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ON EXISTENCE OF OPTIMAL SOLUTION FOR SUBLINEAR PROGRAMS*

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Dedicted to Professor Guang Ya Chen on his 70th birthday.

Abstract: A sublinear program (P), which involves a sublinear objective function and a constrained set defined by a cone-sublinear function and a closed convex cone, is considered. We show that the existence of optimal solutions for (P) is closely related to zero solution and that a condition for the existence can be expressed in terms of subdifferentials of the functions involved in (P).

Key words: *sublinear program, optimal solutions, existence, subdifferential*

Mathematics Subject Classification: *90C36, 90C46*

1 Introduction and Preliminaries

Consider the following sublinear program:

$$(\mathrm{P}) \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \in -S, \end{array}$$

where X and Y are Banach spaces, S is a closed convex cone in Y, which does not necessarily have nonempty interior, and the mappings $f : X \to \mathbb{R}$ and $g : X \to Y$ are a continuous sublinear function and a continuous S-sublinear function, respectively. Recall that g is said to be S-sublinear if

(i) $\forall x \in X \quad \forall \lambda \geqq 0,\ g(\lambda x) = \lambda g(x)$,

(ii) $\forall x, y \in X, \quad g(x) + g(y) - g(x+y) \in S$.

We assume that the feasible set $A := \{x \in X \mid g(x) \in -S\}$ is nonempty. We denote the set of all solutions of (P) by sol(P). The continuous dual space of Y is denoted by Y^* and is endowed with the weak* topology. The (positive) polar of the cone $S \subseteq Y$ is the cone $S^+ = \{\theta \in Y^* \mid \theta(k) \geq 0 \ \ \forall k \in S\}$.

Recently, many authors [2, 4–7] extended Farkas Lemma ([1]) to convex systems.

In this brief paper, we show that the existence of optimal solutions for (P) is closely related to zero solution and that a condition for the existence can be expressed in terms of subdifferentials of the functions involved in (P). The condition is obtained from a generalized Farkas Theorem in [6].

Now we give notations and preliminary results that will be used later.

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Definition 1.1 ([3, 8]). Let X be a Banach space, let X^* be the continuous dual space of X and let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower-semicontinuous convex function.

(1) The conjugate function of $h, h^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$h^*(v) = \sup\{v(x) - h(x) \mid x \in \text{dom}h\},$$

where the domain of h, dom$h$, is given by

$$\text{dom}h = \{x \in X \mid h(x) < +\infty\}.$$

(2) The epigraph of h, epi$h$, is defined by

$$\text{epi}h = \{(x, r) \in X \times \mathbb{R} \mid x \in \text{dom}h, h(x) \leq r\}.$$

(3) The subdifferential of h at $a \in \text{dom}h$ is defined as the non-empty weak* compact convex set

$$\partial h(a) := \{v \in X^* \mid h(x) - h(a) \geq v(x - a), \quad \forall x \in \text{dom}h\}.$$

It is well-known that if h is sublinear (i.e., convex and positively homogeneous of degree one), then

$$\text{epi}h^* = \partial h(0) \times \mathbb{R}_+.$$

For a closed convex subset D of X, the indicator function δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The support function δ_D^* is defined by $\delta_D^*(u) = \sup_{x \in D} u(x)$. Then $\partial\delta_D(x) = N_D(x)$, which is known as the normal cone of D of x.

Let $S \subseteq Y$ be a closed convex cone. Then we say that the mapping $g : X \to Y$ is $S-$convex if for any $x_1, x_2 \in X$ and any $\lambda \in [0, 1]$,

$$g(\lambda x_1 + (1 - \lambda)x_2) \in \lambda g(x_1) + (1 - \lambda)g(x_2) - S.$$

Now we give a generalized Farkas Lemma to convex systems.

Lemma 1.2 ([6]). *Let $S \subseteq Y$ be a closed convex cone, let $u : X \to \mathbb{R}$ be a continuous linear mapping, and let $g : X \to Y$ be a continuous S-convex mapping. Suppose that the system $g(x) \in -S$ is consistent. Let $\alpha \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\{x \in X \mid g(x) \in -S\} \subseteq \{x \in X \mid u(x) \leq \alpha\}$.

