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AN EXAMPLE TO DEMONSTRATE THE IMPORTANCE OF USING ELLIPSOIDAL NORM IN LATTICE BASIS REDUCTION FOR BRANCHING ON HYPERPLANE ALGORITHMS*

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Abstract: The use of lattice basis reduction has been proposed in reformulating instances of hard integer programs. We construct integer programming instances showing that branching on hyperplane algorithms may benefit from using the geometrical information of the feasible set using an ellipsoidal approximation of the linear relaxation while using lattice basis reduction methods. Feasible as well as infeasible instances of these problems in four dimensions are given.

Key words: *integer programming, lattice basis reduction,*

Mathematics Subject Classification: *90C10, 11Y50*

1 Introduction

Lenstra [5] established a seminal result showing that linear integer programs can be solved in time that is polynomial in fixed dimension. His algorithm branches on general hyperplanes instead of elementary hyperplanes defined by single variables. Central to Lenstra's branching-on-hyperplane algorithm is the use of a reduced lattice basis computation at every node of the branching-on-hyperplane tree. Aardal *et al.* [1, 2, 3] and Aardal and Lenstra [4] have proposed and computationally studied reformulation techniques for pure integer problems using a kernel lattice that is Lenstra, Lenstra, Lovász [7] reduced (LLL-reduced) in the Euclidean norm. Although their technique is a heuristic, computationally they showed that branching on single variables in the reformulated problem requires significantly fewer branches than those required to solve the original problem using CPLEX version 6.5.3 for difficult knapsack and market share problems known in the literature. Their reformulation generates a full dimensional problem using an LLL reduced kernel lattice and a short solution satisfying the equality constraints. The LLL basis reduction method is used to obtain such a basis and short solution. The LLL basis reduction is performed using the Euclidean norm to measure the length of each basis vector.

Mehrotra and Li [9] recently studied the problem of generating branching hyperplanes, and showed that the branching hyperplanes can be generated without a full dimensional

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reformulation. They also suggest a reformulation technique, which performs the lattice basis reduction using an ellipsoidal norm at the root node. The matrix used to define the ellipsoidal norm is obtained by taking an ellipsoidal approximation of the feasible region. We present a generic four dimensional integer program to show the difference between the method in Mehrotra and Li [9] and the reformulation method used in [1, 2, 3, 4]. The constructed examples show that when LLL-reduced lattice in the Euclidean-norm is used to find the branching direction, the longest lattice vector may not give the smallest width branching Hyperplane. This is because the Euclidean-norm LLL-reduction ignores the polyhedral geometry of linear relaxation by approximating it with a sphere. The situation may be improved by incorporating the geometric information by using an ellipsoidal approximation of the feasible region and finding an LLL-reduced basis under a more general ellipsoidal-norm. We give examples illustrating that the reduced lattice basis which incorporates the information on the linear relaxation region at the root node via its ellipsoidal approximation may significantly reduce the required number of branching nodes, hence has potential practical value. In particular, this example shows that the number of branches grow with the problem data magnitude (see dependence on k_2 in (2.7) below) if Euclidean norm reduced lattice basis is used. The number of branches is constant for the example if the ellipsoidal norm is used. Remarkably, despite having only four variables, a flagship mixed integer programming solver package (CPLEX) which branches on single variable reached an incorrect conclusion when a slightly harder numerical example was constructed based on the general technique given in this paper.

This paper is organized as follows. The notation, terminology and definitions used in this paper are given in the Appendix. The next section gives our general technique for constructing a four dimensional example with equality constraints. This section also gives reformulations of the basic model using an LLL-reduced basis when the basis reduction is performed using the Euclidean norm, and an ellipsoidal norm. It also describes an interior point approach [10] for finding an ellipsoidal rounding of the linear relaxation region to define the ellipsoidal norm for the LLL-reduced basis. Section 3 gives numerical examples based on the general technique. Section 3.1 compares the number of branches required using the Euclidean and Ellipsoidal-norm reformulations, as well as the number of branches required by CPLEX. The example in Section 3.2 demonstrates that for more 'difficult' four variable problems CPLEX-MIP solver fails. Section 4 gives infeasible versions of problems considered in Sections 2–3. Some concluding remarks are made at the end.

2 A Four Dimensional Feasible Integer Program

We consider the following integer program:

$$\begin{aligned} \min \quad & e^T x \\ s.t. \quad & Ax = b \\ & x \in \mathbb{Z}^4_+, \end{aligned} \tag{2.1}$$

where $e \in \mathbb{R}^4$ is a vector of all ones, and A and b are given as

$$A = \begin{bmatrix} 1 & 0 & -k_1(k_2k_3-1)+k_1k_2 & -k_1(k_2k_3-1) \\ 0 & 1 & (k_1k_2-1)(k_3-1)-k_1 & k_3(k_1k_2-1)-k_1 \end{bmatrix} \tag{2.2}$$

and

$$b = \begin{bmatrix} k_1k_2k_3(-2k_2k_3+k_2+2)+k_1-1 \\ k_2k_3((k_1k_2-1)(2k_3-1)-2k_1)-1 \end{bmatrix}. \tag{2.3}$$

