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CONNECTEDNESS OF THE GLOBAL EFFICIENT AND THE HENIG EFFICIENT SOLUTION SETS FOR THE SET-VALUED VECTOR EQUILIBRIUM PROBLEMS*

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Dedicated to professor Guang-Ya Chen in honour of his 70th birthday.

Abstract: In this paper, we introduce the concepts of global efficient solution, and Henig efficient solution for set-valued vector equilibrium problems, and study the existence and connectedness of the global efficient and the Henig efficient solution sets for a kind of the set-valued vector equilibrium problems and the set-valued Hartman-Stampacchia variational inequality in normed linear space.

Key words: *set-valued vector equilibrium problems, global efficient solution, Henig efficient solution, existence; connectedness*

Mathematics Subject Classification: *90C30*

1 Introduction

Vector variational inequality was introduced by Giannessi [10] in 1980. Later on, vector variational inequality and its various extensions have been studied by Chen and Cheng [5], Chen and Yang [6], Chen [7], and other authors (see [8, 16, 17, 21, 24, 25]). Lee, Kim, Lee and Yun [18], Cheng [9] both have studied the connectedness of weak efficient solution sets for single-valued vector variational inequalities in finite dimensional Euclidean space. Gong [12–14] introduced the concepts of f-efficient solution, Henig efficient solution, global efficient solution, super efficient solution, cone-super efficient solution, Benson efficient solution for vector equilibrium problems, and studied the connectedness and scalarization of the solution sets in infinite dimension space. Ansari, Oettli and Schläger [1] introduced the set-valued vector equilibrium problems. Later on, Fu [11], Hou, Yu and Chen [15], Tan [22], Peng, Lee and Yang [20], and Long, Huang and Teo [19] have studied the existence of solutions for set-valued vector equilibrium problems and set-valued vector variational inequalities. Chen, Gong, and Yuan [4] studied the connectedness and compactness of weak efficient solutions for set-valued vector equilibrium problems.

Because that the concepts of Henig efficient solution and the global efficient solution are very important concepts for vector equilibrium problems. In this paper, we will introduce the concepts of global efficient solution, and Henig efficient solution for set-valued vector equilibrium problems, and study the existence and connectedness of the Henig efficient and the global efficient solution sets for set-valued vector equilibrium problems and the set-valued vector Hartman-Stampacchia variational inequality in normed linear space.

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2 Preliminaries

Throughout this paper, let X, Y be two normed linear spaces, let Y^* be the topological dual space of Y, and let C be a closed convex pointed cone in Y. Let A be a nonempty subset of X, and $F : A \times A \to 2^Y$ be a set-valued map.

We consider the following set-valued vector equilibrium problem (in short:SVEP): find $\bar{x} \in A$, such that

$$F(\bar{x}, y) \cap (-K) = \emptyset \quad \text{for all } y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y .

Let

$$C^* = \{f \in Y^* : f(y) \geq 0 \quad \text{for all } y \in C\}$$

be the dual cone of C.

Denote the quasi-interior of C^* by $C^\#$; i.e.

$$C^\# = \{f \in Y^* : f(y) > 0 \quad \text{for all } y \in C\setminus\{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\text{cone}(D) = \{td : t \geq 0, d \in D\}.$$

A nonempty convex subset B of the convex cone C is called a base of C if

$$C = \text{cone}(B) \quad \text{and} \quad 0 \notin \text{cl}(B).$$

It is easy to see that $C^\# \neq \emptyset$ if and only if C has a base.

If C has a base B, we can associate C with another closed convex pointed cone $C_\varepsilon(B)$, defined by

$$C_\varepsilon(B) = \text{cl}(\text{cone}(B + \varepsilon U)),$$

where

$$0 < \varepsilon < \delta = \inf\{\|b\| : b \in B\},$$

and U is the closed unit ball of Y. The notion δ and U will be used in the rest of this paper. By [3], if $0 < \varepsilon < \varepsilon' < \delta$, then $C_\varepsilon(B)$ is a closed convex pointed cone,

$$\text{C}\setminus\{0\} \subset \text{int}C_\varepsilon(B),$$

and

$$C_\varepsilon(B) \subset \text{cone}(B + \varepsilon' U).$$

Let

$$C^\Delta(B) = \left\{f \in C^\#\text{: there exists } t > 0, \text{such that } f(b) \geq t \text{ for all } b \in B\right\}.$$

By the separation theorem of convex sets, we know that $C^\Delta(B) \neq \emptyset$.

Now we introduce the concepts of global efficient solution, Henig efficient solution for SVEP.