(ii) $\begin{pmatrix} u \\ \alpha \end{pmatrix}^T \in cl\Big(\bigcup_{\lambda \in S^+} \text{epi}(\lambda \circ g)^*\Big)$.

2 Existence of Optimal Solution

The following proposition is needed to prove the main results.

Proposition 2.1. *If $sol(P) \neq \emptyset$, then $sol(P) = \{x \in A \mid f(x) = 0\}$.*

Proof. Suppose that $sol(\text{P}) \neq \emptyset$. Let $\bar{x} \in A$ be a solution of (P). Then for any $x \in A$, $f(\bar{x}) \leqq f(x)$. Since $\bar{x} \in A$ and g is S-sublinear,

$$g(\alpha\bar{x}) = \alpha g(\bar{x}) \in -S \quad \text{for any } \alpha \geqq 0.$$

Thus $\alpha\bar{x} \in A$ for any $\alpha \geqq 0$, and hence $f(\bar{x}) \leqq f(\alpha\bar{x}) = \alpha f(\bar{x})$ for any $\alpha \geqq 0$. Since $0 \in A$, $f(\bar{x}) \leqq f(0) = 0$. If $f(\bar{x}) < 0$, $f(\alpha\bar{x}) = \alpha f(\bar{x}) \to -\infty$ as $\alpha \to \infty$. This is impossible since (P) has a solution. Therefore $f(\bar{x}) = 0$. So, we have,

$$sol(\text{P}) = \{x \in A \mid f(x) = 0\}.$$

□

Now we give an example to illustrate Proposition 2.1.

Example 2.2. Let $f(x) = g(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ x & \text{if} \quad x > 0 \end{cases}$. Consider the following sublinear program:

$$\text{(P)} \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & x \in A := \{x \mid g(x) \in -\mathbb{R}_+ \}. \end{array}$$

Then it is clear that (P) has a solution. Moreover,

$$\begin{aligned} sol(\mathrm{P}) &= \{x \in A \mid f(x) = 0\} \\ &= \{x \in (-\infty, 0] \mid f(x) = 0\} \\ &= (-\infty, 0]. \end{aligned}$$

So, Proposition 2.1 holds. □

Now we give a well-known formula for normal cone ([2]). For the completeness, we give a proof for the formula.

Proposition 2.3. *Let $g : X \to Y$ be a continuous S-sublinear function and $A := \{x \in X \mid g(x) \in -S\}$. Then $N_A(0) = cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)\Big)$.*

Proof. $v \in N_A(0)$

$$\begin{aligned} &\iff \quad \forall x \in A,\ v(x) \leqq 0 \\ &\iff \quad A \subset \{v \mid v(x) \leqq 0\} \\ &\iff \quad \text{(by Lemma 1.2)} \quad \begin{pmatrix} v \\ 0 \end{pmatrix}^T \in cl\Big(\bigcup_{\lambda \in S^+} \mathrm{epi}(\lambda \circ g)^*\Big) \end{aligned}$$

$$\begin{aligned} &\iff \quad \text{(since } g \text{ is } S-\text{sublinear)} \quad \begin{pmatrix} v \\ 0 \end{pmatrix}^T \in cl \bigcup_{\lambda \in S^+} \Big(\partial(\lambda \circ g)(0) \times \mathbb{R}_+\Big) \\ &\iff \quad \begin{pmatrix} v \\ 0 \end{pmatrix}^T \in cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)\Big) \times \mathbb{R}_+ \\ &\iff \quad v \in cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)\Big). \end{aligned}$$

□

We give a theorem which shows that the existence of optimal solutions for (P) is closely related to zero solution and that a condition for the existence can be expressed in terms of subdifferentials of the functions involved in (P).

Theorem 2.4. *The following statements are equivalent:*

(i) *$sol(P) \neq \emptyset$.*
(ii) *$0 \in sol(P)$.*
(iii) *$0 \in \partial f(0) + cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)\Big)$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $sol(\text{P}) \neq \emptyset$. Then by Proposition 2.1, $sol(\text{P}) = \{x \in A \mid f(x) = 0\}$. Since f is sublinear, $f(0) = 0$ and hence $0 \in sol(\text{P})$.

(ii) $\Rightarrow$ (i): It is trivial.

