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OPTIMALITY CONDITIONS AND DUALITY IN MULTIOBJECTIVE GENERALIZED FRACTIONAL PROGRAMMING WITH GENERALIZED CONVEXITY*

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Dedicated to Professor Guangya Chen on the occation of his 70 th birthday.

Abstract: In this paper, we are concerned with a nondifferentiable multiobjective generalized fractional programming problem. We present the Kuhn-Tucker type necessary condition for a weakly efficient solution, under the assumption of a kind of generalized Abadie Constraint Qualification. And then the Kuhn-Tucker type sufficient condition for weakly efficient solution is given under the assumption of $(C, \alpha, \rho, d)-$convexity. Subsequently, we apply the optimality conditions to formulate a kind of duality model and prove some duality theorems.

Key words: *multiobjective generalized fractional programming, optimality conditions, duality theorems, constraint qualification, generalized convexity*

Mathematics Subject Classification: *90C32, 90C46, 90C47*

1 Introduction

We consider the following multiobjective generalized fractional programming problem

$$\text{(FVP)} \qquad \begin{cases} \min \ E(x) = (E_1(x), \cdots, E_p(x))^T, \\ \text{s.t. } \ g(x) = (g_1(x), \cdots, g_r(x))^T \leq 0, \\ \quad x \in X, \end{cases}$$

where

$$E_i(x) = \max_{y \in Y} \ \frac{f_i(x, y) + \Phi_i(x)}{h_i(x, y) - \Psi_i(x)}, \ i = 1, \cdots, p.$$

In addition

(a) X is a nonempty convex subset of R^n and Y is a compact subset of R^m ,

(b) $f_i(x, y) : X \times Y \longrightarrow R, h_i(x, y) : X \times Y \longrightarrow R,\ g : R^n \to R^r$ is continuously differentiable,

(c) $\bigtriangledown_x f_i(x, y)$ and $-\bigtriangledown_x h_i(x, y)$ exist and continuous with respect to (x, y), $i = 1, \cdots, p$,

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(d) $f_i(x,y)$ and $-h_i(x,y)$ are upper semicontinuous functions with respect to y on Y, $i=1,\cdots,p$,

(e) $\Phi_i(x), \Psi_i(x) : R^n \longrightarrow R$ ($i=1,\cdots,p$) are convex functions on X,

(f) $f_i(x,y)+\Phi_i(x)\geq 0, h_i(x,y)-\Psi_i(x)>0, \forall(x,y)\in R^n\times Y, i=1,\ldots,p.$

Since Schmitendorf [1] introduced necessary and sufficient optimality conditions for generalized minimax programming, much attention has been paid to optimality conditions and duality theorems for generalized minimax programming problems, for example, see $[1-9]$. Yadav and Mukherjee [2] employed the optimality conditions of [1] to construct two kinds of dual problems and they derived duality theorems for convex differentiable minimax fractional programming problem, in other words $p=1, \Phi_1(x)=\Psi_1(x)=0$ in (FVP). In [3], Chandra and Kumar pointed out that the formulation of [2] has some omissions and inconsistencies, and they constructed two modified dual problems and prove duality theorems. Later on, Liu and Wu [4, 5], Liang and Shi [6] and Yang and Hou [7] relaxed the convexity assumption in the sufficient optimality conditions in [3], and they employed the optimality conditions to construct dual problems, and they established duality theorems. In a recent paper, Yuan and Liu [9] presented $(C,\alpha,\rho,d)-$ convex function, which extend these generalized convexity assumptions given in $[4-6]$, and they derived sufficient conditions and constructed dual problems, and derived duality theorems.

In this paper, we will derive optimality conditions for (FVP), and apply the optimality conditions to construct dual problem, and establish the duality theorems. Some definitions and notations are given in Section 2. In Section 3, we derive Kuhn-Tucker type necessary condition for a weakly efficient solution of (FVP) under a kind of generalized Abadie Constraint Qualification. And then the sufficient condition for a weakly efficient solution is given under the assumption of $(C,\alpha,\rho,d)-$ convexity. When the optimality conditions are utilized, a kind of dual problems may be formulated and duality results are presented in Section 4.