Definition 2.1. A vector $x \in A$ is called a global efficient solution to the SVEP if there exists a point convex cone $H \subset Y$, with $C\setminus\{0\} \subset \text{int}H$, such that

$$F(x, y) \cap ((-H)\setminus\{0\}) = \emptyset \quad \text{for all } y \in A.$$

The set of global efficient solutions to the SVEP is denoted by $V_G(A, F)$.

Definition 2.2. Let B be a base of C . A vector $x \in A$ is called a Henig efficient solution to the SVEP if there exists a $0 < \varepsilon < \delta$, such that

$$F(x, y) \cap (-\text{int}C_\varepsilon(B)) = \emptyset \quad \text{for all } y \in A.$$

The set of Henig efficient solutions to the SVEP is denoted by $V_H(A, F)$.

Definition 2.3. Let $f \in C^* \setminus \{0\}$. A vector $x \in A$ is called a f-efficient solution to the SVEP if

$$f(F(x, y)) \geq 0 \text{ for all } y \in A,$$

where $f(F(x, y)) \geq 0$ means that $f(z) \geq 0$, for all $z \in F(x, y)$. The set of f-efficient solutions to the SVEP is denoted by $V_f(A, F)$.

Definition 2.4. Let A be a nonempty convex subset in X. A set-valued map $F : A \times A \to 2^Y$ is called to be C-convex in its second variable if, for each fixed $x \in A$, for every $y_1, y_2 \in A$, $t \in [0, 1]$, the following property holds:

$$tF(x, y_1) + (1 - t)F(x, y_2) \subset F(x, ty_1 + (1 - t)y_2) + C.$$

Definition 2.5. Let A be a nonempty convex subset in X. A set-valued map $F : A \times A \to 2^Y$ is called to be C-concave in its first variable if, for each fixed $y \in A$, for every $x_1, x_2 \in A$, $t \in [0, 1]$, the following property holds:

$$F(tx_1 + (1 - t)x_2, y) \subset tF(x_1, y) + (1 - t)F(x_2, y) + C .$$

Definition 2.6. Let G be a set-valued map from a topological space W to another topological space Q.

(i) We say that $G : W \to 2^Q$ is upper semicontinuous at $x_0 \in W$ if, for any neighborhood $U(G(x_0))$ of $G(x_0)$, there is a neighborhood $U(x_0)$ of x_0 such that

$$G(x) \subset U(G(x_0)) \quad \text{for all } x \in U(x_0).$$

G is said to be upper semicontinuous on W if it is upper semicontinuous at each $x \in W$.

(ii) G is said to be lower semicontinuous at $x_0 \in W$ if, for any $y_0 \in G(x_0)$ and any neighborhood $U(y_0)$ of y_0, there exists a neighborhood $U(x_0)$ of x_0 such that

$$G(x) \cap U(y_0) \neq \emptyset, \quad \text{for all } x \in U(x_0).$$

G is said to be lower semicontinuous on W if it is lower semicontinuous at each $x \in W$. G is said to be continuous on W if it is both upper semicontinuous and lower semicontinuous on W.

(iii) G is said to be closed, if $\text{Graph}(G) = \{(x, y) : x \in W,\ y \in G(x)\}$ is a closed subset in $W \times Q$.

Definition 2.7. Let $T : A \to 2^{L(X,Y)}$ be a set-valued map, where $L(X, Y)$ is the space of all bounded linear operators from X into Y [let $L(X, Y)$ be equipped with operator norm topology].

(i) Let $f \in C^* \setminus \{0\}$. T is said to be f-pseudomonotone on A if, for every pair of points $x, y \in A$, $f((s, y - x)) \geq 0$, for all $s \in Tx$, then $f((t, y - x)) \geq 0$, for all $t \in Ty$.

(ii) T is said to be v-hemicontinuous if, for every pair of points $x, y \in A$, the set-valued map

$$J(\alpha) := (T(\alpha y + (1-\alpha)x), y - x), \ \alpha \in [0,1],$$

is lower semicontinuous at 0 .

Definition 2.8. Let X be a Hausdorff topological vector space, let $K \subset X$ be a nonempty set. $G : K \to 2^X$ is called to be a KKM map, if for any finite set $\{x_1, \cdots, x_n\} \subset K$, the relation

$$\mathrm{co}\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$$

holds, where $\mathrm{co}\{x_1, \cdots, x_n\}$ denoted the convex hull of $\{x_1, \cdots, x_n\}$.

The following FKKM theorem plays a crucial role in this paper.

