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BENSON EFFICIENCY OF A MULTI-CRITERION NETWORK EQUILIBRIUM MODEL*

Yunan Wu and T.C. Edwin Cheng

Dedicated to Guangya Chen on the occasion of his 70th birthday.

Abstract: We consider Benson efficiency of a multi-product network equilibrium model based on Wardrop's principle in this paper. We show that in both the single and multiple criteria cases, such proper efficiency can be recast as vector variational inequalities. In the multiple criteria case we derive a sufficient and a necessary condition for Benson efficiency in terms of vector variational inequalities by using the Gerstewitz's function.

Key words: *Benson efficiency, network equilibrium, variational inequality, Wardrop's equilibrium principle*

Mathematics Subject Classification: *90C29, 90C35; 90B06, 90B10*

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1 Introduction

The earliest network equilibrium model was proposed by Wardrop (1952) for a transportation network, which asserts that the traffic flow along a path joining an origin-destination (OD) pair is positive only if the delay (cost) for this path is the minimum possible amongst all the paths joining the same OD pair. Since then, many other equilibrium models have been proposed in the economics literature. Until only recently, all these equilibrium models are based on a single cost or utility function. In practice, the choice of paths based on a single criterion by all the road users may not be reasonable. Minimum delay paths are sometimes not the cheapest ones to travel on. Recently, equilibrium models based on multi-criterion consideration or vector-valued cost functions have been proposed. In Chen and Yen (1993), a multi-criterion traffic equilibrium model was proposed. Other papers that consider multi-criterion equilibrium models can be found in Yang and Goh (1997), Chen, Goh and Yang (1999), Cheng and Wu (2006) and Li, Teo and Yang (2008).

In many multi-criterion decision-making problems, the common practice is to obtain the set of efficient decisions, i.e., decisions that are not dominated by any others. Kuhn and Tucker (see Kuhn and Tucker (1951)), and later Geoffrion (see Geoffrion (1968)), observed that a subset of efficient set may be "improper". Practically, this means that points in the subset cannot be satisfactorily characterized by a scalar minimization problem, even if the

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decision set is convex. So, the concept of *proper efficiency* was introduced by Kuhn and Tucker (1951), Geoffrion (1968), and modified and formulated into a more general framework by Borwein (1977), Benson (1979), Henig (1982), and Borwein and Zhuang (1993), among many other researchers. The motivation for introducing proper efficiency is that it enables one to eliminate certain anomalous efficient decisions and to prove the existence of equivalent scalar problems whose solutions produce most of the efficient decisions at least, namely the proper ones. It has been amply demonstrated that proper efficiency is a natural concept in vector optimization.

In this paper we consider a kind of proper efficiency – Benson efficiency – of a multi-product network equilibrium model with a vector-valued cost function. We establish a sufficient and a necessary condition for a Benson equilibrium pattern flow for a multi-product network equilibrium problem in terms of vector variational inequalities for the single criterion case and the multiple criteria case.

The organization of the paper is as follows. In Section 2 we introduce some notation and preliminaries. The relation between Benson efficiency of a multi-product network equilibrium model with a single criterion and vector variational inequalities is established in Section 3. In Section 4 we deduce a sufficient and a necessary condition for Benson efficiency of a multi-product network equilibrium model with multiple criteria in terms of vector variational inequalities by using Gerstewitz's scalarization function. We conclude the paper in Section 5.

2 Notations and Preliminaries

We consider a network in which q products traverse with a typical product denoted by j. Consider a general network $G = [N, A, I]$, where N denotes the set of nodes representing manufacturers and retailers, as well as distributing centers and warehouses, and A the set of directed arcs. Let $a \in A$ denote an arc connecting a pair of nodes. Let I denote the set of all the OD pairs associated with each pair of manufacturer and retailer, and $|I| = l$. We denote by K_i the set of paths that connect an OD pair $i \in I$ associated with a given pair of manufacturer and retailer and let $m = \sum_{i\in I} |K_i|$. Let $k \in K_i$ denote a path, assumed to be acyclic, consisting of a sequence of arcs connecting an OD pair i.

For a path $k \in K_i$, let v_k^j denote the flow of product j on path k. A path flow v_k^j induces a flow v_a^j of product j on an arc $a \in A$ as follows:

$$v_a^j = \sum_{i\in I} \sum_{k\in K_i} \delta_{ak} v_k^j,$$

where

$$\Delta = [\delta_{ak}] \in R^{|A|\times m}$$

is the arc path incidence matrix, with

$$\delta_{ak} = \begin{cases} 1, & \text{if } a \in k \\ 0, & \text{otherwise.} \end{cases}$$

A vector $v^j = (v_k^j : \ k \in K_i, \ i \in I)$ such that $v_k^j \geqslant 0, \ \forall k \in K_i, \ i \in I, \ j = 1, 2, \cdots, q$, is said to be a flow of product j on the network and $v = (v^1, v^2, \cdots, v^q)^T$ is called a flow of the network. Let there also be given a vector of demands $d = (d_i^j : \ i \in I, \ j = 1, 2, \cdots, q)$. Each component d_i^j indicates the demand of the OD pair i for product j, i.e., the quantity of product j that needs to go from the manufacturer to the retailer associated with the OD pair i. We say that a flow of the network v satisfies the demands if

$$\sum_{k\in K_i} v_k^j = d_i^j, \quad \forall i \in I,\ j = 1,2,\cdots,q.$$

Then, the set $D = \{v : \sum_{k\in K_i} v_k^j = d_i^j,\ \forall i \in I,\ j = 1,2,\cdots,q\}$ is the feasible set. D is clearly a convex set.

Next, we introduce the notation about Benson efficiency. Let Y be a real normed space ordered by a closed, convex and pointed cone $M \subset Y$ with nonempty interior $int\ M$. We denote ordering as follows:

$$x \leqslant y \quad \text{iff} \quad y - x \in M;$$

$$x < y \quad \text{iff} \quad y - x \in int\ M.$$

We denote the closure of a nonempty subset P of Y by $cl(P)$ and the cone hull of P by $cone(P)$, i.e.,

$$cone(P) := \cup\{\lambda a :\ \lambda \geqslant 0,\ a \in P\}.$$

A point $e^* \in P \subset Y$ is said to be an efficient point of P if $e - e^* \notin -M \setminus \{0\}$ for any $e \in P$. By $Eff(P)$ we denote the set of all the efficient points of P. We also need to introduce the concept of Benson efficient points of the set P. A point $e^* \in P \subset Y$ is said to be a Benson efficient point of P if $cl(cone(P + M - e^*)) \cap (-M) = \{0\}$. We denote the set of all the Benson efficient points of P by $Benson(P)$.