2 Preliminaries

Throughout this paper, we let $S=\{x\in X : g(x)\leq 0\}$ be the set of feasible solutions of problem (FVP). For each $x\in S$, we define

$$
\begin{aligned}
&I(x)=\{i : g_i(x)=0, i=1,\cdots,m\};\\
&Z(x)=\{z\in R^n : \nabla g_i^T(x)(z-x)\leq 0, \quad i\in I(x)\};\\
&Y_i(x)=\{y\in Y : \frac{f_i(x,y)+\Phi_i(x)}{h_i(x,y)-\Psi_i(x)}=\max_{z\in Y}\frac{f_i(x,z)+\Phi_i(x)}{h_i(x,z)-\Psi_i(x)}\}, \quad i=1,\cdots,p\ ;\\
&K(x)=\{(s,t,\bar{y})\in N\times R^{s\times p}\times R^{p\times m\times s} : 1\leq s\leq n+1, t=(t^1,\cdots,t^p),\\
&\qquad t^i=(t_1^i,\cdots,t_s^i)^T\geq 0, \sum_{l=1}^{s}t_l^i=1, \bar{y}=(y^1,\cdots,y^p)^T, y^i=(y_1^i,\cdots,y_s^i),\\
&\qquad y_l^i\in Y_i(x), l=1,\cdots,s, i=1,\cdots,p\}.
\end{aligned}
$$

In addition, we let

$$\Lambda^+=\{\lambda\in R^p : \lambda_i\geq 0, i=1,\cdots,p, \sum_{i=1}^{p}\lambda_i=1\}.$$

In order to establish the necessary condition in section 3, we recall these following results.

Lemma 2.1 ([10]). *Let $A \subset R^n$ be a nonempty convex compact set, $C \subset R^n$ be a convex set, If for any $d \in C$, there exists $\xi_d \in A$ such that $\xi_d^T d \geq 0$, then there exists $\xi \in A$, such that $\xi^T d \geq 0,\ \forall d \in C$.*

Lemma 2.2 ([11]). *Let $Y \subset R^m$ be a nonempty compact set, $f(x,y) : R^n \times Y \longrightarrow R$, $\nabla_x f(x,y)$ exists and be a continuous function with respect to (x,y), if let $r(x) = \max_{y \in Y} f(x,y)$, then*

$$\partial r(x) = \mathrm{co}\big\{\nabla_x f(x,y) : y \in M(x) = \{y \in Y : f(x,y) = r(x)\}\big\}.$$

Lemma 2.3 ([12]). *Let $A_i \subset R^n$, $i = 1, \cdots, m$ be convex sets, then*

$$\mathrm{co}\{A_1 \cup A_2 \cup \cdots \cup A_m\} = \{\sum_{i=1}^{m} \lambda_i \xi_i :\ \xi_i \in A_i, \lambda_i \geq 0, i = 1, \cdots, m, \sum_{i=1}^{m} \lambda_i = 1\}.$$

Definition 2.4 ([9]). A functional $C : X \times X \times R^n \longrightarrow R$ is convex on R^n with respect to the third argument if for any fixed $(x, x_0) \in X \times X$, the inequality $C_{(x,x_0)}(\lambda\alpha_1 + (1-\lambda)\alpha_2) \leq \lambda C_{(x,x_0)}(\alpha_1) + (1-\lambda)C_{(x,x_0)}(\alpha_2),\ \forall \lambda \in (0,1)$ holds for all $\alpha_1, \alpha_2 \in R^n$.

Remark 2.5. By the definition, we have that for any fixed $(x, x_0) \in X \times X$, if $\lambda_i \in (0,1), \alpha_i \in R^n, i = 1, \cdots, n$, and $\sum_{i=1}^{n} \lambda_i = 1$, then

$$C_{(x,x_0)}(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n) \leq \lambda_1 C_{(x,x_0)}(\alpha_1) + \lambda_2 C_{(x,x_0)}(\alpha_2) + \cdots + \lambda_n C_{(x,x_0)}(\alpha_n).$$

Throughout this paper, we assume that $C_{(x,x_0)}(0) = 0,\ \forall (x, x_0) \in X \times X$, and for any fixed $\bar{x} \in S$, we have $N_X(\bar{x}) \subset \{\varepsilon \in R^n : C_{(x,\bar{x})}(\varepsilon) \leq 0,\ \ \forall x \in X\}$. Where, $N_X(\bar{x})$ is the normal cone of the convex set X at $\bar{x}$.

Now, we introduce a generalized convexity based on the convex functional $C_{(x,x_0)}$ as follows.

We let $f : X \to R$ be a locally Lipschitzian function, $\alpha : X \times X \to R_+ \setminus \{0\}$, $\rho \in R$, $d : X \times X \to R_+$ be a function with the property that $d(x, x_0) = 0 \Longleftrightarrow x = x_0$.