Lemma 2.9. *Let X be a Hausdorff topological vector space. Let K be a nonempty convex subset of X, and let $G : K \to 2^K$ be a KKM map. If for each $x \in K$, $G(x)$ is closed in X, and if there exists a point $x_0 \in K$ such that $G(x_0)$ is compact, then $\underset{x\in K}{\cap} G(x) \neq \emptyset$.*

By definition, we can get the following lemma.

Lemma 2.10. *Let A be a nonempty convex subset of X. Let $F : A \times A \to 2^Y$ be a set-valued map, and let $C \subset Y$ be a closed convex pointed cone. Moreover, suppose that $F(x, y)$ is C-convex in its second variable. Then, for each $x \in A$, $F(x, A) + C$ is convex, where*

$$F(x, A) = \bigcup_{y \in A} F(x, y).$$

3 Scalarization

In this section, we extend the scalarization results of the global efficient and the Henig efficient solution sets in [14] to set-valued map.

Theorem 3.1. *Suppose C has a base B. Then*

(i) $\bigcup_{f \in C^\#} V_f(A, F) \subset V_G(A, F)$.

(ii) *if for each $x \in A$, the set $F(x, A) + C$ is a convex set, then*

$$V_G(A, F) = \bigcup_{f \in C^\#} V_f(A, F).$$

Proof. (i) Let $x \in \bigcup_{f \in C^\#} V_f(A, F)$, then exists some $f \in C^\#$, such that $x \in V_f(A, F)$. Hence

$$f(F(x, y)) \geq 0 \quad \text{for all } y \in A. \tag{3.1}$$

Let

$$D = \{u \in C, f(u) = 1\},$$

and

$$U = \left\{u \in Y, |f(u)| < 1/2\right\}.$$

We have

$$D+U\subset\left\{u\in Y,f(u)\geq {}^1\!/\!_2\right\},$$

$D+U$ is a convex set and $0\notin cl\,(D+U)$. Let

$$C_U(D)=cone\,(D+U)\,.$$

It is clear that $C_U(D)$ is a convex pointed cone, and

$$C\setminus\{0\}\subset \mathrm{int}C_U(D)\,. \tag{3.2}$$

By (3.1), we have

$$F(x,A)\cap(-C_U(D)\setminus\{0\})=\emptyset. \tag{3.3}$$

By definition, we know $x\in V_G(A,F)$. Thus

$$\bigcup\nolimits_{f\in C^{\#}}V_f(A,F)\subset V_G(A,F)\,.$$

(ii) By (i), we only need to prove that $V_G(A,F)\subset\bigcup_{f\in C^{\#}}V_f(A,F)$. Let $x\in V_G(A,F)$, then there exists a point convex cone $H\subset Y$, with $C\setminus\{0\}\subset\mathrm{int}H$, and

$$F(x,y)\cap((-H)\setminus\{0\})=\emptyset\quad\text{ for all } y\in A.$$

Hence, we have $F(x,A)\cap((-H)\setminus\{0\})=\emptyset$. Because H is a convex cone,

$$(F(x,A)+C)\cap(-\mathrm{int}H)=\emptyset.$$

By assumption, $F(x,A)+C$ is a convex set, by the separation theorem of convex sets, there exists some $f\in Y^*\setminus\{0\}$, such that

$$\inf\{f(F(x,y)+c):y\in A,c\in C\}\geq\sup\{f(-z):z\in\mathrm{int}H\}\,. \tag{3.4}$$

By (3.4), we obtain that $f\in H^*$ and

$$f(F(x,y))\geq 0\quad\text{for all } y\in A.$$

Because $C\setminus\{0\}\subset\mathrm{int}H$, we know that for each $x\in C\setminus\{0\}$, $f(x)>0$. Hence $f\in C^{\#}$. Therefore, $x\in\bigcup_{f\in C^{\#}}V_f(A,F)$. Hence, $V_G(A,F)\subset\bigcup_{f\in C^{\#}}V_f(A,F)$. Thus, we have

$$V_G(A,F)=\bigcup\nolimits_{f\in C^{\#}}V_f(A,F).$$

□

Theorem 3.2. *Suppose C has a base B. Then*

(i) $\bigcup_{f\in C^{\Delta}(B)}V_f(A,F)\subset V_H(A,F)$

(ii) *if for each $x\in A$, the set $F(x,A)+C$ is a convex set, then*

$$\bigcup\nolimits_{f\in C^{\Delta}(B)}V_f(A,F)=V_H(A,F)\,.$$

Proof. (i) Let $x \in \bigcup_{f\in C^{\Delta}(B)} V_f(A,F)$, then exists some $f \in C^{\Delta}(B)$, such that $x \in V_f(A,F)$. That is

$$f(F(x,y)) \geq 0 \quad \text{for all } y \in A.$$

Thus,

$$F(x,y) \cap \{u \in Y : f(u) < 0\} = \emptyset \quad \text{for all } y \in A. \tag{3.5}$$