3 Benson Efficiency of a Network Equilibrium Model with a Single Criterion

In this section the function $c_a^j(v) : R^{q\times m} \to R_+$ is interpreted as the cost of product j on arc $a \in A$. Then the cost function of product j on a path k ($k \in K_i,\ i \in I$) depending on the flow of the network is defined by the formula

$$c_k^j(v) = \sum_{a\in k} c_a^j(v).$$

Then, the vector function $c^j(v) = (c_k^j(v) :\ k \in K_i,\ i \in I)$ and $c(v) = (c^1(v), c^2(v), \cdots, c^q(v))^T$ are called the cost function of product j on the network and the cost function of the network, respectively.

For each $i \in I$, we define the minimum cost function of product j for the OD pair i by putting

$$m_i^j(v) = \min_{k\in K_i} c_k^j(v).$$

Set $m_i(v) = (m_i^1(v), m_i^2(v), \cdots, m_i^q(v))^T$. We group the $q \times m$ matrix v into a q-dimensional column vector v_k ($\forall\ k \in K_i,\ i \in I$) with components $v_k = (v_k^1, v_k^2, \cdots, v_k^q)^T$, where $v = (v_k :\ k \in K_i,\ i \in I)$. Also, group the vector $c(v)$ into a q-dimensional column vector $c_k(v),\ k \in K_i,\ i \in I$, with components $c_k(v) = (c_k^1(v), c_k^2(v), \cdots, c_k^q(v))^T$, where $c(v) = (c_k(v) :\ k \in K_i,\ i \in I)$. For the q-dimensional Euclidean space R^q, by $\leqslant$ we denote the ordering induced by R_+^q :

$$x \leqslant y \quad \text{iff} \quad y - x \in R_+^q;$$

$$x < y \quad \text{iff} \quad y - x \in int\ R_+^q.$$

The ordering $\geqslant$ and $>$ are defined similarly.

Applying Wardrop's equilibrium principle (Wardrop (1952)), we see that the equilibrium principle (user-optimizing principle) in a multi-product network equilibrium problem takes on the following form.

Definition 3.1. A vector $v \in D$ is called an equilibrium pattern flow iff

$$c_k(v) - m_i(v) \left\{ \begin{array}{ll} = 0 & \text{if } v_k \in R^q_+ \setminus \{0\} \\ \geqslant 0 & \text{if } v_k = 0. \end{array} \right. \tag{3.1}$$

for each $i \in I$ and each $k \in K_i$.

The above equilibrium principle involves no explicit optimization concept because the network users act independently, in a noncooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, condition (3.1) means that only those paths connecting an OD pair that have minimal user travel costs in terms of vector ordering will be used. Otherwise, the network users could improve upon their situations by switching to a path with a lower cost. That is, for any OD pair of manufacturer and retailer i, if the transportation cost of all the products on a path $k \in K_i$ is greater than the minimum cost of the OD pair i in terms of vector ordering, then the flow of all the products on k is zero.

For the sake of convenience, the equilibrium condition (3.1) can be expressed in the following equivalent form.

Proposition 3.2 (see Cheng and Wu (2006)). *The network equilibrium condition (3.1) is equivalent to the following statement:*

$$c_r(v) - c_k(v) \in R^q_+ \setminus \{0\} \Rightarrow v_r = 0, \tag{3.2}$$

for each $i \in I$ and any $k, r \in K_i$.

It seems that the left hand side of (3.2) is defined in a way similar to the definition of strong efficiency (see Liu and Gong (2000)). Next, we introduce a kind of proper efficiency – Benson efficiency – of a network equilibrium model.

Definition 3.3. A vector $v \in D$ is called a Benson equilibrium pattern flow iff, for each $i \in I$ and $k \in K_i$, the following statement holds:

$$\left. \begin{array}{l} cl(cone(c_{K_i}(v) + R^q_+ - c_k(v))) \cap (-R^q_+) = \{0\} \\ c_r(v) - c_k(v) \neq 0 \end{array} \right\} \Rightarrow v_r = 0, \quad r \in K_i, \ r \neq k.$$

Variational inequality theory is a powerful tool in the qualitative analysis of equilibrium theory (see, for example, Nagurney (1999)). Now, let us introduce the concept of Benson efficient solution to a vector variational inequality problem.

Definition 3.4. A vector $v \in D$ is called a Benson efficient solution to a vector variational inequality problem iff

$$cl(cone(c(v)(D - v)^T + R^{q\times q}_+)) \cap (-R^{q\times q}_+) = \{0\}.$$

We need the following assumption.

Assumption 3.5.

$$c_r(v) - c_k(v) \neq 0, \qquad \text{if} \quad r \neq k,$$

for any $i \in I$ and $r, k \in K_i$.

Next, we will establish a sufficient and a necessary condition for Benson efficiency of a network equilibrium problem in terms of vector variational inequality problems. Specifically, we wish to prove the following two theorems.

Theorem 3.6. *Under Assumption 3.1, if a vector $v \in D$ is a Benson equilibrium pattern flow, then v is a Benson efficient solution to the vector variational inequality problem: to find $v \in D$ such that*

$$cl(cone(c(v)(D-v)^T + R_+^{q\times q})) \cap (-R_+^{q\times q}) = \{0\}.$$

Proof. Let a vector $v \in D$ be a Benson equilibrium pattern flow for a network equilibrium problem. By Definition 3.3, we have the following statement:

$$\left.\begin{array}{l} cl(cone(c_{K_i}(v) + R_+^q - c_k(v))) \cap (-R_+^q) = \{0\} \\ c_r(v) - c_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0, \quad r \in K_i,\ r \neq k,$$

for each $i \in I$ and any $k \in K_i$.
For any $u \in D$, we have

$$\begin{aligned} &\langle c(v), (u-v)^T\rangle \\ =&(c_1(v), c_2(v), \cdots, c_m(v))(u_1 - v_1, u_2 - v_2, \cdots, u_m - v_m)^T \\ =&\sum_{t=1}^{m} c_t(v)(u_t - v_t)^T \\ =&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t(v)(u_t - v_t)^T]. \end{aligned}$$

We know $c_t(v)(u_t - v_t)^T$ is a $q \times q$ matrix whose components are $c_t^{\alpha}(v)(u_t^{\beta} - v_t^{\beta})$, where $\alpha, \beta \in \{1, 2, \cdots, q\}$. Hence, $\langle c(v), (u-v)^T\rangle$ is also a $q \times q$ matrix whose components are $\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\alpha}(v)(u_t^{\beta} - v_t^{\beta})]$, where $\alpha, \beta = 1, 2, \cdots, q$.