Definition 2.6 ([9]). f is said to be $(C, \alpha, \rho, d)-$convex at $x^* \in X$, if there exist C, α, d and $\rho \in R$, such that for any $x \in X$ and $\xi \in \partial f(x^*)$, the following inequality holds:

$$\frac{f(x) - f(x^*)}{\alpha(x, x^*)} \geq C_{(x,x^*)}(\xi) + \rho\frac{d(x, x^*)}{\alpha(x, x^*)}. \tag{2.1}$$

Particularly, f is said to be a strictly $(C, \alpha, \rho, d)-$ convex at $x^* \in X$, if the inequality (2.1) is a strictly inequality when $x \neq x^*$.

Definition 2.7. $\bar{x} \in S$ is said to be a weakly efficient solution of (FVP), if there is no other $x \in S$ such that $E(x) < E(\bar{x})$.

Definition 2.8 ([13]). The function g satisfy generalized Abadie Constraint Qualification (EACQ) at $x^* \in S$, if

$$(Z(x^*) \cap riX) - \{x^*\} \subset T(S, x^*).$$

Where, riX is the relative interior of X, in other words, $riX = \{x \in \mathrm{aff}X : \exists \epsilon > 0, (x + \epsilon B) \cap (\mathrm{aff}X) \subset X\}$. B is the Euclidean closed units ball in R^n.

Remark 2.9. We know that, the relation $T(S,x^*) \subset Z(x^*) - \{x^*\}$ always hold. But the relation $T(S,x^*) \subset (Z(x^*) \cap riX) - \{x^*\}$ can not always hold. Now, we give an example to show it.

Example 2.10. Consider the following problem (P1)

$$\begin{cases} \min & x + 2|x|, \\ \text{s.t.} & x^3 \leq 0, \\ & x \in X = (-\infty, 0]. \end{cases}$$

Obviously, $x^* = 0$ is an optimal solution of (P1). Let the set of feasible solution of (P1) be S, then $S = X$. On the other hand $Z(x^*) = R, riX = (-\infty, 0)$, and $T(S,x^*) = (-\infty, 0]$, so the relation $T(S,x^*) \subset (Z(x^*) \cap riX) - \{x^*\}$ can not hold.

3 Optimality Conditions

In this section, we derive necessary condition for weakly efficient solution of problem (FVP), under the assumption of EACQ , and sufficient condition for weakly efficient solution is given under the assumption of generalized convexity.

First, we introduce the following auxiliary programming problem:

$$(\text{FVP})_e \qquad \min_{x \in S} \ \Omega(x,e) = (\Omega_1(x,e_1), \cdots, \Omega_p(x,e_p))^T,$$

where, $e = (e_1, \cdots, e_p)^T \in R^P$, $\Omega_i(x,e_i) = \max_{y \in Y}\{f_i(x,y) + \Phi_i(x) - e_i(h_i(x,y) - \Psi_i(x))\}$, $i = 1, \cdots, p$.

It is easy to prove the following Lemma.

Lemma 3.1. *$\bar{x}$ is a weakly efficient solution of* (FVP) *if and only if $\bar{x}$ is a weakly efficient solution of* $(\text{FVP})_{\bar{e}}$*, where $\bar{e} = E(\bar{x})$.*

Theorem 3.2. *Let $\bar{x}$ be a weakly efficient solution of* (FVP)*, g satisfies EACQ at $\bar{x}$. If*

$$riX \cap Z(x^*) \neq \emptyset,$$

then there exist $(s,t,y) \in K(\bar{x}), \lambda \in \Lambda^+, \bar{u} \in R^r_+$, and $\bar{e} \in R^p_+$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \{ \sum_{l=1}^{s} t_l^i \{\nabla_x f_i(\bar{x}, y_l^i) - \bar{e}_i \nabla_x h_i(\bar{x}, y_l^i)\} + \partial\Phi_i(\bar{x}) + \bar{e}_i \partial\Psi_i(\bar{x})\} + \sum_{j=1}^{r} \bar{u_j} \nabla g_j(\bar{x}) + N_X(\bar{x}); \tag{3.1}$$

$$f_i(\bar{x}, y_l^i) - \bar{e}_i h_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) + \bar{e}_i \Psi_i(\bar{x}) = 0, \quad i = 1, \cdots, p, l = 1, \cdots, s; \tag{3.2}$$

$$\sum_{j=1}^{r} \bar{u_j} g_j(\bar{x}) = 0; \tag{3.3}$$

$$\sum_{l=1}^{s} t_l^i = 1, t_l^i \geq 0, \quad i = 1, \cdots, p, l = 1, \cdots, s. \tag{3.4}$$