Since $f \in C^{\Delta}(B)$, there exists $t > 0$ such that

$$f(b) \geq t \quad \text{for all } b \in B.$$

Set

$$V = \{u \in Y : f(u) < t\}.$$

Then V is a neighborhood of zero. Choose $0 < \varepsilon < \delta$, such that $\varepsilon U \subset V$. We have

$$(\varepsilon U - B) \subset \{u \in Y : f(u) < 0\}. \tag{3.6}$$

Pick $0 < \varepsilon' < \varepsilon$. By [3], $C_{\varepsilon'}(B)$ is a closed convex pointed cone and $C_{\varepsilon'}(B) \subset \mathit{cone}(B + \varepsilon U)$. Let $u \in -\mathrm{int}C_{\varepsilon'}(B)$. Then

$$u \in -\mathrm{int}C_{\varepsilon'}(B) \subset -C_{\varepsilon'}(B) \subset -\mathit{cone}(B + \varepsilon U).$$

We have

$$u = -\lambda(v + b) = \lambda(-v - b),$$

where $\lambda > 0, b \in B, v \in \varepsilon U$. It follows from (3.6) that $f(u) < 0$. We obtain that

$$-\mathrm{int}C_{\varepsilon'}(B) \subset \{u \in Y : f(u) < 0\}, \tag{3.7}$$

which combining with (3.5), we have

$$F(x,y) \cap (-\mathrm{int}C_{\varepsilon'}(B)) = \emptyset \quad \text{for all } y \in A.$$

Thus,

$$x \in V_H(A,F),$$

and hence

$$\bigcup_{f\in C^{\Delta}(B)} V_f(A,F) \subset V_H(A,F).$$

(ii) Let $x \in V_H(A,F)$. By the definition, there exists $0 < \varepsilon < \delta$ such that

$$F(x,y) \cap (-\mathrm{int}C_{\varepsilon}(B)) = \emptyset \quad \text{for all } y \in A.$$

It is clear that

$$(F(x,A) + C) \cap (-\mathrm{int}C_{\varepsilon}(B)) = \emptyset.$$

By assumption, $F(x,A) + C$ is a convex set. By the separation theorem of convex sets, there exists some $f \in Y^* \setminus \{0\}$, such that

$$\inf\{f(F(x,y) + c) : y \in A, c \in C\} \geq \sup\{f(-z) : z \in \mathrm{int}C_{\varepsilon}(B)\}. \tag{3.8}$$

From this, we get

$$f(F(x,y)) \geq 0 \quad \text{for all } y \in A,$$

and $f \in (C_\varepsilon(B))^*$. It follows that

$$f(\varepsilon U + B) \geq 0.$$

Since $f \neq 0$, there exists $u \in \varepsilon U$such that $f(u) < 0$. Thus

$$f(u+b) \geq 0 \quad \text{for all } b \in B.$$

This implies that $f \in C^\Delta(B)$. Hence, $V_H(A,F) \subset \bigcup_{f \in C^\Delta(B)} V_f(A,F)$. Thus,

$$V_H(A,F) = \bigcup_{f \in C^\Delta(B)} V_f(A,F).$$

□

4 Existence of Solutions

In this section, we present the existence of solutions of the global efficient and Henig efficient for set-valued vector equilibrium problems.

Theorem 4.1. *Let A be a nonempty compact convex subset of X, and let $C \subset Y$ be a closed convex pointed cone with a base. Let $F : A \times A \to 2^Y$ be a set-valued map with $F(x,x) \subset C$ for all $x \in A$. Suppose that $F(x,y)$ is lower semicontinuous in its first variable, and that $F(x,y)$ is C- convex in its second variable. Then, for any $f \in C^\#$, $V_f(A,F) \neq \emptyset$, therefore, $V_G(A,F) \neq \emptyset$.*

