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GLOBAL SOLVER FOR NONLINEAR BILEVEL VECTOR OPTIMIZATION PROBLEMS

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Dedicated to Guangya Chen on the occasion of his 70th birthday.

Abstract: Optimality conditions are given for nonlinear bilevel vector optimization problems in infinite dimensions. A new numerical method based on a multiobjective search algorithm with subdivision technique, is introduced for the determination of global solutions. Numerical results are presented for low dimensional nonlinear problems.

Key words: *bilevel optimization, vector optimization, global solvers*

Mathematics Subject Classification: *90C29, 90C30, 90-08*

1 Introduction

In bilevel optimization one investigates two coupled optimization problems. The problem on the lower level is a parametric optimization problem with a solution set used for the definition of the feasible set of the problem on the upper level (for details see [1], [7], [8]). With bilevel optimization problems certain leader-follower optimization problems can be formulated. Bilevel optimization has important applications (see [1], [7], [8]); among others, Stackelberg games (see [23]) are a special case of these bilevel problems. Using the so-called optimistic approach bilevel problems can be simplified because the problem on the upper level considers only optimal solutions of the problem on the lower level being the best for the upper level (for instance, see [18]).

In this paper we investigate nonlinear bilevel vector optimization problems, i.e. the optimization problems on the two levels are nonlinear problems, and we assume that the two objectives are vector functions. Problems in bilevel vector optimization are much more complicated than scalar bilevel problems because the solution set of the vector optimization problem on the lower level generally consists of infinitely many elements. Since this solution set is needed for different parameters, these bilevel vector optimization problems are highly complex and they require a lot of computation time for its numerical solution. Up to now there are only two numerical methods for these nonlinear bilevel vector optimization problems given by Eichfelder in [10], [11] and described by Deb and Sinha in [5], [6]. Eichfelder's method is some kind of path-following technique for smooth nonlinear bilevel problems whereas Deb and Sinha use an evolutionary approach for the solution of these bilevel problems. The method in this paper is based on a multiobjective search algorithm with subdivision technique.

To be more specific we use the following assumption.

Assumption 1.1. Let X, Y, U, V, W be real linear spaces, and let the linear spaces U, V, W be partially ordered by pointed convex cones C_U, C_V and C_W, respectively. Let $F : X \times Y \to U$, $f : X \times Y \to V$ and $g : X \times Y \to W$ be given maps, and let $S_X \subset X$ and $S_Y \subset Y$ be nonempty sets.

Under this assumption we investigate the nonlinear vector optimization problem using the optimistic approach

$$\begin{array}{c} \min\limits_{x,y} \; F(x,y) \\ \text{subject to the constraints} \\ x \in \Psi(y) \\ y \in S_Y \end{array} \tag{1.1}$$

where $\Psi(y)$, for an arbitrary $y \in S_Y$, denotes the solution set of the problem of the lower level

$$\begin{array}{c} \min\limits_{x} \; f(x,y) \\ \text{subject to the constraints} \\ g(x,y) \in -C_W \\ x \in S_X \end{array} \tag{1.2}$$

(an explicit definition of $\Psi(y)$ is given later). The two problems (1.1) and (1.2) are vector optimization problems. Obviously, we assume that $\Psi(y) \neq \emptyset$ for every $y \in S_Y$. For an arbitray element $y \in S_Y$ let

$$S(y) := \{x \in S_X \mid g(x,y) \in -C_W\}$$

denote the feasible set of problem (1.2). Then an element $\bar{x} \in S(y)$ is called a *minimal solution of problem (1.2)*, if the image $f(\bar{x}, y)$ is a minimal element of the image set $f(S(y), y)$, i.e.

$$(\{f(\bar{x}, y)\} - C_V) \cap f(S(y), y) = \{f(\bar{x}, y)\}$$

(compare [13], [2]). Then $\Psi(y)$ is defined as set of these minimal solutions or the set-valued map $\Psi : S_Y \rightrightarrows X$ is given by

$$\Psi(y) := \{x \in S(y) \mid x \text{ is a minimal solution of problem (1.2)}\} \text{ for all } y \in S_Y.$$

This set-valued map Ψ then defines the constraints w.r.t. the variable x in problem (1.1) of the upper level. As pointed out before, the set $\Psi(y)$ generally consists of infinitely many elements for every $y \in S_Y$ being difficult to determine. Minimal solutions of problem (1.1) are also defined as preimages of minimal elements of the image set of F. If

$$S := \{(x,y) \in X \times Y \mid x \in \Psi(y) \text{ and } y \in S_Y\}$$

denotes the feasible set of problem (1.1) assumed to be nonempty, then a pair $(\bar{x}, \bar{y}) \in S$ is called a *minimal solution of problem (1.1)*, if

$$(\{F(\bar{x}, \bar{y})\} - C_U) \cap F(S) = \{F(\bar{x}, \bar{y})\}.$$

Finally, we are interested in these minimal solutions $(\bar{x}, \bar{y})$.