Proof. First, we prove the system

$$\begin{cases} \xi_i^T(x-\bar{x})<0, \ \ \forall \xi_i \in \partial\Omega_i(\bar{x},\bar{e}_i), \ i=1,\cdots,p \ , \\ \nabla g_j^T(\bar{x})(x-\bar{x}) \le 0, \ \ j \in I(\bar{x}), \end{cases}$$

has no solution $x \in riX$. Suppose to the contrary that this system has solutions. Observe by the definition of $Z(\bar{x})$ that there exists $x \in riX \cap Z(\bar{x})$ such that

$$\xi_i^T(x-\bar{x})<0, \ \ \forall \xi_i \in \partial\Omega_i(\bar{x},\bar{e}_i), \ i=1,\cdots,p.$$

Hence, $\Omega_i^0(\bar{x},\bar{e}_i;x-\bar{x})<0, \ \ i=1,\cdots,p$, where $\Omega_i^0(\bar{x},\bar{e}_i;x-\bar{x})$ are the generalized directional derivatives. For each $i \in \{1,\cdots,p\}$, similar to the proof of theorem 1 in [8], we can obtain that there exists a sequence $\{x_k\} \subset S$, such that $\Omega_i(x_k,\bar{e}_i)<\Omega_i(\bar{x},\bar{e}_i)$, when k is large enough. Obviously, this is a contradiction to the assumption that $\bar{x}$ is a weakly efficient solution of (FVP). This contradiction implies that the following system

$$\xi_i^T(x-\bar{x})<0, \ \ \forall \xi_i \in \partial\Omega_i(\bar{x},\bar{e}_i), i=1,\cdots,p,$$

has no solution $x \in riX \cap Z(\bar{x})$. So for each $x \in riX \cap Z(\bar{x})$, there exist $i \in \{1,\cdots,p\}$ and $\bar{\xi}_i \in \partial\Omega_i(\bar{x},\bar{e}_i)$, such that $\bar{\xi}_i^{\,T}(x-\bar{x}) \ge 0$. If we let $\bar{\lambda}_i=1$, $\bar{\lambda}_j=0$ $(j \ne i, j \in \{1,\cdots,p\})$, then for each $x \in riX \cap Z(\bar{x})$, there exist $\bar{\lambda} \in \Lambda^+$ and $\bar{\xi}_x=\bar{\xi}_i=\sum\limits_{i=1}^{p}\bar{\lambda}_i\bar{\xi}_i$ (where $\bar{\xi}_i \in \partial\Omega_i(\bar{x},\bar{e}_i), i=1,\cdots,p$), such that

$$\bar{\xi}_x^{\,T}(x-\bar{x})=(\sum_{i=1}^{p}\bar{\lambda}_i\bar{\xi}_i)^{\ T}(x-\bar{x})=\bar{\xi}_i^{\,T}(x-\bar{x}) \ge 0.$$

Let $A_i=\partial\Omega_i(\bar{x},\bar{e}_i), \ i=1,\cdots,p, \ A=\mathrm{co}\{A_1 \cup \cdots \cup A_p\}$. Obviously, A is a nonempty convex compact set. By lemma 2.3, we have $\bar{\xi}_x \in A$. So we have, for all $x \in riX \cap Z(\bar{x})$, there exists $\bar{\xi}_x \in A$ such that $\bar{\xi}_x^{\,T}(x-\bar{x}) \ge 0$. On the other hand , $riX \cap Z(\bar{x})$ is a nonempty convex set. By lemma 2.1, there exists $\bar{\xi} \in A$, such that

$$\bar{\xi}^{\ T}(x-\bar{x}) \ge 0, \ \forall x \in riX \cap Z(\bar{x}).$$

Combining this with the definition of A and lemma 2.3, we have, there exist $\xi_i \in \partial\Omega_i(\bar{x},\bar{e}_i)$, $i=1,\cdots,p$ and $\lambda \in \Lambda^+$, such that $\bar{\xi}=\sum\limits_{i=1}^{p}\lambda_i\xi_i$. So the following system

$$\begin{cases} (\sum\limits_{i=1}^{p}\lambda_i\xi_i)^T(x-\bar{x})<0, \\ \nabla g_j^T(\bar{x})(x-\bar{x}) \le 0, \ \ j \in I(\bar{x}), \end{cases}$$

has no solution $x \in riX$.