Proof. Let $f \in C^\#$. Define the set-valued map $G : A \to 2^A$ by

$$G(y) = \{x \in A : f(F(x,y)) \geq 0\} \text{ for all } y \in A.$$

By assumption, $y \in G(y)$, for all $y \in A$, so $G(y) \neq \emptyset$. We claim that G is a KKM map. Suppose to the contrary that there exists a finite subset $\{y_1, \cdots, y_n\}$ of A, and there exists $\bar{x} \in \mathrm{co}\{y_1, \cdots, y_n\}$ such that $\bar{x} \notin \bigcup\limits_{i=1}^{n} G(y_i)$. Then, $\bar{x} = \sum\limits_{i=1}^{n} t_i y_i$ for some $t_i \geq 0$, $1 \leq i \leq n$, with $\sum\limits_{i=1}^{n} t_i = 1$, and $\bar{x} \notin G(y_i)$ for all $i = 1, \cdots, n$. Then, there exist $z_i \in F(\bar{x}, y_i)$, such that

$$f(z_i) < 0, \text{for all } i = 1, \cdots, n \tag{4.1}$$

As $F(x,y)$ is C-convex in its second invariable, we can get that

$$t_1 F(\bar{x}, y_1) + t_2 F(\bar{x}, y_2) + \cdots + t_n F(\bar{x}, y_n) \subset F(\bar{x}, \bar{x}) + C. \tag{4.2}$$

By (4.2), we know that there exist $z \in F(\bar{x}, \bar{x})$, $c \in C$, such that

$$t_1 z_1 + t_2 z_2 + \cdots t_n z_n = z + c.$$

Hence, $f(z+c) = f(t_1 z_1 + t_2 z_2 + \cdots t_n z_n)$. By assumption, we have $f(z+c) \geq 0$. By (4.1), however, we have $f(t_1 z_1 + t_2 z_2 + \cdots t_n z_n) < 0$. This is a contradiction. Thus, G is a KKM map. Now, we show that for each $y \in A$, $G(y)$ is closed. For any sequence $\{x_n\} \subset G(y)$ and $x_n \to x_0$. Because A is a compact set, we have $x_0 \in A$. By assumption, $F(x,y)$ is lower semicontinuous in its first variable, then by [2], for each fixed $y \in A$, and for each $z_0 \in F(x_0, y)$, there exist $z_n \in F(x_n, y)$, such that $z_n \to z_0$. Because $\{x_n\} \subset G(y)$, we have

$$f(F(x_n, y)) \geq 0.$$

Thus $f(z_n) \geq 0$. By the continuity of f, and $z_n \to z_0$, we have $f(z_0) \geq 0$. By the arbitrary of $z_0 \in F(x_0, y)$, we have $f(F(x_0, y)) \geq 0$. That is, $x_0 \in G(y)$. Hence, $G(y)$ is closed, since A is compact, $G(y)$ is compact. By Lemma 2.9, we have $\bigcap_{y \in A} G(y) \neq \emptyset$. Thus, there exists $x \in \bigcap_{y \in A} G(y)$. This means that

$$f(F(x, y)) \geq 0 \text{ for all } y \in A.$$

Therefore, $x \in V_f(A, F)$. It follows from Theorem 3.1 that $V_f(A, F) \subset V_G(A, F)$, thus, $V_G(A, F) \neq \emptyset$. □

In the same way, we can get the existence theorem of Henig efficient solution for set-valued vector equilibrium problems.

Theorem 4.2. *Let A be a nonempty compact convex subset of X, and let $C \subset Y$ be a closed convex pointed cone with a base. Let $F : A \times A \to 2^Y$ be a set-valued map with $F(x, x) \subset C$ for all $x \in A$. Suppose that $F(x, y)$ is lower semicontinuous in its first variable, and that $F(x, y)$ is C- convex in its second variable. Then, for any $f \in C^{\Delta}(B)$, $V_f(A, F) \neq \emptyset$, therefore, $V_H(A, F) \neq \emptyset$.*

Now we give the existence theorem of global efficient solution for set-valued vector Hartman-Stampacchia variational inequality.

Similarly to the proof of Theorem 4.2 of [4], we can get the following theorems.

Theorem 4.3. *Let A be a nonempty compact convex subset of X, and $C \subset Y$ be a closed convex pointed cone with a base. Let $f \in C^{\#}$. Assume that $T : A \to 2^{L(X,Y)}$ is a v-hemicontinuous, f-pseudomonotone mapping. Moreover, assume that the set-valued map $F : A \times A \to 2^Y$ defined by $F(x, y) = (Tx, y - x)$ is C-convex in its second variable. Then $V_f(A, F) \neq \emptyset$, that is, there exists $x \in A$, for all $s \in Tx$,*

$$f((s, y - x)) \geq 0 \textit{ for all } y \in A$$

holds. Hence, $V_G(A, F) \neq \emptyset$.