This paper is now organized as follows. In Section 2 we present some optimality conditions for problem (1.1) using the Lagrange multiplier rule for problem (1.2) and the contingent cone. Section 3 is devoted to a new method for the determination of global solutions of the nonlinear bilevel vector optimization problem in $\mathbb{R}^n$ for small $n \in \mathbb{N}$. Numerical results for this method are presented in the last part.

2 Optimality Conditions

In this section we investigate the nonlinear bilevel vector optimization problem (1.1) under Assumption 1.1. In order to simplify the bilevel problem we describe the minimal solutions of problem (1.2) using the generalized Lagrange multiplier rule. The following theorem recalls this optimality condition.

Recall that the *dual cone* C_{V^*} of the ordering cone C_V is defined by

$$C_{V^*} := \{l \in V^* \mid l(c) \geq 0 \text{ for all } c \in C_V\}$$

and its *quasi-interior* $C^{\#}_{V^*}$ is given by

$$C^{\#}_{V^*} := \{l \in V^* \mid l(c) > 0 \text{ for all } c \in C_V \backslash \{0_V\}\}.$$

Theorem 2.1. *Let Assumption 1.1 be satisfied and, in addition, let the feasible set $S(y)$ be nonempty for every $y \in S_Y$.*
(a) Let X be a real Banach space, let V and W be real normed spaces, let the ordering cones $C_V \neq V$ and C_W have a nonempty interior and let the set S_X be convex with a nonempty interior. Let $\bar{x} \in S(y)$ be a minimal solution of problem (1.2) for an arbitrary $y \in S_Y$. If the maps $f(\cdot, y)$ and $g(\cdot, y)$ are Fréchet differentiable at $\bar{x}$, then there are continuous linear functionals $t \in C_{V^*}$, $u \in C_{W^*}$ with $(t, u) \neq 0_{V^* \times W^*}$ so that

$$\left(t \circ f'_x(\bar{x}, y) + u \circ g'_x(\bar{x}, y)\right)(x - \bar{x}) \geq 0 \text{ for all } x \in S_X \tag{2.1}$$

and

$$(u \circ g)(\bar{x}, y) = 0 \tag{2.2}$$

(f'_x and g'_x denote the Fréchet derivative w.r.t. the variable x). If, in addition, there is an $\hat{x} \in int(S_X)$ with $g(\bar{x}, y) + g'_x(\bar{x}, y)(\hat{x} - \bar{x}) \in -int(C_W)$, then $t \neq 0_{Y^*}$.
(b) Let X, V, W be real normed spaces and let $f(\cdot, y)$ and $g(\cdot, y)$ be Fréchet differentiable at some $\bar{x} \in S(y)$ for an arbitrary $y \in S_Y$. If there are continuous linear functionals $t \in C^{\#}_{V^*}$, $u \in C_{W^*}$ satisfying the system (2.1),(2.2) and if $(t \circ f)(\cdot, y)$ is pseudoconvex at $\bar{x}$ (i.e. for arbitrary $x \in S(y)$: $\left(t \circ f'_x(\bar{x}, y)\right)(x - \bar{x}) \geq 0 \;\Rightarrow\; (t \circ f)(\bar{x}, y) \leq (t \circ f)(x, y)$) and $(u \circ g)(\cdot, y)$ is quasiconvex at $\bar{x}$ (i.e. for arbitrary $x \in S(y)$: $(u \circ g)(x, y) \leq (u \circ g)(\bar{x}, y) \;\Rightarrow\; \left(u \circ g'_x(\bar{x}, y)\right)(x - \bar{x}) \leq 0$), then $\bar{x}$ is a minimal solution of problem (1.2).

Proof. (a) Because of $C_V \neq V$ every minimal solution $\bar{x}$ of problem (1.2) is also a so-called weakly minimal solution, i.e.

$$(\{f(\bar{x}, y)\} - \text{int}(C_V)) \cap f(S(y), y) = \emptyset$$

(see [13, Lemma 4.14]), and then Theorem 7.4 in [13] leads to the assertion.
(b) Let $x \in S(y)$ be arbitrarily chosen. Then we have $g(x, y) \in -C_W$ and with $u \in C_{W^*}$ we obtain $(u \circ g)(x, y) \leq 0$. With the equality (2.2) and the quasiconvexity of $(u \circ g)(\cdot, y)$ at $\bar{x}$ we then get $\left(u \circ g'_x(\bar{x}, y)\right)(x - \bar{x}) \leq 0$. The inequality (2.1) then implies $\left(t \circ f'_x(\bar{x}, y)\right)(x - \bar{x}) \geq 0$ and with the pseudoconvexity of $(t \circ f)(\cdot, y)$ at $\bar{x}$ we conclude $(t \circ f)(\bar{x}, y) \leq (t \circ f)(x, y)$. Since this inequality holds for arbitrary $x \in S(y)$ and the functional t belongs to the quasi-interior of the dual cone, we obtain with a known scalarization result (compare [13, Thm. 5.18,(b)]) that $\bar{x}$ is a minimal solution of problem (1.2). □