Set

$$J_i(v) := \{\bar{r} \in K_i :\ c_{\bar{r}}(v) \in Benson\{c_r(v) :\ r \in K_i\}\} \subset K_i.$$

Then, for any $\bar{r} \in J_i(v) \subset K_i$,

$$cl(cone(c_{K_i}(v) + R_+^q - c_{\bar{r}}(v))) \cap (-R_+^q) = \{0\}.$$

By Assumption 3.5 and Definition 3.3, we have $v_r = 0$ for any $r \in K_i,\ i \in I$ and $r \neq \bar{r}$. Since, $Benson\{c_r(v) :\ r \in K_i\} \subset Eff\{c_r(v) :\ r \in K_i\}$, we know that

$$c_{\bar{r}}(v) \in Eff\{c_r(v) :\ r \in K_i\}.$$

That is,

$$c_r(v) - c_{\bar{r}}(v) \notin -R_+^q \setminus \{0\}, \qquad \forall\, r \in K_i,\ i \in I \text{ and } r \neq \bar{r}.$$

By $c_r(v) - c_{\bar{r}}(v) \neq 0$, we obtain

$$c_r(v) - c_{\bar{r}}(v) \notin -R_+^q, \qquad \forall\, r \in K_i,\ i \in I \text{ and } r \neq \bar{r}.$$

It means that there exists an $\bar{\alpha} \in \{1, 2, \cdots, q\}$ such that

$$c_r^{\bar{\alpha}}(v) - c_{\bar{r}}^{\bar{\alpha}}(v) > 0.$$

Thus, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\bar{\alpha}}(v)(u_t^{\beta}-v_t^{\beta})] \\
=&\sum_{i=1}^{l}[\sum_{t\in K_i\setminus\{\bar{r}\}} c_t^{\bar{\alpha}}(v)(u_t^{\beta}-v_t^{\beta})+c_{\bar{r}}^{\bar{\alpha}}(v)(u_{\bar{r}}^{\beta}-v_{\bar{r}}^{\beta})].
\end{aligned}
$$

Since $t\in K_i\setminus\{\bar{r}\}$, we have $t\in K_i$ and $t\neq\bar{r}$. Hence, $v_t=0$. That is, for any $\beta\in\{1,2,\cdots,q\}$, $v_t^{\beta}=0$. So, we get

$$
\begin{aligned}
&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\bar{\alpha}}(v)(u_t^{\beta}-v_t^{\beta})] \\
=&\sum_{i=1}^{l}[\sum_{t\in K_i\setminus\{\bar{r}\}} c_t^{\bar{\alpha}}(v)u_t^{\beta}+c_{\bar{r}}^{\bar{\alpha}}(v)(u_{\bar{r}}^{\beta}-v_{\bar{r}}^{\beta})].
\end{aligned}
$$

Also since $t\in K_i\setminus\{\bar{r}\}$, we have $c_t^{\bar{\alpha}}(v)>c_{\bar{r}}^{\bar{\alpha}}(v)$. By $u\in D$, we know that there must exist $\bar{\beta}\in\{1,2,\cdots,q\}$ such that $u_t^{\bar{\beta}}>0$. Hence, we get

$$
\begin{aligned}
&\sum_{i=1}^{l}[\sum_{t\in K_i\setminus\{\bar{r}\}} c_t^{\bar{\alpha}}(v)u_t^{\bar{\beta}}+c_{\bar{r}}^{\bar{\alpha}}(v)(u_{\bar{r}}^{\bar{\beta}}-v_{\bar{r}}^{\bar{\beta}})] \\
>&\sum_{i=1}^{l}[c_{\bar{r}}^{\bar{\alpha}}(v)(\sum_{t\in K_i} u_t^{\bar{\beta}}-v_{\bar{r}}^{\bar{\beta}})].
\end{aligned}
$$

Since $v\in D$, by $v_r=0$ for any $r\in K_i,\ i\in I$ and $r\neq\bar{r}$, we know

$$
\sum_{t\in K_i} v_t^{\bar{\beta}}=\sum_{t\in K_i\setminus\{\bar{r}\}} v_t^{\bar{\beta}}+v_{\bar{r}}^{\bar{\beta}}=v_{\bar{r}}^{\bar{\beta}}=d_i^{\bar{\beta}}.
$$

Hence, we derive that there exists an $\bar{\alpha}\in\{1,2,\cdots,q\}$ and a $\bar{\beta}\in\{1,2,\cdots,q\}$such that

$$
\begin{aligned}
&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}}-v_t^{\bar{\beta}})] \\
>&\sum_{i=1}^{l}[c_{\bar{r}}^{\bar{\alpha}}(v)(d_i^{\bar{\beta}}-d_i^{\bar{\beta}})] \\
=&0.
\end{aligned}
$$

Thus, we get

$$
\langle c(v),(u-v)^T\rangle\notin -R_+^{q\times q},\quad \forall\ u\in D. \tag{3.3}
$$