(ii) $\Rightarrow$ (iii): Suppose that $0 \in sol(\text{P})$. Then $f(x) + \delta_A(x) \geqq f(0) + \delta_A(0) + 0(x - 0)$ for any $x \in A$ and hence $0 \in \partial(f + \delta_A)(0) = \partial f(0) + \partial\delta_A(0) = \partial f(0) + N_A(0)$. Thus by Proposition 2.3, $0 \in \partial f(0) + cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)\Big)$.

(iii) $\Rightarrow$ (ii): Suppose that (iii) holds. Then it follows from Propostion 2.3 that there exists $v \in \partial f(0)$ such that $-v \in N_A(0) = \partial\delta_A(0)$. Hence $0 \in \partial f(0) + \partial\delta_A(0) = \partial(f + \delta_A)(0)$. So, $f(x) + \delta_A(x) \geqq f(0) + \delta_A(0) + 0(x - 0)$ for any $x \in X$ and hence $f(x) \geqq f(0)$ for any $x \in A$. This means that $0 \in sol(\text{P})$. ☐

Remark 2.5. The equivalence between (ii) and (iii) in Theorem 2.4 was already established by Glover ([2]). However, we give its proof in order to show that a condition for the existence of solution of (P) can be expressed in terms of subdifferentials of the functions involved in (P).

Consider the following linear program:

$$(\text{LP}) \qquad \text{Minimize} \quad c(x) \qquad \text{subject to} \quad A(x) \in -S,$$

where X and Y are Banach spaces, S a closed convex cone in Y, which does not necessarily have nonempty interior, and the mappings $c : X \to \mathbb{R}$ and $A : X \to Y$ are continuous and linear functions. We denote the adjoint operator of the linear mapping A by A^T. We denote the set of all solutions of (LP) by sol(LP). Then we can easily obtain the following corollary from Theorem 2.4.

Corollary 2.6. *The following statements are equivalent:*

(i) $sol(LP) \neq \emptyset$.

(ii) $0 \in sol(LP)$.

(iii) $0 \in c + cl\Big(A^T(S^+)\Big)$.

Now we give two examples illustrating Theorem 2.4.

Example 2.7. Let $f(x) = \begin{cases} x & \text{if } x \leqq 0 \\ 2x & \text{if } x > 0 \end{cases}$ and $g(x) = \begin{cases} 0 & \text{if } x \leqq 0 \\ x & \text{if } x > 0 \end{cases}$. Consider the following sublinear program:

$$(\text{P}) \qquad \text{Minimize} \quad f(x) \qquad \text{subject to} \quad x \in A := \{x \in \mathbb{R} \mid g(x) \in -\mathbb{R}_+ \}.$$

Then $f(0) = 0$, but (P) has no solution. Moreover, $\partial(\lambda g)(0) = [0, \lambda]$ for any $\lambda \geqq 0$ and hence $\bigcup_{\lambda \geqq 0} \partial(\lambda g)(0) = \mathbb{R}_+$. But $\partial f(0) = [1, 2]$ and hence $\partial f(0) + cl\Big(\bigcup_{\lambda \geqq 0} \partial(\lambda g)(0)\Big) = [1, \infty)$. So, Theorem 2.4 does not hold. ☐

Example 2.8. Let $f(x, y) = x$ and $g(x, y) = (x^2 + y^2)^{\frac{1}{2}} - y$. Consider the following sublinear program:

$$(\text{P}) \qquad \text{Minimize} \quad f(x, y) \qquad \text{subject to} \quad (x, y) \in A := \{(x, y) \in \mathbb{R}^2 \mid g(x, y) \in -\mathbb{R}_+ \}.$$

Then $A = \{(0, y) \mid y \geqq 0\}$ and sol(P)=A. Let $(\bar{x}, \bar{y}) = (0, 1)$. Then $(\bar{x}, \bar{y}) \in$ sol(P) and $f(\bar{x}, \bar{y}) = 0$. Moreover, $\partial f(0,0) = \{(1,0)\}$ and $cl\Big(\bigcup_{\lambda \in \mathbb{R}_+} \partial(\lambda g)(0,0)\Big) = \{(v_1, v_2) \mid v_1 \in \mathbb{R},\ v_2 \leqq 0\}$. Thus

$$(0,0) \in \partial f(0,0) + cl\Big(\bigcup_{\lambda \in \mathbb{R}^+} \partial(\lambda g)(0,0)\Big).$$

So, Theorem 2.4 holds.

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