If we recall the *normal cone* to S_X at $\bar{x}$

$$N_{S_X} := \{l \in X^* \mid l(x - \bar{x}) \leq 0 \text{ for all } x \in S_X\},$$

then the inequality (2.1) can be rewritten as

$$t \circ f'_x(\bar{x}, y) + u \circ g'_x(\bar{x}, y) \in -N_{S_X}(\bar{x}). \tag{2.3}$$

Notice that it is not possible to give a complete characterization of minimal solutions because in part (a) of the preceding theorem the Lagrange multiplier t belongs to the dual cone C_{W^*} whereas in part (b) t is an element of the quasi-interior of C_{W^*}. This theoretical gap is well-known in vector optimization (see [13, p. 133]) and it is characteristic for the multiobjective case. This gap disappears, if we work with the so-called weak minimality concept. But this notion may lead to a larger set $\Psi(y)$ which means that one gets another solution set for problem (1.1), i.e. in general, the feasible set of problem (1.1) becomes larger than desired. In this case the bilevel problem (1.1) is replaced by another one. It is known that under strong assumptions the so-called set of properly minimal elements obtained with the Lagrange multiplier $t \in C^{\#}_{V^*}$ is dense in the set of minimal elements (compare [12], [21], [3]). Therefore, from a computational point of view it makes sense to work with the quasi-interior $C^{\#}_{V^*}$. This is the main reason why we consider a map $\tilde{\Psi} : S_Y \rightrightarrows X$ with

$$\tilde{\Psi}(y) := \{x \in S(y) \mid \exists (t, u) \in C^{\#}_{V^*} \times C_{W^*} \text{ with (2.3) and (2.2)}\} \text{ for all } y \in S_Y.$$

Since we cannot give a complete characterization of minimal solutions of problem (1.2) in the multiobjective case, we replace the set $\Psi(y)$ by $\tilde{\Psi}(y)$ in the problem on the upper level. Problem (1.1) is then modified by the problem

$$\begin{gathered}\min\ F(x, y)\\ \text{subject to the constraints}\\ t \circ f'_x(x, y) + u \circ g'_x(x, y) \in -N_{S_X}(\bar{x})\\ (u \circ g)(x, y) = 0\\ g(x, y) \in -C_W\\ x \in S_X,\ y \in S_Y,\ t \in C^{\#}_{v^*},\ u \in C_{W^*}.\end{gathered}$$

This problem is closely related to optimization problems with equilibrium constraints (see [19], [20]).

In order to avoid a modification of the original problem (1.1) we now present another type of optimality condition. For the used concept of contingent cones we refer to [15]. Recall that a pair $(\bar{x}, \bar{y}) \in S$ is called a *weakly minimal solution of problem (1.1)*, if

$$(\{F(\bar{x}, \bar{y})\} - \text{int}(C_U)) \cap F(S) = \emptyset.$$

Moreover, the set S is said to be *starshaped* w.r.t. $\bar{z} := (\bar{x}, \bar{y})$ if for arbitrary $z \in S$

$$\lambda z + (1 - \lambda)\bar{z} \in S \text{ for all } \lambda \in [0, 1].$$

Theorem 2.2. *Let Assumption 1.1 be satisfied and, in addition, let X, Y, U be real normed spaces, let the ordering cone $C_U \neq U$ have a nonempty interior, and let the feasible set S of problem (1.1) be nonempty.*

(a) If $(\bar{x}, \bar{y}) \in S$ is a minimal solution of problem (1.1), the map F is Gâteaux differentiable at $(\bar{x}, \bar{y})$ and S is starshaped w.r.t. $(\bar{x}, \bar{y})$, then

$$F'(\bar{x}, \bar{y})(h) \notin -int(C_U) \text{ for all } h \in T(S, (\bar{x}, \bar{y})). \tag{2.4}$$

(b) Let the ordering cone C_U be closed and let the map F be convex. If the condition (2.4) is fulfilled at some $(\bar{x}, \bar{y}) \in S$ and S is starshaped w.r.t. $(\bar{x}, \bar{y})$, then the pair $(\bar{x}, \bar{y})$ is a weakly minimal solution of problem (1.1).

Proof. (a) Assume that the condition (2.4) does not hold, i.e. there exists some tangent vector $h \in T(S, (\bar{x}, \bar{y})$ with $F'(\bar{x}, \bar{y})(h) \in -\mathrm{int}(C_U)$. Then there are a sequence $(x_n, y_n)_{n\in I\!N}$ of elements in S and a sequence $(\lambda_n)_{n\in I\!N}$ of positive real numbers with $(\bar{x}, \bar{y}) = \lim_{n\to\infty} (x_n, y_n)$ and $h = \lim_{n\to\infty} \lambda_n(x_n - \bar{x}, y_n - \bar{y})$. Since Gâteaux derivatives are continuous and linear, we obtain

$$F'(\bar{x}, \bar{y})(h) = F'(\bar{x}, \bar{y}) \left(\lim_{n\to\infty} \lambda_n(x_n - \bar{x}, y_n - \bar{y}) \right) = \lim_{n\to\infty} \lambda_n F'(\bar{x}, \bar{y})(x_n - \bar{x}, y_n - \bar{y})$$

and

$$F'(\bar{x}, \bar{y})(x_n - \bar{x}, y_n - \bar{y}) \in -\mathrm{int}(C_U) \text{ for a sufficiently large } n \in I\!N.$$