We assume that $cl(cone(c(v)(D-v)^T+R_+^{q\times q}))\cap(-R_+^{q\times q})\neq\{0\}$. Since $0\in cl(cone(c(v)(D-v)^T+R_+^{q\times q}))\cap(-R_+^{q\times q})$, we know that there must exist an $\bar{x}\in cl(cone(c(v)(D-v)^T+R_+^{q\times q}))\cap(-R_+^{q\times q})$ such that $\bar{x}\neq 0$. Obviously, $cone(c(v)(D-v)^T+R_+^{q\times q})$ is closed. So, we have $\bar{x}\in cone(c(v)(D-v)^T+R_+^{q\times q})\cap(-R_+^{q\times q})$. We set $\bar{x}=\tilde{\lambda}\tilde{x}$, where $\tilde{x}\in c(v)(D-v)^T+R_+^{q\times q}$ and $\tilde{\lambda}>0$ since $\bar{x}\neq 0$. Thus, we get $\tilde{\lambda}\tilde{x}\in -R_+^{q\times q}$. Since $-R_+^{q\times q}$ is also a cone, we obtain

$\tilde{x} \in -R_+^{q\times q}$. Hence, we know that there exists an $\tilde{x} \in (c(v)(D-v)^T + R_+^{q\times q}) \cap (-R_+^{q\times q})$ and $\tilde{x} \neq 0$. So, there also exist a $\tilde{u} \in D$ and $\tilde{a} \in R_+^{q\times q}$ such that $\tilde{x} = \langle c(v), (\tilde{u}-v)^T\rangle + \tilde{a}$. Hence,

$$\langle c(v), (\tilde{u}-v)^T\rangle = \tilde{x} - \tilde{a} \in -R_+^{q\times q}.$$

This contradicts (3.3). Therefore,

$$cl(cone(c(v)(D-v)^T + R_+^{q\times q})) \cap (-R_+^{q\times q}) = \{0\}.$$

The proof is completed. □

Theorem 3.7. *A vector $v \in D$ is a Benson equilibrium pattern flow if v is a solution to the strong vector variational inequality problem: to find $v \in D$ such that*

$$\langle c(v), (u-v)^T\rangle \in R_+^{q\times q}, \quad \forall\, u \in D.$$

Proof. Suppose that $v \in D$ is a solution to the strong variational inequality problem. Also, assume that $cl(cone(c_{K_i}(v) + R_+^q - c_k(v))) \cap -R_+^q = \{0\}$ and $c_r(v) - c_k(v) \neq 0$ for any $i \in I$, $k, r \in K_i$ and $k \neq r$. We want to deduce that $v_r = 0$.

We consider the vector u whose components are such that

$$u_t = \{ \begin{array}{ll} v_t & \text{if } t \neq r, k \\ 0 & \text{if } t = r \\ v_r + v_k & \text{if } t = k. \end{array}$$

Since $v \in D$, i.e., $\sum\limits_{t\in K_i} v_t^j = d_i^j$ for any $i \in I$ and any $j = 1, 2, \cdots, q$, we have

$$\begin{aligned} \sum_{t\in K_i} u_t^j &= \sum_{t\in K_i\setminus\{r,k\}} u_t^j + u_r^j + u_k^j \\ &= \sum_{t\in K_i\setminus\{r,k\}} v_t^j + 0 + v_r^j + v_k^j \\ &= \sum_{t\in K_i} v_t^j \\ &= d_i^j. \end{aligned}$$

So, $u \in D$. By the above proof, we know

$$\begin{aligned} &\langle c(v), (u-v)^T\rangle \\ =&\sum_{t=1}^{m} c_t(v)(u_t - v_t)^T \\ =&\sum_{t\neq r,k} c_t(v)(v_t - v_t)^T - c_r(v)v_r^T + c_k(v)v_r^T \\ =&(c_k(v) - c_r(v))v_r^T \in R_+^{q\times q}. \end{aligned}$$

If $(c_k(v) - c_r(v))v_r^T \neq 0$, then for any $\alpha, \beta \in \{1, 2, \cdots, q\}$, it holds that

$$(c_k^\alpha(v) - c_r^\alpha(v))v_r^\beta \geqslant 0,$$

where the inequality holds strictly for some $\alpha, \beta \in \{1, 2, \cdots, q\}$.

By $v_r^\beta \geqslant 0$, we know

$$(c_k^\alpha(v) - c_r^\alpha(v)) \geqslant 0.$$

Also, the inequality holds strictly for some $\alpha, \beta \in \{1, 2, \cdots, q\}$ in the statement above. That is,

$$c_k(v) - c_r(v) \in R_+^q \setminus \{0\}.$$

Hence,

$$c_r(v) - c_k(v) \in cl(cone(c_{K_i}(v) + R_+^q - c_k(v)))$$

and

$$c_r(v) - c_k(v)) \in -R_+^q.$$

Thus, we derive that there exists $c_r(v) - c_k(v)) \neq 0$ such that

$$c_r(v) - c_k(v) \in cl(cone(c_{K_i}(v) + R_+^q - c_k(v))) \cap (-R_+^q).$$

It is a contradiction. So, we obtain $(c_k(v) - c_r(v))v_r^T = 0$. Since $c_k(v) - c_r(v) \neq 0$ when $r \neq k$, we deduce that $v_r = 0$, for any $r \in K_i$ and $k \neq r$.
We complete the proof. □

4 Benson Efficiency of a Network Equilibrium Model with Multi-criteria

The study of vector equilibrium models is only recent. It is more reasonable to assume that no network user will choose a path which incurs both a higher cost as well as a longer delay than some other path. In other words, a vector equilibrium should be based on the principle that traffic flow along a path joining an OD pair is positive only if the vector cost of this path is not dominated by the cost of some other path joining the same OD pair.

Let Z be a Hausdorff topological vector space ordered by a pointed, closed convex cone $S \subset Z$ with nonempty interior $int\ S$. For the network $G = [N, A, I]$, if we define the cost function of product j on an arc $a \in A$ as a vector-valued function of the flow v : $C_a^j(v) : R^{q\times m} \to Z$ and $C_a^j(v) \geqslant 0$, then the cost function $C_k^j(v)$ of product j on a path $k \in K_i,\ i \in I$, is also a vector-valued function, which is defined as above: $C_k^j(v) = \sum_{a\in k} C_a^j(v)$. The vector-valued function $C^j(v) = (C_k^j(v) :\ k \in K_i,\ i \in I) \in Z^m$ and $C_k(v) = (C_k^1(v), C_k^2(v), \cdots, C_k^q(v))^T \in Z^q$ are the cost function of product j in the network and the cost function on the path $k \in K_i,\ i \in I$, respectively. Then, the vector-valued cost function of the network is $C(v) = (C^1(v), C^2(v), \cdots, C^q(v))^T \in Z^{q\times m}$ or $C(v) = (C_k(v) :\ k \in K_i,\ i \in I)$.