By the alternative theorem [14], there exists $\bar{u} \in R_+^r$, such that $\sum\limits_{j=1}^{r}\bar{u}_j g_j(\bar{x})=0$,

$$0 \in \sum_{i=1}^{p}\lambda_i\xi_i+\sum_{j=1}^{r}\bar{u}_j\nabla g_j(\bar{x})+N_X(\bar{x}). \tag{3.5}$$

For each $i \in \{1,\cdots,p\}$, we calculate $\partial\Omega_i(\bar{x},\bar{e}_i)$, we have

$$\partial\Omega_i(\bar{x},\bar{e}_i)=\sum_{l \in J_{i0}(\bar{x})} t_l^i\{\nabla_x f_i(\bar{x},y_l^i)-\bar{e}_i\nabla_x h_i(\bar{x},y_l^i)\}+\partial\Phi_i(\bar{x})+\bar{e}_i\partial\Psi_i(\bar{x}),$$

where $J_{i0}(\bar{x})$ is some finite subset of index sets $J_i(\bar{x}) = \{l: \ y_l^i \in Y_i(\bar{x})\}$, $t_l^i \geq 0, l \in J_{i0}(\bar{x})$, and $\sum_{l \in J_{i0}(\bar{x})} t_l^i = 1$. By the Caratheodory theorem, we have, there exists $s \in N, 1 \leq s \leq n+1$, such that

$$\partial\Omega_i(\bar{x}, \bar{e}_i) = \sum_{l=1}^{s} t_l^i \{\nabla_x f_i(\bar{x}, y_l^i) - \bar{e}_i \nabla_x h_i(\bar{x}, y_l^i)\} + \partial\Phi_i(\bar{x}) + \bar{e}_i \partial\Psi_i(\bar{x}).$$

Combining this with (3.5), we obtain (3.1). The proof is completed. □

Remark 3.3. Suppose now that the set X in problem (FVP) is open (not necessary convex), then $N_X(\bar{x}) = \{0\}$. Choose an open ball X^1 included in X , with center $\bar{x}$, then X^1 is a convex open set, and $Z(\bar{x}) \cap riX^1 = Z(\bar{x}) \cap intX^1 \neq \emptyset$. Replacing X by X^1 in problem (FVP) to get a new problem, denoted by (FVP1), that $\bar{x}$ is a weakly efficient solution implies that it also a weakly efficient solution of (FVP1).We see that, via writing a theorem for (FVP1) similar to Theorem 3.2, $N_X(\bar{x})$ in (3.1) is vanish, and the constraint qualification extend these following constraint qualifications: Abadie Constraint Qualification, the two kinds of constraint qualifications in [15], which are weaker than Kuhn-Tucker and Arrow-Hurwicz-Uzawa Constraint Qualifications, respectively. So Theorem 3.2 derive the Kuhn-Tucker type of necessary condition for weakly efficient solution of multiobjective generalized fractional programming problem (FVP), under the more weaker constraint qualification. This result extend the optimality necessary conditions in [8].

Now, we derive the sufficient condition for weakly efficient solution of (FVP), under the assumption of $(C, \alpha, \rho, d)-$convexity.

Theorem 3.4. *Let $\bar{x} \in S$, and there exist $(s, t, y) \in K(\bar{x}), \lambda \in \Lambda^+, u \in R_+^r, e \in R_+^p$ and $K > 0$ to satisfy the relations* (3.1) – (3.4)*, If the following conditions hold: $f_i(x, y_l^i) + \Phi_i(x)$ are $(C, \alpha, \rho_l^i, d_l^i)-$convex and $-h_i(x, y_l^i) + \Psi_i(x)$ are $(C, \alpha, \bar{\rho}_l^i, \bar{d}_l^i)-$convex for $l = 1, \cdots, s, i = 1, \cdots, p$ at $\bar{x}$, $g_j(x)$ are $(C, \beta_j, \varepsilon_j, c_j)-$convex for $j = 1, \cdots, r$ at $\bar{x}$, and*

$$\frac{\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{\rho_l^i d_l^i + e_i \bar{\rho}_l^i \bar{d}_l^i\}}{\alpha(x, \bar{x})} + \sum_{j=1}^{r} \frac{u_j \varepsilon_j c_j}{\beta_j(x, \bar{x})} \geq 0, \tag{3.6}$$

then $\bar{x}$ is a weakly efficient solution of (FVP).