Because of the equation

$$\begin{aligned} F'(\bar{x}, \bar{y})(x_n - \bar{x}, y_n - \bar{y}) &= \lim_{\lambda\to 0} \frac{1}{\lambda} \left(F(\bar{x} + \lambda(x_n - \bar{x}), \bar{y} + \lambda(y_n - \bar{y})) - F(\bar{x}, \bar{y}) \right) \\ &= \lim_{\lambda\to 0_+} \frac{1}{\lambda} \left(F(\bar{x} + \lambda(x_n - \bar{x}), \bar{y} + \lambda(y_n - \bar{y})) - F(\bar{x}, \bar{y}) \right) \end{aligned}$$

we get for some sufficiently small $\lambda > 0$

$$F(\bar{x} + \lambda(x_n - \bar{x}), \bar{y} + \lambda(y_n - \bar{y})) - F(\bar{x}, \bar{y}) \in -\mathrm{int}(C_U).$$

Hence, the pair $(\bar{x}, \bar{y})$ is not a weakly minimal solution of problem (1.1) and, therefore, it is not a minimal solution of problem (1.1).

(b) Since F is assumed to be convex for all $(x, y) \in S$ and $\lambda \in (0, 1]$, we have

$$\lambda F(x, y) + (1 - \lambda) F(\bar{x}, \bar{y}) - F(\lambda x + (1 - \lambda)\bar{x}, \lambda y + (1 - \lambda)\bar{y}) \in C_U$$

implying

$$\frac{1}{\lambda} \big(F(\bar{x} + \lambda(x - \bar{x}), \bar{y} + \lambda(y - \bar{y})) - F(\bar{x}, \bar{y}) \big) \in \{F(x, y) - F(\bar{x}, \bar{y})\} - C_U.$$

Since F is Gâteaux differentiable at $(\bar{x}, \bar{y})$ and the ordering cone C_U is closed, we conclude

$$F'(\bar{x}, \bar{y})(x - \bar{x}, y - \bar{y}) \in \{F(x, y) - F(\bar{x}, \bar{y})\} - C_U. \tag{2.5}$$

If we assume that the pair $(\bar{x}, \bar{y})$ is not a weakly minimal solution of problem (1.1), there exists some $(\tilde{x}, \tilde{y}) \in S$ with $F(\tilde{x}, \tilde{y}) - F(\bar{x}, \bar{y}) \in -\mathrm{int}(C_U)$. Together with the condition (2.5) we get

$$F'(\bar{x}, \bar{y})(\tilde{x} - \bar{x}, \tilde{y} - \bar{y}) \in -\mathrm{int}(C_U) - C_U = -\mathrm{int}(C_U) \tag{2.6}$$

(see Lemma 1.12,(b) and Lemma 1.32,(a) in [15] for the last equality). Since the set S is starshaped w.r.t. $(\bar{x}, \bar{y})$ the set $S - \{(\bar{x}, \bar{y})\}$ is contained in the contingent cone $T(S, (\bar{x}, \bar{y}))$ and, therefore, we have $(\tilde{x} - \bar{x}, \tilde{y} - \bar{y}) \in T(S, (\bar{x}, \bar{y}))$. Then the condition (2.6) implies that the condition (2.4) is not fulfilled. □

In the preceding theorem we have assumed that the feasible set S is starshaped w.r.t. $(\bar{x}, \bar{y})$. Then it is well-known that the contingent cone $T(S, (\bar{x}, \bar{y}))$ equals the closure of the cone generated by the set $S - \{(\bar{x}, \bar{y})\}$ (for instance, see [13, Cor. 4.11]). The assumption that the feasible set S is starshaped w.r.t. $(\bar{x}, \bar{y})$ is a strong assumption. In general, it is not fulfilled for nonlinear problems. Notice that even a linear vector optimization problem may not have a starshaped solution set.

In the single-objective case Thm. 2.2 can be proved under weaker assumptions. This result extends Thm. 2.1 in [9] to the multiobjective case in infinite dimensions. Similar to the result in Thm. 2.1 of this section the condition (2.4) is not a complete characterization of minimal solutions. This fact comes in by the partial ordering in the linear space U. Therefore, it only seems to be suitable to use optimality conditions for bilevel vector optimization problems with special structure. For the numerical solution of nonlinear bilevel problems we work with a global solution method instead of using the presented optimality conditions.

3 Global Solver

In this section we present a new numerical method for the solution of nonlinear bilevel vector optimization problems. Up to now, only Eicherfelder's method (see [10], [11]) and the approach by Deb and Sinha (see [5], [6]) are known for solving these problems. Eichfelder uses a self-adaptive scalarization technique for a concise and representative approximation of the minimal solutions of the problem on the lower level. Deb and Sinha work with an evolutionary algorithm. Here we now describe an approach based on a multiobjective search algorithm with subdivision technique published in [14].