In this section we consider Z as a finite-dimensional Euclidean space R^p with the special ordering cone $S = R_+^p$, which is more realistic than an abstract topological vector space from a practical viewpoint. Now we can generalize Wardrop's equilibrium principle to a network equilibrium problem with a vector-valued cost function with respect to Benson efficiency.

Definition 4.1. A vector $v \in D$ is said to be a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function iff

$$\left.\begin{array}{l} cl(cone(C_{K_i}(v) + R_+^{q\times p} - C_k(v))) \cap (-R_+^{q\times p}) = \{0\} \\ C_r(v) - C_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0,$$

for each $i \in I$, $k, r \in K_i$ and $r \neq k$.

A useful approach to analyzing vector-valued problems is to reduce it to a scalarized problem. In general, the linear scalarization method appears to be popular. But such kind of methods rely heavily on some underlying convexity assumptions, which are hardly valid for many real problems. In our paper, by using Gerstewitz's function, which was used in Gerth and Weidner (1990) to establish a useful non-convex separation theorem, we develop another scalarization method for the vector-valued Wardrop's network equilibrium problem without any convexity assumptions.

Definition 4.2. Given a fixed $e \in int\ R^p_+$, the Gerstewitz's function $\xi_e:\ R^p \to R$ is defined by:

$$\xi_e(y) = min\{\lambda \in R:\ y \in \lambda e - R^p_+\}, \quad \forall\ y \in R^p.$$

Obviously, there are some salient properties of this function that we will use later.

Lemma 4.3 (see Chen and Yang (2002)). *Let $e \in int\ R^p_+$. For each $\eta \in R$ and $y \in R^p$, we have the following results:*

(i) $\xi_e\,(y) < \eta \Leftrightarrow y \in \eta e - int\ R^p_+$;
(ii) $\xi_e\,(y) \leqslant \eta \Leftrightarrow y \in \eta e - R^p_+$;
(iii) $\xi_e\,(y) \geqslant \eta \Leftrightarrow y \notin \eta e - int\ R^p_+$;
(iv) $\xi_e\,(y) > \eta \Leftrightarrow y \notin \eta e - R^p_+$;
(v) $\xi_e\,(y) = \eta \Leftrightarrow y \in \eta e - \partial R^p_+$, *where ∂R^p_+ is the topological boundary of R^p_+.*

Lemma 4.4 (see Cheng and Wu (2006)). *For an $e \in int\ R^p_+$ and $\eta \in R$,*

$$\xi_e(-\eta e) = -\xi_e(\eta e) = -\eta.$$

We denote

$$\xi_e \circ C^j_k(v) = \xi_e(C^j_k(v)) = min\{\lambda \in R:\ C^j_k(v) \in \lambda e - R^p_+\},$$

for any $v \in D,\ k \in K_i,\ i \in I, j = 1, 2, \cdots, q$;

$$\xi_e \circ C_k(v) = (\xi_e \circ C^j_k(v):\ j = 1, 2, \cdots, q)^T \in R^q;$$

and

$$\xi_e(v) = \xi_e \circ C(v) = (\xi_e \circ C_k(v):\ k \in K_i,\ i \in I) \in R^{q\times m}.$$

Definition 4.5. A vector $v \in D$ is said to be an ξ_e-Benson equilibrium pattern flow in a vector-valued network equilibrium problem if there exists an $e \in int\ R^p_+$ such that for any $i \in I,\ k \in K_i$,

$$\left.\begin{array}{l} cl(cone(\xi_e \circ C_{K_i}(v) + R^q_+ - \xi_e \circ C_k(v))) \cap (-R^q_+) = \{0\} \\ \xi_e \circ C_r(v) - \xi_e \circ C_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0,$$

for any $r \in K_i$ and $r \neq k$.

We assume that $C^j_k(v):\ R^{q\times m} \to R^p_+$ is in the following form:

$$C^j_k(v) = f^j_k(v)k_0, \quad \forall\ k \in K_i,\ i \in I \text{ and } j \in \{1, 2, \cdots, q\}. \tag{4.1}$$

where $f^j_k(v):\ R^{q\times m} \to R_+$ and $k_0 \in int\ R^p_+$. It is realistic from a practical viewpoint since the transportation cost function is made up of elementary costs. We see that k_0 is a vector of elementary costs, i.e., it is vector-valued, and each $C^j_k(v)$ is its real-valued multiple, i.e., the multiple $f^j_k(v)$ is a real-valued function of flow v.

Now we will scalarize the vector-valued network equilibrium problem. It is important to note that we do not require any convexity assumptions since we use Gerstewitz's function in our scalarization method.

Theorem 4.6. *Let $C_k^j(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. A vector $v \in D$ is a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function if and only if v is an ξ_{k_0}-Benson equilibrium pattern flow.*

Proof. Necessity: Let $v \in D$ be a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function. Next we will prove

$$\left.\begin{array}{l} cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R_+^q - \xi_{k_0} \circ C_k(v))) \cap (-R_+^q) = \{0\} \qquad (4.2) \\ \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0,$$

for any $r \in K_i$ and $r \neq k$.
First, we prove

$$\left.\begin{array}{c} cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R_+^q - \xi_{k_0} \circ C_k(v))) \cap (-R_+^q) = \{0\} \\ \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0 \end{array}\right\}$$

$$\Rightarrow \left\{ \begin{array}{l} cl(cone(C_{K_i}(v) + R_+^{q\times p} - C_k(v))) \cap (-R_+^{q\times p}) = \{0\} \\ C_r(v) - C_k(v) \neq 0. \end{array}\right.$$

By $\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0$, we know that

$$C_r(v) - C_k(v) \neq 0, \quad \text{for } k,\ r \in K_i \text{ and } r \neq k.$$