In the same way, we can get the existence theorem of Henig efficient solution for set-valued vector Hartman-Stampacchia variational inequality.

Theorem 4.4. *Let A be a nonempty compact convex subset of X, and $C \subset Y$ be a closed convex pointed cone with a base. Let $f \in C^{\Delta}(B)$. Assume that $T : A \to 2^{L(X,Y)}$ is a v-hemicontinuous, f-pseudomonotone mapping. Moreover, assume that the set-valued map $F : A \times A \to 2^Y$ defined by $F(x, y) = (Tx, y - x)$, is C-convex in its second variable. Then $V_f(A, F) \neq \emptyset$, that is, there exists $x \in A$, for all $s \in Tx$,*

$$f((s, y - x)) \geq 0 \textit{ for all } y \in A$$

holds. Hence, $V_H(A, F) \neq \emptyset$.

$\boxed{5}$ Connectedness of the Solutions Set

In this section, we present the connectedness of the global efficient and the Henig efficient solution sets for set-valued vector equilibrium problems.

Theorem 5.1. *Let A be a nonempty compact convex subset of X, $C \subset Y$ be a closed convex pointed cone with a base, and let $F : A \times A \to 2^Y$ be a set-valued map. Assume that the following conditions are satisfied:*

(i) *$F(x, y)$ is lower semi-continuous in its first variable.*

(ii) *$F(x, y)$ is C-concave in its first variable and C-convex in its second variable.*

(iii) *$F(x, x) \subset C$, for all $x \in A$.*

(iv) *$\{F(x, y) : x, y \in A\}$ is a bounded subset in Y.*

Then $V_G(A, F)$ is a nonempty connected set.

Proof. We define the set-valued map $H : C^{\#} \to 2^A$ by

$$H(f) = V_f(A, F), f \in C^{\#}.$$

By Theorem 4.1, for each $f \in C^{\#}$, we have $H(f) \neq \emptyset$. So $V_G(A, F)$ is a nonempty set. It is clear that $C^{\#}$ is convex, so it is a connected set. Now we prove that, for each $f \in C^{\#}$, $H(f)$ is a connected set. Let $x_1, x_2 \in H(f)$, for $i = 1, 2$, we have

$$f(F(x_i, y)) \geq 0 \quad \text{for all } y \in A. \tag{5.1}$$

Because $F(x, y)$ is C-concave in its first variable, for each fixed $y \in A$, and for every $x_1, x_2 \in A$, $t \in [0, 1]$, we have

$$F(tx_1 + (1 - t) x_2, y) \subset tF(x_1, y) + (1 - t) F(x_2, y) + C.$$

Hence, for each $y \in A$, $z \in F(tx_1 + (1 - t) x_2, y)$, there exist $z_1 \in F(x_1, y)$, $z_2 \in F(x_2, y)$, $c \in C$, such that $z = tz_1 + (1 - t)z_2 + c$. As $f \in C^{\#}$ and by (5.1), we have

$$f(z) = tf(z_1) + (1 - t)f(z_2) + f(c) \geq 0$$

Thus,

$$f(F(tx_1 + (1 - t)x_2,\ y)) \geq 0 \quad \text{for all } y \in A.$$

That is $tx_1 + (1 - t) x_2 \in H(f)$. So $H(f)$ is a convex set, therefore it is a connected set.

Now we show that H is upper semicontinuous on $C^{\#}$. Since A is a nonempty compact set, by [2], we just need to prove that H is a closed map. Let the sequence $\{(f_n, x_n)\} \subset \text{Graph}(H)$, and $(f_n, x_n) \to (f_0, x_0)$, where $\{f_n\}$ converge to f_0 with respect to the norm topology. As $(f_n, x_n) \in \text{Graph}(H)$, we have

$$x_n \in H(f_n) = V_{f_n}(A, F).$$

That is, $f_n(F(x_n, y)) \geq 0$, for all $y \in A$. As $x_n \to x_0$ and A is compact, then $x_0 \in A$. Since $F(x, y)$ is lower semi-continuous in its first variable, for each fixed $y \in A$, and each $z_0 \in F(x_0, y)$, there exist $z_n \in F(x_n, y)$, such that $z_n \to z_0$. From $f_n(F(x_n, y)) \geq 0$, we have

$$f_n(z_n) \geq 0. \tag{5.2}$$

By the continuity of f_0 and $z_n \to z_0$, we have

$$f_0(z_n) \to f_0(z_0). \tag{5.3}$$

Let $D = \{F(x,y) : x, y \in A\}$. By assumption, D is a bounded set in Y, then, there exists some $M > 0$, such that for each $z \in D$, we have $\|z\| \leq M$. Because $f_n - f_0 \to 0$ with respect to norm topology, for any $\varepsilon > 0$, there exists $n_0 \in N$, when $n \geq n_0$, we have $\|f_n - f_0\| < \varepsilon$. Therefore, there exists $n_0 \in N$, when $n \geq n_0$, we have

$$|f_n(z_n) - f_0(z_n)| = |(f_n - f_0)(z_n)| \leq \|f_n - f_0\| \|z_n\| \leq M\varepsilon.$$