In Assumption 1.1 we now set $X = \mathbb{R}^n$, $Y = \mathbb{R}^1$, $U = \mathbb{R}^k$, $V = \mathbb{R}^l$, $W = \mathbb{R}^m$, $C_U = \mathbb{R}^k_+$, $C_V = \mathbb{R}^l_+$, $C_W = \mathbb{R}^m_+$ and $S_Y = [a, b]$ for $-\infty < a < b < \infty$. The case that the set S_Y is the union of closed intervals could also be considered. We assume that the problems (1.1) and (1.2) are given with nonempty feasible sets, i.e. $S \neq \emptyset$ and $S(y) \neq \emptyset$ for all $y \in S_Y$. For simplicity let the feasible set $S(y)$ be bounded for every $y \in S_Y$.

Under these assumptions the problem on the lower level is a finite-dimensional one-parametric vector optimization problem with a parameter varying in the interval $[a, b]$. Since we need a representative approximation of the whole set of minimal solutions, so-called interactive methods cannot be applied. We use a multiobjective search algorithm being briefly recalled.

For an arbitrary $y \in [a, b]$ the multiobjective search algorithm with subdivision technique (MOSAST) [14, Alg. 2] can be applied for the solution of problem (1.2). Since the feasible set $S(y)$ is bounded, it is contained in a box in which minimal solutions can be determined using random vectors. Then this box is partitioned into small boxes, but we take only those boxes containing at least one computed minimal solution (for subdivision techniques in vector optimization see [22]). For each of these selected small boxes we determine minimal solutions using random vectors. This system of boxes is reduced again and again. The iteration process is stopped, if the diameter of these boxes is less than a given small positive number. At each iteration minimal solutions in a certain box are determined with the Graef-Younes method with backward iteration (GYMBI) [14, Alg. 1]. Obviously, this method makes only sense for problems with non-expensive functions, i.e. function values can be rapidly computed. It can be used for highly nonlinear and nonsmooth functions because only randomly generated points are considered.

The method described in the present paper determines for some few numbers

$$a = y_1 < y_2 < \ldots < y_p = b \tag{3.1}$$

an approximation M_i ($i \in \{1, \ldots, p\}$) of the set of minimal solutions of problem (1.2). These p vector optimization problems are solved with the multiobjective search algorithm (MOSAST). For every y_i-layer ($i \in \{1, \ldots, p\}$) the set M_i is reduced to only those points x so that (x, y_i) is a minimal solution of problem (1.1) w.r.t. the determined discrete points. This reduced set is again called M_i. In order to refine our discretization scheme we choose new discretization points as mean values $\tilde{y}_i := \frac{y_i + y_{i+1}}{2}$ for $i = 1, \ldots, p-1$. For every new parameter $\tilde{y}_i$ we choose the union $M_i \cup M_{i+1}$ as a starting set for the subdivision technique used in the multiobjective search algorithm. This means that we consider a system of boxes containing at least one element of the set $M_i \cup M_{i+1}$. For each of these boxes we apply the Graef-Younes method with backward iteration (GYMBI) leading to a first approximation of the set of minimal solutions of problem (1.2) for the new parameter $\tilde{y}_i$. These boxes can become smaller and smaller until we have an acceptable approximation of minimal solutions. This set of minimal solutions is again reduced using the afore-mentioned reduction technique. Using this approach we have approximations for $2p - 1$ problems on the lower level. Again, with mean values of these discretization points we can repeat the projection procedure in connection with the subdivision technique.

Now we present the global solver for the bilevel vector optimization problem.

Algorithm 3.1.

Input: Choose $\delta > 0$ and $\varepsilon > 0$ % (stopping criteria)
  Choose $p \in I\!N$ with $p \geq 2$ % (number of discretization points)
Select a discretization $y_1, \ldots, y_p$ with (3.1)
for $i = 1 : 1 : p$ **do**
  Apply MOSAST and determine an approximation M_i of the set of minimal
    solutions of problem (1.2) w.r.t. y_i
end for
Using GYMBI select all minimal solutions of problem (1.1) restricted to the discrete
  set $\bigcup_i M_i \times \{y_i\}$ and reduce M_i to the corresponding components of the
  determined minimal solutions
while $\max_i\{y_{i+1} - y_i\} > \delta$ **do**
  for $i = 1 : 1 : p - 1$ **do**
    $\tilde{y}_i := \frac{y_i + y_{i+1}}{2}$
    Select a system $\mathcal{B}_1$ of boxes in $I\!R^n$
    $s := 1$
    $\tilde{M} := M_i \cup M_{i+1}$
    $\hat{M} := \emptyset$
    while diameter$(\mathcal{B}_s) > \varepsilon$ **do**
      $i_s := \#(\tilde{M})$ % (magnitude of the set $\tilde{M}$)
      $j_s := \#(\mathcal{B}_s)$ % (magnitude of the system $\mathcal{B}_s$)
      for $j = 1 : 1 : j_s$ **do**
        $B_j := (\mathcal{B}_s)_j$ % (j-th set in the system $\mathcal{B}_s$)
        for $t = 1 : 1 : i_s$ **do**
          $m_t := (\tilde{M})_t$ % (t-th element in the set $\tilde{M}$)
          if $m_t \in B_j$ **then**
            Apply GYMBI w.r.t. the set $S(\tilde{y}_i) \cap B_j$ and determine an
              approximation $\hat{M}_i$
            $\hat{M} := \hat{M} \cup \hat{M}_i$
          end if