By (3.1), we also know $C_{K_i}(v) = f_{K_i}(v) \circ k_0$, where $f_{K_i}(v) := \{f_r(v) :\ r \in K_i\}$ and $f_r(v) = (f_r^1(v), f_r^2(v), \cdots, f_r^q(v))$. From Lemma 4.4, we get

$$\begin{aligned} \xi_{k_0} \circ C_{K_i}(v) &= \{\xi_{k_0} \circ C_r(v) :\ r \in K_i\} \\ &= \{(\xi_{k_0} \circ C_r^1(v), \xi_{k_0} \circ C_r^2(v), \cdots, \xi_{k_0} \circ C_r^q(v)) :\ r \in K_i\} \\ &= \{(f_r^1(v), f_r^2(v), \cdots, f_r^q(v)) :\ r \in K_i\} \\ &= f_{K_i}(v). \end{aligned}$$

Thus, (4.2) becomes

$$cl(cone(f_{K_i}(v) + R_+^q - f_k(v))) \cap (-R_+^q) = \{0\}.$$

That is, $f_k(v) \in Benson\{f_r(v) :\ r \in K_i\}$. Since $Benson\{f_r(v) :\ r \in K_i\} \subset Eff\{f_r(v) :\ r \in K_i\}$, we obtain $f_k(v) \in Eff\{f_r(v) :\ r \in K_i\}$, i.e.,

$$f_r(v) - f_k(v) \notin -R_+^q \setminus \{0\}, \qquad \forall r \in K_i. \tag{4.3}$$

We assume that $cl(cone(C_{K_i}(v)+R_+^{q\times p}-C_k(v)))\cap(-R_+^{q\times p}) \neq \{0\}$. Since $0 \in cl(cone(C_{K_i}(v)+R_+^{q\times p}-C_k(v)))\cap(-R_+^{q\times p})$, there must exist an $\bar{x} \in cl(cone(C_{K_i}(v)+R_+^{q\times p}-C_k(v)))\cap(-R_+^{q\times p})$ such that $\bar{x} \neq 0$. Obviously, $cone(C_{K_i}(v) + R_+^{q\times p} - C_k(v))$ is closed. So, we have $\bar{x} \in cone(C_{K_i}(v) + R_+^{q\times p} - C_k(v)) \cap (-R_+^{q\times p})$. We set $\bar{x} = \tilde{\lambda}\tilde{x}$, where $\tilde{x} \in C_{K_i}(v) + R_+^{q\times p} - C_k(v)$ and $\tilde{\lambda} > 0$ since $\bar{x} \neq 0$. Thus, we get $\tilde{\lambda}\tilde{x} \in -R_+^{q\times p}$. Since $-R_+^{q\times p}$ is also a cone, we obtain $\tilde{x} \in -R_+^{q\times p}$. Hence, we know that there exists an $\tilde{x} \in (C_{K_i}(v) + R_+^{q\times p} - C_k(v)) \cap (-R_+^{q\times p})$ and $\tilde{x} \neq 0$. So, there also exist a $\tilde{r} \in K_i$ and $\tilde{a} \in R_+^{q\times p}$ such that

$$\tilde{x} = C_{\tilde{r}}(v) + \tilde{a} - C_k(v).$$

Hence,

$$C_{\tilde{r}}(v) - C_k(v) = \tilde{x} - \tilde{a} \in -R_+^{q\times p} - R_+^{q\times p} = -R_+^{q\times p}.$$

That is, for any $j \in \{1, 2, \cdots, q\}$,

$$C^j_{\tilde{r}}(v) - C^j_k(v)) \in -R^p_+.$$

By Lemma 4.3, we derive that

$$\xi_{k_0}(C^j_{\tilde{r}}(v) - C^j_k(v)) \leqslant 0.$$

Also by (4.1) and Lemma 4.4, it holds that

$$f^j_{\tilde{r}}(v) - f^j_k(v) \leqslant 0, \qquad \forall\, j \in \{1, 2, \cdots, q\},$$

i.e.,

$$f_{\tilde{r}}(v) - f_k(v) \in -R^q_+.$$

If $f_{\tilde{r}}(v) - f_k(v) = 0$, then we know that $C_{\tilde{r}}(v) - C_k(v) = 0$. Hence, $\tilde{x} = \tilde{a}$. Since $\tilde{x} \in -R^{q\times p}_+$ and $\tilde{a} \in R^{q\times p}_+$, we get $\tilde{x} = 0$. It is a contradiction with $\tilde{x} \neq 0$. Thus, we have

$$f_{\tilde{r}}(v) - f_k(v) \in -R^q_+ \setminus \{0\}.$$

This contradicts (4.3). Therefore, we derive that

$$\left.\begin{array}{c} cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R^q_+ - \xi_{k_0} \circ C_k(v))) \cap (-R^q_+) = \{0\} \\ \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0 \end{array}\right\}$$
$$\Rightarrow \left\{\begin{array}{l} cl(cone(C_{K_i}(v) + R^{q\times p}_+ - C_k(v))) \cap (-R^{q\times p}_+) = \{0\} \\ C_r(v) - C_k(v) \neq 0. \end{array}\right.$$

Since $v \in D$ is a Benson equilibrium pattern flow, we know

$$\left.\begin{array}{l} cl(cone(C_{K_i}(v) + R^{q\times p}_+ - C_k(v))) \cap (-R^{q\times p}_+) = \{0\} \\ C_r(v) - C_k(v) \neq 0. \end{array}\right\} \Rightarrow v_r = 0,$$

for any $r, k \in K_i, r \neq k$.
Therefore, we have

$$\left.\begin{array}{c} cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R^q_+ - \xi_{k_0} \circ C_k(v))) \cap (-R^q_+) = \{0\} \\ \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0,$$

for any $r, k \in K_i, r \neq k$.