Problem (2.1) has a unique integral feasible solution $[k_1-1, 0, 2k_2k_3-1, k_2+1]^T$, where we assume that $k_1, k_2, k_3 \in \mathbb{Z}$ satisfy the following conditions

$$k_1 > 2,\ k_2 >> k_1,\ gcd(k_1, k_2) = 1, \text{ and } k_3 >> k_1k_2, \tag{2.4}$$

where $gcd(k_1, k_2)$ is the greatest common divisor of k_1 and k_2. A particular short solution to $Ax = b$ is given by $v = [k_1-1, -1, k_2k_3, k_2k_3]^T$.

Using Hermite Normal Form (HNF) calculation a Kernel lattice basis of A is

$$Z = \begin{bmatrix} -k_1(k_2k_3-1)+k_1k_2 & -k_1(k_2k_3-1) \\ (k_1k_2-1)(k_3-1)-k_1 & k_3(k_1k_2-1)-k_1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and from the LLL basis reduction algorithm [7], an LLL-reduced kernel lattice basis of A is given by

$$Z_{LLL} = \begin{bmatrix} k_1k_2 & -k_1 \\ -(k_1k_2-1) & k_1 \\ -1 & k_3 \\ 1 & -(k_3-1) \end{bmatrix}.$$

It is easy to verify that under the conditions in (2.4) no further reduction of the second column of Z_{LLL} by adding an integer multiple of the first column is possible.

The technique in Aardal *et al.* [1, 2, 3] reformulates (2.1) using $x = Z_{LLL}y + v$ as an equivalent integer program:

$$\begin{array}{lll} \min & y_1 + y_2 & \\ s.t. & k_1k_2y_1 - k_1y_2 & \geq -(k_1-1) \\ & -(k_1k_2-1)y_1 + k_1y_2 & \geq 1 \\ & -y_1 + k_3y_2 & \geq -k_2k_3 \\ & y_1 - (k_3-1)y_2 & \geq -k_2k_3 \\ & y_1, y_2 \in \mathbb{Z}. & \end{array} \tag{2.5}$$

The unique solution $[k_1-1, 0, 2k_2k_3-1, k_2+1]^T$ of (2.1) is transformed to $[1, k_2]^T$ for (2.5). It is an exercise to verify that the relaxed optimal solution of (2.5) is $\left[2-k_1, k_2(2-k_1)+\frac{k_1-1}{k_1}\right]^T$ (for $k_1 = 3$), or $\left[-\frac{k_3(k_1k_2+k_1-1)}{k_1k_2k_3-k_1}, -k_2-\frac{(k_1k_2+k_1-1)}{k_1k_2k_3-k_1)}\right]^T$ (if $k_1 > 3$).

Denote the relaxed feasible set of Problem (2.5) by

$$\begin{aligned} \mathcal{Q} = \{y \in \mathbb{R}^2 \mid \mathrm{k_1k_2y_1 - k_1y_2 \geq -(k_1-1), -(k_1k_2-1)y_1 + k_1y_2 \geq 1}, \\ -y_1 + k_3y_2 \geq -k_2k_3,\ y_1 - (k_3-1)y_2 \geq -k_2k_3\}. \end{aligned} \tag{2.6}$$

We calculate $W_I(e_1, \mathcal{Q})$ and $W_I(e_2, \mathcal{Q})$, which are the integer widths of $\mathcal{Q}$ along e_1, e_2. Note that $W_I(u, \mathcal{Q}) = 0$ means that $\mathcal{Q}$ does not contain any integer solution.

Using the conditions in (2.4), it is an exercise to show that

$$
\begin{aligned}
W_I(e_2, \mathcal{Q}) &= \left\lfloor \frac{k_2k_3(k_1k_2-1)-1}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor \\
&\quad - \left\lceil \max\left\{ k_2(2-k_1) + \frac{k_1-1}{k_1}, -k_2 - \frac{(k_1k_2+k_1-1)}{k_1k_2k_3-k_1)} \right\} \right\rceil + 1 \\
&= \left\lfloor k_2 + \frac{k_1k_2+k_2(k_1k_2-1)-1}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor + \min\{k_2(k_1-2)-1, k_2\} + 1 \\
&= k_2 + \min\{k_2(k_1-2)-1, k_2\} + 1, \\
W_I(e_1, \mathcal{Q}) &= \left\lfloor \frac{k_1k_2k_3-(k_3-1)}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor \\
&\quad - \left\lceil \max\left\{ 2-k_1, -\frac{k_3(k_1k_2+k_1-1)}{k_1k_2k_3-k_1} \right\} \right\rceil + 1, \\
&= \left\lfloor 1 + \frac{k_1(k_2+1)}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor \\
&\quad - \left\lceil \max\left\{ 2-k_1, -1-\frac{k_3(k_1-1)+k_1}{k_1(k_2k_3-1)} \right\} \right\rceil + 1 \\
&= 1 + \min\{k_1-2, 1\} + 1 = 3.
\end{aligned} \tag{2.7}
$$

Therefore, at the root node branching on $e_2 = (0,1)$ will generate $2k_2$ (for $k_1 = 3$) or $2k_2+1$ (if $k_1 > 3$) subproblems to declare optimality. Branching on $e_1 = (1,0)$ first generates 3 subproblems. Aardal *et al.* [1, 2, 3] propose to branch on e_2 first because the second column of Z_{LLL} has a larger norm.