Proof. Suppose on the contrary that $\bar{x}$ is not a weakly efficient solution for (FVP), then there exists $x \in S$ such that $E(x) < E(\bar{x}) = e$. By $\lambda \in \Lambda^+$, we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x)\} < 0 =$$
$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i h_i(\bar{x}, y_l^i) + e_i \Psi_i(\bar{x})\}.$$

Using the feasibility $x \in S$ for (FVP), and inequality (3.3), we have

$$\frac{\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x)\}}{\alpha(x, \bar{x})} - \frac{\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i h_i(\bar{x}, y_l^i) + e_i \Psi_i(\bar{x})\}}{\alpha(x, \bar{x})} + \sum_{j=1}^{r} \frac{u_j(g_j(x) - g_j(\bar{x}))}{\beta_j(x, \bar{x})} < 0. \quad (3.7)$$

On the other hand, by the generalized convexity assumptions, we have

$$\frac{f_i(x, y_l^i) + \Phi_i(x) - f_i(\bar{x}, y_l^i) - \Phi_i(\bar{x})}{\alpha(x, \bar{x})} \geq C_{(x,\bar{x})}(\xi_l^i) + \frac{\rho_l^i d_l^i(x, \bar{x})}{\alpha(x, \bar{x})},$$

$$\frac{-h_i(x, y_l^i) + \Psi_i(x) + h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x})}{\alpha(x, \bar{x})} \geq C_{(x,\bar{x})}(\eta_l^i) + \frac{\bar{\rho}_l^i \bar{d}_l^i(x, \bar{x})}{\alpha(x, \bar{x})},$$

$$\frac{g_j(x) - g_j(\bar{x})}{\beta_j(x, \bar{x})} \geq C_{(x,\bar{x})}(\nabla g_j(\bar{x})) + \frac{\varepsilon_j \gamma_j(x, \bar{x})}{\beta_j(x, \bar{x})}.$$

$$\forall \xi_l^i \in \nabla_x f_i(\bar{x}, y_l^i) + \partial \Phi_i(\bar{x}), \ \ \forall \eta_l^i \in -\nabla_x h_i(\bar{x}, y_l^i) + \partial \Psi_i(\bar{x}),$$
$$l = 1, \cdots, s, \ i = 1, \cdots, p, \ j = 1, \cdots, r.$$

Combining this with (3.7), we obtain

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i C_{(x,\bar{x})}(\xi_l^i) + \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i e_i C_{(x,\bar{x})}(\eta_l^i) + \sum_{j=1}^{r} u_j C_{(x,\bar{x})}(\nabla g_j(\bar{x})) + \frac{\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{\rho_l^i d_l^i + e_i \bar{\rho}_l^i \bar{d}_l^i\}}{\alpha(x, \bar{x})} + \sum_{j=1}^{r} \frac{u_j \varepsilon_j \gamma_j}{\beta_j(x, \bar{x})} < 0$$

From the relation (3.6) and convex functional C satisfies $C_{(x,\bar{x})}(\eta) \leq 0, \ \forall \eta \in N_X(\bar{x})$, we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i C_{(x,\bar{x})}(\xi_l^i) + \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i e_i C_{(x,\bar{x})}(\eta_l^i) + \sum_{j=1}^{r} u_j C_{(x,\bar{x})}(\nabla g_j(\bar{x})) + C_{(x,\bar{x})}(\eta) < 0.$$

By using Remark 2.5, we obtain

$$C_{(x,\bar{x})}\Big(\frac{1}{\tau}\Big(\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (\xi_l^i + e_i \eta_l^i) + \sum_{j=1}^{r} u_j \nabla g_j(\bar{x}) + \eta\Big)\Big) < 0,$$

where, $\tau = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i + \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i e_i + \sum_{j=1}^{r} u_j + 1 = 1 + \sum_{i=1}^{p} \lambda_i e_i + \sum_{j=1}^{r} u_j + 1$. Obviously, it contradicts the relation (3.1). Hence the proof is completed. □