Hence,

$$\lim_{n\to\infty} (f_n(z_n) - f_0(z_n)) = 0. \tag{5.4}$$

Consequently, by (5.3), (5.4), we have

$$\begin{aligned} \lim_{n\to\infty} f_n(z_n) &= \lim_{n\to\infty} (f_n(z_n) - f_0(z_n) + f_0(z_n)) \\ &= \lim_{n\to\infty} (f_n(z_n) - f_0(z_n)) + \lim_{n\to\infty} (f_0(z_n)) = f_0(z_0). \end{aligned}$$

By (5.2), we have $f_0(z_0) \geq 0$. So for any $y \in A$ and for each $z_0 \in F(x_0, y)$, we have $f_0(z_0) \geq 0$. Hence

$$f_0(F(x_0, y)) \geq 0 \quad \text{for all } y \in A.$$

This means that

$$x_0 \in V_{f_0}(A, F) = H(f_0).$$

Hence, the graph of H is closed. Therefore, H is a closed map. By [2], H is upper semi-continuous on $C^{\#}$. Because $F(x,y)$ is C-convex in its second variable, by Lemma 2.10, for each $x \in A$, $F(x, A) + C$ is convex. It follows from Theorem 3.1 that

$$V_G(A, F) = \bigcup\nolimits_{f \in C^{\#}} V_f(A, F).$$

Thus, by the Theorem 3.1 in [23], $V_G(A, F)$ is a connected set. $\square$

In the same way, we can show the connectedness theorem of Henig efficient solutions set for set-valued vector equilibrium problems.

Theorem 5.2. *Let A be a nonempty compact convex subset of X, $C \subset Y$ be a closed convex pointed cone with a base, and let $F : A \times A \to 2^Y$ be a set-valued map. Assume that the following conditions are satisfied:*

(i) *$F(x,y)$ is lower semi-continuous in its first variable.*

(ii) *$F(x,y)$ is C-concave in its first variable and C-convex in its second variable.*

(iii) *$F(x,x) \subset C$, for all $x \in A$.*

(iv) *$\{F(x,y) : x, y \in A\}$ is a bounded subset in Y.*

Then $V_H(A, F)$ is a nonempty connected set.

Theorem 5.3. *Let A be a nonempty compact convex subset of X, and $C \subset Y$ be a closed convex pointed cone with a base. Assume that for each $f \in C^{\#}$, $T : A \to 2^{L(X,Y)}$ is a v-hemicontinuous, f-pseudomonotone mapping. Moreover, assume that the set-valued map $F : A \times A \to 2^{Y}$ defined by $F(x,y) = (Tx, y-x)$ is C-convex in its second variable, and the set $\{F(x,y) : x, y \in A\}$ is a bounded set in Y. Then, $V_G(A,F)$ is a nonempty connected set.*

Proof. We define the set-valued map $H : C^{\#} \to 2^{A}$ by

$$H(f) = V_f(A,F), f \in C^{\#}.$$

By Theorem 4.3, for each $f \in C^{\#}$, we have $H(f) = V_f(A,F) \neq \emptyset$, hence $V_G(A,F) \neq \emptyset$. Clearly, $C^{\#}$ is a convex set, hence it is a connected set. Define the set-valued maps E, $G : A \to 2^{A}$ by

$$E(y) = \{x \in A | \forall s \in Tx, f((s, y-x)) \geq 0\}, y \in A.$$

$$G(y) = \{x \in A | \forall s \in Ty, f((s, y-x)) \geq 0\}, y \in A.$$

Now, we prove that for each $f \in C^{\#}$, $H(f)$ is a connected set. Let $x_1, x_2 \in H(f) = V_f(A,F)$, then $x_1, x_2 \in \bigcap\limits_{y \in A} E(y)$. By assumption, we can see that $\bigcap\limits_{y \in A} G(y) = \bigcap\limits_{y \in A} E(y)$, so $x_1, x_2 \in \bigcap\limits_{y \in A} G(y)$. Hence, for $i = 1, 2$, and for each $y \in A$, $s \in Ty$, we have

