 89 (1998) 713–730.

Manuscript received 13 February 2008
revised 15 December 2008
accepted for publication 18 December 2008

X.W. Xue
College of Mathematics and Science, Chongqing University
Chongqing, 400044, China
E-mail address: xuexw1@126.com

S.J. Li
College of Mathematics and Science, Chongqing University
Chongqing, 400044, China
E-mail address: lisj@cqu.edu.cn