end for
end for
Select a new system $\mathcal{B}_{s+1}$ of boxes with diameter($\mathcal{B}_{s+1}$) < diameter($\mathcal{B}_s$)
$s := s + 1$
$\tilde{M} := \tilde{M} \cup \hat{M}$
$\hat{M} := \emptyset$
end while
Apply GYMBI w.r.t. $\tilde{M}$ and determine an approximation $\tilde{M}_i$
end for
$p := 2p - 1$
Rename the discretization points as $y_1, y_2, \ldots$
Rename the approximations of the set of minimal solutions as $M_1, M_2, \ldots$
Using GYMBI select all minimal solutions of problem (1.1) restricted to the discrete set $\bigcup_i M_i \times \{y_i\}$ and reduce M_i to the corresponding components of the determined minimal solutions
end while
Let M denote the set of computed minimal solutions of problem (1.1)
Output: M

Theorem 3.1. *Under the assumptions of this section Algorithm 3.1 is well-defined and it determines all (global) minimal solutions of the bilevel optimization problem (1.1),(1.2) among the randomly generated vectors.*

Proof. Obviously, Algorithm 3.1 is well-defined. It is proved in [16, Thm. 1] that the Graef-Younes Method with backward iteration [16, Alg. 1] determines all (global) minimal solutions of a vector optimization problem, if the constraints are restricted to the randomly generated feasible points. Since we only work with this method in appropriate boxes, we cannot loose potentially minimal solutions among the randomly generated vectors. □

Remark 3.2. Algorithm 3.1 is really a global solver because it determines all minimal solutions among a discrete set of points. This is an essential advantage in contrast to standard methods applied to optimality conditions. The examples in the next section illustrate that these solution sets are very complex so that one needs a global solver in bilevel vector optimization.

Remark 3.3. It does not seem to be desirable to work with systems of very small boxes because this would be time-consuming. But it is important to have a fine partition of the interval $[a, b]$. So, one should carry out several projection steps. Then one gets a better characterization of the feasible set of the problem on the upper level.

4 Numerical Results

Now we apply Algorithm 3.1 to various bilevel biobjective optimization problems in low dimensional spaces. For all examples we consider the linear spaces $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $U = V = \mathbb{R}^2$ and the set $S_X = \mathbb{R}$. The vector functions $F, f : \mathbb{R}^3 \to \mathbb{R}^2$ are sometimes highly nonlinear so that one needs a global solver for the solution of the bilevel problem (1.1).

Example 4.1. This problem introduced by Eichfelder is taken from [10], [11]. For the problems (1.1) and (1.2) we have the objectives F, f with

$$F(x_1, x_2, y) = \begin{pmatrix} x_1 + x_2^2 + y + \sin^2(x_1 + y) \\ (\cos x_2) \cdot (0.1 + y) \cdot \exp(-\frac{x_1}{0.1+x_2}) \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3$$

and

$$f(x_1, x_2, y) = \begin{pmatrix} \frac{(x_1-2)^2+(x_2-1)^2}{4} + \frac{x_2 y+(5-y)^2}{16} + \sin\frac{x_2}{10} \\ \frac{x_1^2+(x_2-6)^4-2x_1 y-(5-y)^2}{80} \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3$$

and the constraint map $g : I\!R^3 \to I\!R^4$ with

$$g(x_1, x_2, y) = \begin{pmatrix} x_1^2 - x_2 \\ 5x_1^2 + x_2 - 10 \\ x_2 - 5 + y/6 \\ -x_1 \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3.$$

The ordering cone C_W is chosen as $I\!R^4_+$ and the set S_Y equals $[0, 10]$. An application of Algorithm 3.1 gives an approximation of the set of minimal solutions of the bilevel problem illustrated on the left hand of Fig. 1. The right side of this figure shows the images of these minimal solutions. The left picture of Fig. 1 shows only a rough estimate of the set of

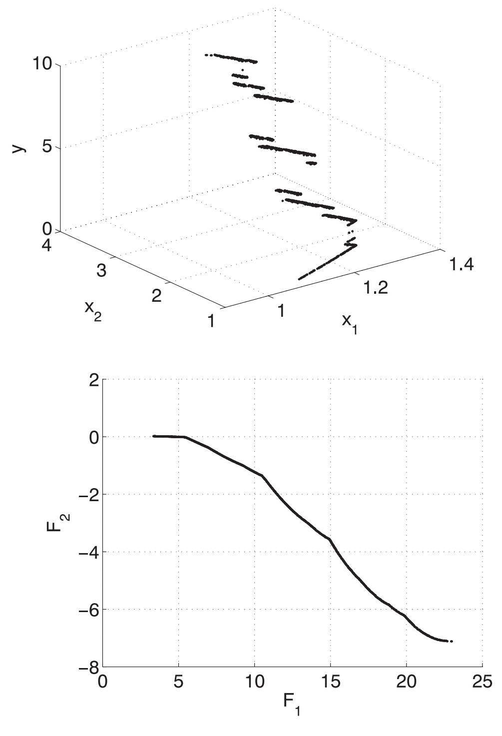

Figure 1: Minimal solutions and its images of Example 4.1.