$$f((s, y - x_i)) \geq 0.$$

Then, for each $y \in A$, $s \in Ty$, and $t \in [0,1]$, we have

$$f((s, y - (tx_1 + (1-t)x_2))) \geq 0.$$

Hence, $tx_1 + (1-t)x_2 \in \bigcap\limits_{y \in A} G(y) = \bigcap\limits_{y \in A} E(y)$. Thus, $tx_1 + (1-t)x_2 \in H(f)$. Consequently, for each $f \in C^{\#}$, $H(f)$ is a convex set. Therefore it is a connected set. The following is to prove that H is upper semicontinuous on $f \in C^{\#}$. Since A is a nonempty compact set, we only need to show that H is a closed map. Let sequence $\{(f_n, x_n)\} \subset \mathrm{Graph}(H)$, and $(f_n, x_n) \to (f_0, x_0)$, where $\{f_n\}$ converges to $f_0 \in C^{\#}$ with respect to the norm topology of Y^*. As $(f_n, x_n) \in Graph(H)$, we have

$$x_n \in H(f_n) = V_{f_n}(A,F).$$

Then, for each $s' \in Tx_n$, we have that

$$f_n((s', y - x_n)) \geq 0, \quad \text{for all } y \in A.$$

By assumption, for each n, $T : A \to 2^{L(X,Y)}$is f_n-pseudomonotone, hence, for each $y \in A$, for above x_n, and for each $s \in Ty$, we have

$$f_n((s, y - x_n)) \geq 0, \quad \text{for all } y \in A. \tag{5.5}$$

As $x_n \to x_0$, we have $(s, y - x_n) \to (s, y - x_0)$, and $f_0((s, y - x_n)) \to f_0((s, y - x_0))$. As $x_n \to x_0$, and A is compact, we have $x_0 \in A$. Let $D = \{F(x,y) : x, y \in A\}$. By assumption, D is a bounded set in Y. Then, there exists $M > 0$, such that for each $z \in D$, we have $\|z\| \leq M$. Because $f_n - f_0 \to 0$ with respect to the norm topology, for any $\varepsilon > 0$, there exists $n_0 \in N$, when $n \geq n_0$, we have $\|f_n - f_0\| < \varepsilon$. Therefore, there exists $n_0 \in N$, when $n \geq n_0$, we have

$$|f_n((s, y - x_n)) - f_0((s, y - x_n))| = |(f_n - f_0)((s, y - x_n))| \leq M\varepsilon.$$

Hence,

$$\lim_{n\to\infty} (f_n((s, y - x_n)) - f_0((s, y - x_n))) = 0.$$

Then,

$$\begin{aligned} \lim_{n\to\infty} f_n((s, y - x_n)) &= \lim_{n\to\infty} (f_n((s, y - x_n)) - f_0((s, y - x_n)) + f_0((s, y - x_n))) \\ &= f_0((s, y - x_0)). \qquad (5.6) \end{aligned}$$

Then by (5.5), (5.6), for each $y \in A$, and for each $s \in Ty$, we have $f_0((s, y - x_0)) \geq 0$. Since T is f_0-pseudomonotone, for each $s^* \in Tx_0$, we have $f_0((s^*, y - x_0)) \geq 0$. Hence, $x_0 \in H(f_0) = V_{f_0}(A, F)$. Therefore, the graph of H is closed, and hence H is a closed map. By [2], we know that, H is upper semicontinuous on $C^{\#}$.

Because $F(x, y)$ is C-convex in its second variable, for each $x \in A$, $F(x, A) + C$ is convex. It follows from Theorem 3.1 that

$$V_G(A, F) = \bigcup_{f\in C^{\#}} V_f(A, F).$$

Then, by Theorem 3.1 in [23], we know that $V_G(A, F)$ is a connected set. □

In the same way, we can get the connectedness theorem of Henig efficient solutions set for set-valued vector Hartman-Stampacchia variational inequality.

Theorem 5.4. *Let A be a nonempty compact convex subset of X, and $C \subset Y$ be a closed convex pointed cone with a base. Assume that for each $f \in C^{\Delta}(B)$, $T : A \to 2^{L(X,Y)}$ is a v-hemicontinuous, f-pseudomonotone mapping. Moreover, assume that the set-valued map $F : A \times A \to 2^Y$ defined by $F(x, y) = (Tx, y - x)$ is C-convex in its second variable, and the set $\{F(x, y) : x, y \in A\}$ is a bounded set in Y. Then, $V_H(A, F)$ is a nonempty connected set.*

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