From Lemma 3.1, we only study the duality of $(\text{FVP})_e$ for some $e \in R_+^n$. For $e \in R_+^n$, we consider the following auxiliary problem:

$$(\text{FMD})_e \quad \max_{(s,t,y) \in K(z)} \ \max_{(z,\lambda,u) \in H(s,t,y)} \ \Big(\Omega_1(z, e_1) + \sum_{j=1}^{m} u_j g_j(z), \cdots, \Omega_p(z, e_p) + \sum_{j=1}^{m} u_j g_j(z)\Big)^T,$$

where

$$H(s,t,y) = \big\{(z,\lambda,u) \in R^n \times R^p \times R^r :$$
$$0 \in \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{\nabla_x f_i(z,y_l^i) + \partial\Phi_i(z) - e_i(\nabla_x h_i(z,y_l^i) - \partial\Psi_i(z))\}$$
$$+ \sum_{j=1}^{r} u_j \nabla g_j(z) + N_X(z);$$
$$\lambda \in \Lambda^+, u \geq 0\big\}.$$

Theorem 3.5 (Weak duality). *Let* (z,λ,u,s,t,y) *and* $\bar{x}$ *be feasible solutions of* $(\text{FMD})_e$ *and* $(\text{FVP})_e$ $(e = E(z))$*, respectively. If the following generalized convexity assumptions at* z *hold:*
$f_i(x,y_l^i) + \Phi_i(x)$ *are* $(C,\alpha,\rho_l^i,d_l^i)-$*convex,* $-h_i(x,y_l^i) + \Psi_i(x)$ *are* $(C,\alpha,\bar{\rho}_l^i,\bar{d}_l^i)-$*convex for* $l = 1,\cdots,s, i = 1,\cdots,p$ *at* z*,* $g_j(x)$ *are* $(C,\alpha,\varepsilon_j,c_j)-$*convex for* $j = 1,\cdots,r$ *at* z*, and*

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{\rho_l^i d_l^i(\bar{x},z) + e_i \bar{\rho}_l^i \bar{d}_l^i(\bar{x},z)\} + \sum_{j=1}^{r} u_j \varepsilon_j c_j(\bar{x},z) \geq 0,$$

then the following inequalities can not simultaneously hold :

$$\Omega_i(\bar{x},e_i) < \Omega_i(z,e_i) + \sum_{j=1}^{m} u_j g_j(z),\ i = 1,\cdots,p.$$

Proof. Suppose on the contrary that

$$\Omega_i(\bar{x},e_i) < \Omega_i(z,e_i) + \sum_{j=1}^{r} u_j g_j(z),\ i = 1,\cdots,p.$$

From $\lambda \in \Lambda^+$ and $\bar{x} \in S$, we have

$$\sum_{i=1}^{p} \lambda_i \Omega_i(\bar{x},e_i) + \sum_{j=1}^{r} u_j g_j(\bar{x}) < \sum_{i=1}^{p} \lambda_i \Omega_i(z,e_i) + \sum_{j=1}^{r} u_j g_j(z).$$

Hence, we obtain

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x},y_l^i) + \Phi_i(\bar{x}) - e_i h_i(\bar{x},y_l^i) + e_i \Psi_i(\bar{x})\} + \sum_{j=1}^{m} u_j g_j(\bar{x}) <$$
$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(z,y_l^i) + \Phi_i(z) - e_i h_i(z,y_l^i) + e_i \Psi_i(z)\} + \sum_{j=1}^{m} u_j g_j(z).$$

By using the generalized convexity assumptions, similar to the proof of Theorem 3.4, we can complete the proof. □

Theorem 3.6 (Strong duality). *Assume that* $\bar{x}$ *is a weakly efficient solution of* $(\text{FVP})_{\bar{e}}$ $(\bar{e} = E(\bar{x}))$*.* $(\text{FVP})_{\bar{e}}$ *satisfies EACQ at* $\bar{x}$*, and* $riX \cap Z(\bar{x}) \neq \emptyset$*, then there exist* $\bar{\lambda} \in R^p, \bar{u} \in R^r$*, and* $(\bar{s},\bar{t},\bar{y}) \in K(\bar{x})$*, such that* $(\bar{x},\bar{\lambda},\bar{u},\bar{s},\bar{t},\bar{y})$ *is a feasible solution of* $(\text{FMD})_{\bar{e}}$*. In addition, if for any feasible solution* (z,λ,u,s,t,y) *of* $(\text{FMD})_{\bar{e}}$*, the generalized convexity assumptions of Theorem 3.5 holds, then* $(\bar{x},\bar{\lambda},\bar{u},\bar{s},\bar{t},\bar{y})$ *is a weakly efficient solution of* $(\text{FMD})_{\bar{e}}$*.*