Sufficiency: Suppose that v is an ξ_{k_0}-Benson equilibrium pattern flow for a vector-valued network equilibrium problem. Next, we will prove

$$\left.\begin{array}{c} cl(cone(C_{K_i}(v) + R^{q\times p}_+ - C_k(v))) \cap (-R^{q\times p}_+) = \{0\} \\ C_r(v) - C_k(v) \neq 0. \end{array}\right\} \tag{4.4}$$
$$\Rightarrow \left\{\begin{array}{c} cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R^q_+ - \xi_{k_0} \circ C_k(v))) \cap (-R^q_+) = \{0\} \\ \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0. \end{array}\right.$$

We assume that $cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R^q_+ - \xi_{k_0} \circ C_k(v))) \cap (-R^q_+) \neq \{0\}$. Similar to the proof of "Necessity", we obtain that there exists $\tilde{y} \in (\xi_{k_0} \circ C_{K_i}(v) + R^q_+ - \xi_{k_0} \circ C_k(v)) \cap (-R^q_+)$ such that $\tilde{y} \neq 0$. Hence, there exist $\tilde{r} \in K_i$ and $\tilde{b} \in R^q_+$ such that

$$\tilde{y} = \xi_{k_0} \circ C_{\tilde{r}}(v) + \tilde{b} - \xi_{k_0} \circ C_k(v).$$

That is,

$$\xi_{k_0} \circ C_{\tilde{r}}(v) - \xi_{k_0} \circ C_k(v) = \tilde{y} - \tilde{b} \in -R_+^q.$$

From (4.1) and Lemma 4.4, we have

$$f_{\tilde{r}}(v) - f_k(v) \in -R_+^q.$$

Thus, it holds that

$$C_{\tilde{r}}(v) - C_k(v) \in -R_+^{q\times p}.$$

If $C_{\tilde{r}}(v) - C_k(v) = 0$, then we have $\tilde{y} = \tilde{b}$. Since $\tilde{y} \in -R_+^q$ and $\tilde{b} \in R_+^q$, we get $\tilde{y} = 0$. It is a contradiction to $\tilde{y} \neq 0$. So, we derive that there exists a $\tilde{r} \in K_i$ such that

$$C_{\tilde{r}}(v) - C_k(v) \in -R_+^{q\times p} \setminus \{0\}.$$

By (4.4), we know that $C_k(v) \in Benson\{C_r(v) : \ r \in K_i\}$. Hence, $C_k(v) \in Eff\{C_r(v) : r \in K_i\}$. That is,

$$C_r(v) - C_k(v) \notin -R_+^{q\times p} \setminus \{0\}, \quad \forall\ r \in K_i.$$

It is also a contradiction. Therefore, we get $cl(cone(\xi_{k_0} \circ C_{K_i}(v) + R_+^q - \xi_{k_0} \circ C_k(v))) \cap (-R_+^q) = \{0\}$. It is easy to see that $\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0$ by $C_r(v) - C_k(v) \neq 0$. Thus, by the definition of ξ_{k_0}-Benson equilibrium pattern flow, we obtain

$$\left.\begin{array}{c} cl(cone(C_{K_i}(v) + R_+^{q\times p} - C_k(v))) \cap (-R_+^{q\times p}) = \{0\} \\ C_r(v) - C_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0,$$

for any $r, k \in K_i, r \neq k$.
That is, $v \in D$ is a Benson equilibrium pattern flow. □

Combining Theorem 3.6 and Theorem 4.6, we obtain the following corollary.

Corollary 4.7. *Let $C_k^j(v)$ be defined as (4.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. If a vector $v \in D$ is a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function, then v is a Benson efficient solution to the vector variational inequality: to find $v \in D$ such that*

$$cl(cone(\xi_{k_0}(v)(D - v)^T + R_+^{q\times q})) \cap (-R_+^{q\times q}) = \{0\}.$$

Corollary 4.8. *Let $C_k^j(v)$ be defined as (4.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. A vector $v \in D$ is a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function if v is an efficient solution to a strong vector variational inequality: to find $v \in D$ such that*

$$\langle \xi_{k_0}(v), (u - v)^T \rangle \in R_+^{q\times q}, \quad \forall\ u \in D.$$

We know that the Gerstewitz's function is difficult to compute. So the best way to proceed is to convert the Benson efficient solution to the variational inequality to the vector form: to find $v \in D$ such that

$$cl(cone(C(v)(D - v)^T + (R_+^p)^{q\times q})) \cap -(R_+^p)^{q\times q} = \{0\}.$$

Now we prove the equivalent relation.

Theorem 4.9. *Let $C_k^j(v)$ be defined as (4.1) for all $k \in K_i$, $i \in I$ and $j \in \{1, 2, \cdots, q\}$. $v \in D$ is a Benson efficient solution to the vector variational inequality (SVI): finding $v \in D$ such that*

$$cl(cone(\xi_{k_0}(v)(D - v)^T + R_+^{q\times q})) \cap -(R_+^{q\times q}) = \{0\}$$

if and only if v is also a Benson efficient solution to another vector variational inequality (VVI): finding $v \in D$ such that

$$cl(cone(C(v)(D - v)^T + (R_+^p)^{q\times q})) \cap -(R_+^p)^{q\times q} = \{0\}.$$

Proof. Necessity: We assume that

$$cl(cone(C(v)(D - v)^T + (R_+^p)^{q\times q})) \cap -(R_+^p)^{q\times q} \neq \{0\}.$$

Since $0 \in cl(cone(C(v)(D - v)^T + (R_+^p)^{q\times q})) \cap -(R_+^p)^{q\times q}$, there must exist a $\bar{z} \in cl(cone(C(v)(D - v)^T + (R_+^p)^{q\times q})) \cap -(R_+^p)^{q\times q}$ such that $\bar{z} \neq 0$. Similar to the proof of "Necessity" in Theorem 4.6, there exists an $\tilde{z} \in (C(v)(D - v)^T + (R_+^p)^{q\times q}) \cap -(R_+^p)^{q\times q}$ such that $\tilde{z} \neq 0$. Hence, there exist $\tilde{u} \in D$ and $\tilde{c} \in (R_+^p)^{q\times q}$ such that

$$\tilde{z} = C(v)(\tilde{u} - v)^T + \tilde{c}.$$

That is,

$$C(v)(\tilde{u} - v)^T = \tilde{z} - \tilde{c} \in -(R_+^p)^{q\times q}. \tag{4.5}$$

We know that

$$\begin{aligned} &C(v)(\tilde{u} - v)^T \\ =&\sum_{i=1}^{l}[\sum_{t\in K_i} C_t(v)(\tilde{u}_t - v_t)^T] \end{aligned}$$

and it is a $q \times q$ matrix whose components are $\sum_{i=1}^{l}[\sum_{t\in K_i} C_t^\alpha(v)(\tilde{u}_t^\beta - v_t^\beta)] \in R^p$, where $\alpha,\ \beta \in \{1, 2, \cdots, q\}$. Hence, we get for any $\alpha,\ \beta \in \{1, 2, \cdots, q\}$,

$$\sum_{i=1}^{l}[\sum_{t\in K_i} C_t^\alpha(v)(\tilde{u}_t^\beta - v_t^\beta)] \in -R_+^p.$$