minimal solutions of the bilevel problem because the discretization of the y-values is not fine

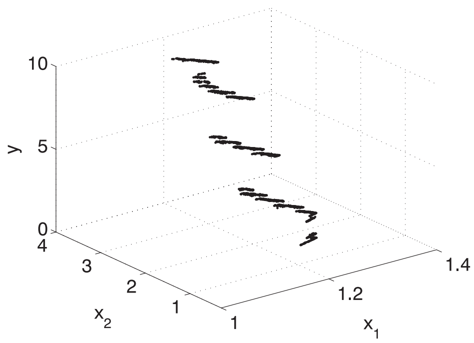

Figure 2: Refinement of minimal solutions of Example 4.1.

enough. Fig. 2 presents a refinement of the set of minimal solutions showing that this set obviously consists of several disconnected parts. These results are based on 41 equidistant discretization points in the interval $[0, 10]$.

Example 4.2. Now we investigate the problems (1.1) and (1.2) with the set $S_Y := [1, 6]$, the objective functions F, f given by

$$F(x_1, x_2, y) = \begin{pmatrix} (x_1 + x_2)^y \\ \sqrt{x_1^2 + \frac{y}{2}} \end{pmatrix} \text{ for all } (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}_+$$

and

$$f(x_1, x_2, y) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ for all } (x_1, x_2, y) \in \mathbb{R}^3$$

and the constraint map $g : \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \times \mathbb{R} \to \mathbb{R}^6$ with

$$g(x_1, x_2, y) = \begin{pmatrix} -x_1^2 - x_2^2 + 1 + \frac{y}{10} \cos \arctan \frac{x_1}{x_2} \\ (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.5 \\ -x_1 \\ x_1 - \pi \\ -x_2 \\ x_2 - \pi \end{pmatrix} \text{ for all } (x_1, x_2, y) \in \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \times \mathbb{R}.$$

We choose the ordering cone $C_W := \mathbb{R}^6_+$. For $y = 1$ the problem on the lower level coincides with a problem proposed by Tanaka et al. [24] (see also [4, p. 366]) used as a test problem for evolutionary algorithms. Fig. 3 illustrates the set of minimal solutions of problem (1.1) on the left; on the right hand one can see the images of these minimal solutions. Although the image set seems to be a smooth curve, the set of preimages is disconnected without special structure. For this example we have worked with 81 equidistant discretization points in the

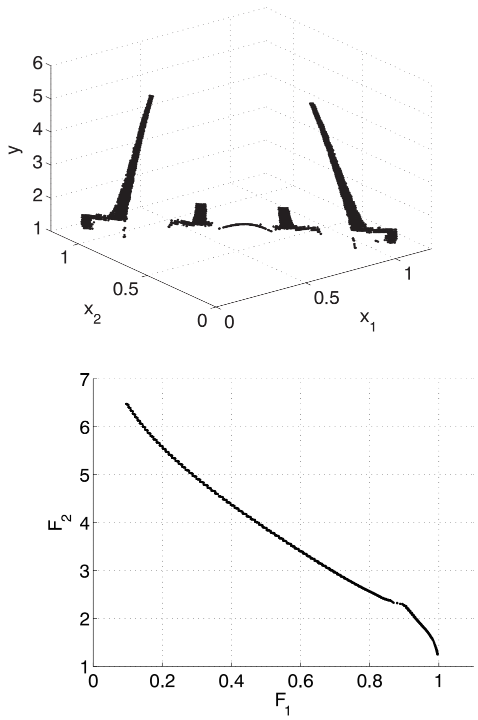

Figure 3: Minimal solutions and its images of Example 4.2.

interval $[1, 6]$.

Example 4.3. This example modifies a (one-level) problem due to Kursawe [17] (see also [4, p. 341]). We consider the problems (1.1) and (1.2) with the objectives F, f given by

$$F(x_1, x_2, y) = \begin{pmatrix} -x_1^2 - 3yx_2 + \sin yx_1 \\ x_2y - \exp(\sin(x_1 + x_2 - y)) \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3$$

and

$$f(x_1, x_2, y) = \begin{pmatrix} \exp(-0.2\sqrt{x_1^2 + x_2^2 + y^2}) \\ |x_1|^{0.8} + |x_2|^{0.8} + \sin y \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3$$

and the constraint map $g : I\!R^3 \to I\!R^2$ with

$$g(x_1, x_2, y) = \begin{pmatrix} x_1^2 - x_2 - \frac{y}{2} \\ x_1 + 2x_2 - 5 \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3.$$

Moreover, we have $S_Y := [0, 4]$ and $C_W := I\!R^2_+$. Notice that the objective function f is not differentiable so that smooth methods cannot be applied. Fig. 4 shows the set of minimal solutions of problem (1.1) on the left hand, the right side illustrates the images of these minimal solutions. Here we have chosen 33 equidistant discretization points in the interval