Proof. From the assumptions, $\bar{x}$ is a weakly efficient solution of (FVP), and (FVP) satisfies EACQ at $\bar{x}$. By using Theorem 3.2, there exist $\bar{\lambda} \in R^p, \bar{u} \in R^r$, and $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{s}, \bar{t}, \bar{y})$ is a feasible solution of $(\text{FMD})_{\bar{e}}$. In addition, for any feasible solution (z, λ, u, s, t, y), the generalized convexity assumptions of Theorem 3.5 hold, then $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{s}, \bar{t}, \bar{y})$ is a weakly efficient solution of $(\text{FMD})_{\bar{e}}$. Otherwise, there exists a feasible solution (z, λ, u, s, t, y) of $(\text{FMD})_{\bar{e}}$ such that

$$\Omega_i(\bar{x}, \bar{e}_i) + \sum_{j=1}^{r} \bar{u}_j g_j(\bar{x}) < \Omega_i(z, \bar{e}_i) + \sum_{j=1}^{r} u_j g_j(z),\ i = 1, \cdots, p.$$

By using Theorem 3.2, we have $\sum_{j=1}^{r} \bar{u}_j g_j(\bar{x}) = 0$. From $\bar{x} \in S$, we have

$$\Omega_i(\bar{x}, \bar{e}_i) + \sum_{j=1}^{r} u_j g_j(\bar{x}) < \Omega_i(z, \bar{e}_i) + \sum_{j=1}^{r} u_j g_j(z),\ i = 1, \cdots, p.$$

By $(s, t, y) \in K(z)$, we obtain

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - \bar{e}_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x}))) + \sum_{j=1}^{r} u_j g_j(\bar{x}) <$$
$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (f_i(z, y_l^i) + \Phi_i(z) - \bar{e}_i(h_i(z, y_l^i) - \Psi_i(z))) + \sum_{j=1}^{r} u_j g_j(z)$$

By using the generalized convexity assumptions, similar to the proof of Theorem 3.5, we can completed the proof. □

Theorem 3.7 (Converse duality). *Let* (z, λ, u, s, t, y) *be a feasible solution of* $(\text{FMD})_e$ $(e = E(z))$*, and there exists* $\bar{x} \in S$*, such that*

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x}))\} \leq \sum_{j=1}^{r} u_j g_j(z). \tag{3.8}$$

If the following generalized convexity assumptions hold: $f_i(x, y_l^i) + \Phi_i(x)$ *are strictly* $(C, \alpha, \rho_l^i, d_l^i)-$*convex at* z, $-h_i(x, y_l^i) + \Psi_i(x)$ *are* $(C, \alpha, \bar{\rho}_l^i, \bar{d}_l^i)-$*convex at* z*, for* $l = 1, \cdots, s, i = 1, \cdots, p$, $g_j(x)$ *are* $(C, \alpha, \varepsilon_j, c_j)-$ *convex at* z *for* $j = 1, \cdots, r$*, and*

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{\rho_l^i d_l^i(\bar{x}, z) + e_i \bar{\rho}_l^i \bar{d}_l^i(\bar{x}, z)\} + \sum_{j=1}^{r} u_j \varepsilon_j c_j(\bar{x}, z) \geq 0.$$

Then $\bar{x} = z$*, and* $\bar{x}$ *is a weakly efficient solution of* $(\text{FVP})_e$.

Proof. We first prove $\bar{x} = z$. Suppose on the contrary that $\bar{x} \neq z$. By $\bar{x} \in S$, $e = E(z)$ and relation (4.1), we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x}))\} + \sum_{j=1}^{r} u_j g_j(\bar{x}) \leq$$
$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(z, y_l^i) + \Phi_i(z) - e_i(h_i(z, y_l^i) - \Psi_i(z))\} + \sum_{j=1}^{r} u_j g_j(z).$$

By using the generalized convexity assumptions and $\bar{x} \neq z$, similar to the proof of Theorem 3.4, we have a contradiction. So $\bar{x} = z$.

Now, we prove $\bar{x}$ is a weakly efficient solution of $(\text{FVP})_e$. If $\bar{x}$ is not a weakly efficient solution , then there exists $x \in S, x \neq \bar{x}$ such that $\Omega(x,e) < \Omega(\bar{x},e) = \Omega(z,e)$. On the other hand, relation (4.1) implies $\sum_{j=1}^{r} u_j g_j(z) \geq 0$, when $\bar{x} = z$. Hence $\Omega(x,e) < \Omega(z,e) + \sum_{j=1}^{r} u_j g_j(z)$. From $x \neq z$, and the generalized convexity assumptions, similar to the proof Theorem 3.5, we can complete the proof. □

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