By Lemma 4.3, we have

$$\xi_{k_0}(\sum_{i=1}^{l}[\sum_{t\in K_i} C_t^\alpha(v)(\tilde{u}_t^\beta - v_t^\beta)]) \leqslant 0.$$

From (4.1) and Lemma 4.4, we obtain

$$\begin{aligned} &\xi_{k_0}(\sum_{i=1}^{l}[\sum_{t\in K_i} C_t^\alpha(v)(\tilde{u}_t^\beta - v_t^\beta)]) \\ =&\xi_{k_0}(\sum_{i=1}^{l}[\sum_{t\in K_i} f_t^\alpha(v)(\tilde{u}_t^\beta - v_t^\beta))] \cdot k_0) \\ =&\sum_{i=1}^{l}[\sum_{t\in K_i} f_t^\alpha(v)(\tilde{u}_t^\beta - v_t^\beta)] \\ =&\sum_{i=1}^{l}[\sum_{t\in K_i} \xi_{k_0}(C_t^\alpha(v))(\tilde{u}_t^\beta - v_t^\beta)] \leqslant 0. \end{aligned}$$

Obviously, $\sum_{i=1}^{l}[\sum_{t\in K_i}\xi_{k_0}(C_t^{\alpha}(v))(\tilde{u}_t^{\beta}-v_t^{\beta})]$ is the component of the matrix $\sum_{i=1}^{l}[\sum_{t\in K_i}\xi_{k_0}(C_t(v))(\tilde{u}_t-v_t)^T]$. Since $\xi_{k_0}(v)(\tilde{u}-v)^T=\sum_{i=1}^{l}[\sum_{t\in K_i}\xi_{k_0}(C_t(v))(\tilde{u}_t-v_t)^T]$, we get

$$\xi_{k_0}(v)(\tilde{u}-v)^T\in -R_+^{q\times q}.$$

Also, we know that

$$\xi_{k_0}(v)(\tilde{u}-v)^T\in \xi_{k_0}(v)(D-v)^T+R_+^{q\times q}.$$

Hence,

$$\xi_{k_0}(v)(\tilde{u}-v)^T\in cl(cone(\xi_{k_0}(v)(D-v)^T+R_+^{q\times q}))\cap(-R_+^{q\times q}).$$

Since $v\in D$ is a Benson efficient solution to (SVI), we know that $\xi_{k_0}(v)(\tilde{u}-v)^T=0$. By (4.1) and Lemma 4.4, we have

$$\begin{aligned}&\xi_{k_0}(v)(\tilde{u}-v)^T\\ =&\xi_{k_0}(f(v)k_0)(\tilde{u}-v)^T\\ =&f(v)(\tilde{u}-v)^T\\ =&0.\end{aligned}$$

Thus,

$$\begin{aligned}&C(v)(\tilde{u}-v)^T\\ =&f(v)k_0(\tilde{u}-v)^T\\ =&0.\end{aligned}$$

By (4.5), we get $\tilde{z}=\tilde{c}$. Since $\tilde{z}\in-(R_+^p)^{q\times q}$ and $\tilde{c}\in(R_+^p)^{q\times q}$, we have $\tilde{z}=0$. It is a contradiction to $\tilde{z}\neq 0$. Hence,

$$cl(cone(C(v)(D-v)^T+(R_+^p)^{q\times q}))\cap-(R_+^p)^{q\times q}=\{0\}.$$

Sufficiency: Since the proof is similar to the "Necessity", we omit it for the sake of conciseness. □

In Cheng and Wu (2006), the following theorem was proved.

Theorem 4.10. *Let $C_k^j(v)$ be defined as (4.1) for all $k\in K_i,\ i\in I$ and $j\in\{1,2,\cdots,q\}$. $v\in D$ is an efficient solution to the strong vector variational inequality (SVI): finding $v\in D$ such that*

$$\langle\xi_{k_0}(v),(u-v)^T\rangle\in R_+^{q\times q},\quad\forall\ u\in D,$$

if and only if v is also an efficient solution to the strong vector variational inequality (VVI):

$$\langle C(v),(u-v)^T\rangle\in(R_+^p)^{q\times q},\quad\forall\ u\in D.$$

Combining with Corollary 4.7, Corollary 4.8, Theorem 4.9 and Theorem 4.10, we have derived the following Theorem 4.11 and Theorem 4.12.

Theorem 4.11. *Let $C_k^j(v)$ be defined as (3.1) for all $k\in K_i,\ i\in I$ and $j\in\{1,2,\cdots,q\}$. If a vector $v\in D$ is a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function, then v is a Benson efficient solution to the vector variational inequality: to find $v\in D$ such that*

$$cl(cone(C(v)(D-v)^T+(R_+^p)^{q\times q}))\cap-(R_+^p)^{q\times q}=\{0\}.$$

Theorem 4.12. *Let $C_k^j(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. A vector $v \in D$ is a Benson equilibrium pattern flow in a network equilibrium problem with a vector-valued cost function if v is an efficient solution to the strong vector variational inequality: to find $v \in D$ such that*

$$\langle C(v), (u - v)^T \rangle \in (R_+^p)^{q \times q}, \quad \forall\ u \in D.$$

5 Conclusions

Based on Wardrop's equilibrium principle, we considered Benson efficiency of a scalar and a vector multi-product network equilibrium model. We established a sufficient and a necessary condition for a Benson equilibrium pattern flow for multi-product network equilibrium models in terms of vector variational inequalities when the cost function is defined in a certain form. We have not been able to derive a condition that is both necessary and sufficient. It is worth noting that there exists no such result in the literature. That is, the question of a solution to what kind of vector variation inequalities is also a Benson equilibrium pattern flow for multi-product network equilibrium models is yet to be answered.

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Yunan Wu
School of International Business, Beijing Foreign Studies University
Beijing 100089, China
E-mail address: wuyunan@bfsu.edu.cn

T.C. Edwin Cheng
Department of Logistics and Maritime Studies, The Hong Kong Polytechnic University
Hung Hom, Kowloon, Hong Kong
E-mail address: edwin.cheng@inet.polyu.edu.hk