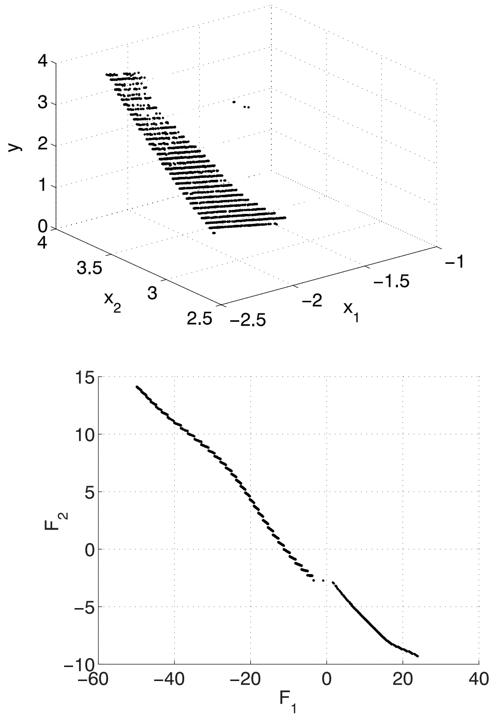

Figure 4: Minimal solutions and its images of Example 4.3.

$[0, 4]$.

Example 4.4. This example is obtained by a modification of a (one-level) problem due to Van Veldhuizen [25, p. 545]. Here we have the objective functions F, f with

$$F(x_1, x_2, y) = \begin{pmatrix} (x_1-1)^4 + (x_2-2)^2 + (\frac{y}{10}-1)^2 \\ (x_1+1)^2 + (x_2+2)^4 + (\frac{y}{10}+1)^2 \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3$$

and

$$f(x_1, x_2, y) = \begin{pmatrix} 1 + \big(y - \frac{1}{\sqrt{3}}\big)^2 - \exp\big(-((x_1-1)^2 + (x_2 - \frac{1}{\sqrt{2}})^2)\big) \\ 1 + \big(y + \frac{1}{\sqrt{3}}\big)^2 - \exp\big(-((x_1+1)^2 + (x_2 + \frac{1}{\sqrt{2}})^2)\big) \end{pmatrix}$$
$$\text{for all } (x_1, x_2, y) \in I\!R^3$$

and the constraint map $g : I\!R^3 \to I\!R^4$ with

$$g(x_1, x_2, y) = \begin{pmatrix} -4 - x_1 \\ x_1 - 4 \\ -4 - x_2 \\ x_2 - 4 \end{pmatrix} \text{ for all } (x_1, x_2, y) \in I\!R^3.$$

Now we have $S_Y := [-4, 4]$ and the ordering cone $C_W := I\!R^4$. The set of minimal solutions of problem (1.1) is illustrated on the left hand of Fig. 5; the right side shows the corresponding set of images. Although the image set is a smooth convex "Pareto curve", the set of preimages is not so clearly drawn and shows the complexity of this problem. 65 equidistant discretization points have been chosen in the interval $[-4, 4]$.

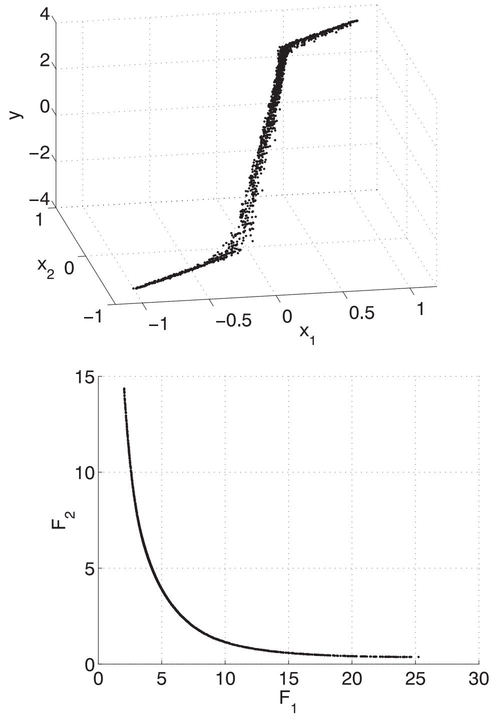

Figure 5: Minimal solutions and its images of Example 4.4.

5 Conclusion

Optimality conditions are formulated for nonlinear bilevel vector optimization problems. Although these conditions are given under differentiability assumptions in a general setting, they do not seem to be suitable for the determination of the complete set of minimal solutions of the bilevel problem. For the approximation of this solution set a new numerical method is presented which is based on a known multiobjective search algorithm with subdivision technique. Even nonsmooth bilevel problems with few variables can be solved with this approach. For the solution of problems with more variables this method can produce good starting points for smooth local solvers. The discussed numerical results show that the set of minimal solutions looks bizarre and for some examples the "Pareto curve" seems to be non-convex. These results also emphasize the complexity of bilevel vector optimization problems.

Recently, Deb and Sinha [6] have proposed test problems which should be used for the comparison of the currently existing methods for the solution of nonlinear bilevel vector optimization problems. This is a topic for future work.

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