Now following the method in Mehrotra and Li [9] we compute an approximate analytic center of the relaxed feasible set of Problem (2.1), denoted by $\mathcal{P} = \{x \in \mathbb{R}^4_+ | \mathrm{Ax} = \mathrm{b}\}$, where the coefficient matrix A and b are given in (2.2) and (2.3), respectively. The analytic center finding problem [10] is formulated as

$$
\begin{aligned}
\min \quad & -\sum_{i=1}^{4} \ln(x_i) \\
s.t. \quad & Ax = b,\ x \in \mathbb{R}^4_+.
\end{aligned} \tag{2.8}
$$

However here for simplicity we construct an approximate analytic center w of $\mathcal{P}$ using the interior-point $w_{\mathcal{Q}} = [\frac{1}{2}, \frac{k_2+1}{2}]^T$ of $\mathcal{Q}$

$$
w = Z_{LLL} w_{\mathcal{Q}} + v = \left[\frac{k_1-2}{2}, \frac{k_1-1}{2}, \frac{3k_2k_3+k_3-1}{2}, \frac{k_2k_3-k_3+k_2+2}{2} \right]^T. \tag{2.9}
$$

An ellipsoid which approximates $\mathcal{P}$ is given by $\mathcal{E}(w, Q) := \{x \in \mathbb{R}^4 | (\mathrm{x}-\mathrm{w})^\mathrm{T} \mathrm{Q} (\mathrm{x}-\mathrm{w}) \le 1\}$, where $Q^{1/2} = \mathrm{diag}\left\{ \frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}, \frac{1}{w_4} \right\}$ is a diagonal matrix. The integral kernel basis Z_{LLL} scaled by $Q^{1/2}$ is given as

$$
Q^{1/2} Z_{LLL} = \begin{bmatrix} \frac{2k_1k_2}{k_1-2} & \frac{-2k_1}{k_1-2} \\ \frac{-2(k_1k_2-1)}{k_1-1} & \frac{2k_1}{k_1-1} \\ \frac{-2}{3k_2k_3+k_3-1} & \frac{2k_3}{3k_2k_3+k_3-1} \\ \frac{2}{k_2k_3-k_3+k_2+2} & \frac{-2(k_3-1)}{k_2k_3-k_3+k_2+2} \end{bmatrix}.
$$

The LLL-reduced basis of $Q^{1/2}Z_{LLL}$ is given as

$$Q^{1/2}Z_{LLL}V = \begin{bmatrix} 0 & \frac{-2k_1}{k_1-2} \\ \frac{2}{k_1-1} & \frac{2k_1}{k_1-1} \\ \frac{-2+2k_2k_3}{3k_2k_3+k_3-1} & \frac{2k_3}{3k_2k_3+k_3-1} \\ \frac{2-2k_2(k_3-1)}{k_2k_3-k_3+k_2+2} & \frac{-2(k_3-1)}{k_2k_3-k_3+k_2+2} \end{bmatrix},$$

where

$$V = \begin{bmatrix} 1 & 0 \\ k_2 & 1 \end{bmatrix}.$$

The corresponding LLL-reduced integer kernel basis is given as

$$Z_{Q,LLL} = Z_{LLL}V = \begin{bmatrix} 0 & -k_1 \\ 1 & k_1 \\ k_2k_3 - 1 & k_3 \\ 1 - k_2(k_3 - 1) & -(k_3 - 1) \end{bmatrix}.$$

Using $x = Z_{Q,LLL}y + v$, the transformed problem is given as

$$\begin{array}{lcl} \min & (1 - k_2)y_1 + y_2 & \\ s.t. & -k_1y_2 & \geq -(k_1 - 1) \\ & y_1 + k_1y_2 & \geq 1 \\ & (k_2k_3 - 1)y_1 + k_3y_2 & \geq -k_2k_3 \\ & (1 - k_2(k_3 - 1))y_1 - (k_3 - 1)y_2 & \geq -k_2k_3 \\ & y_1, y_2 \in \mathbb{Z}. & \end{array} \tag{2.10}$$

Denote its relaxed feasible set by

$$\begin{aligned} \mathcal{R} := \{y \in \mathbb{R}^2 | -\mathrm{k_1y_2} \geq -(\mathrm{k_1} - 1),\ \mathrm{y_1 + k_1y_2} \geq 1,\ (\mathrm{k_2k_3} - 1)\mathrm{y_1} + \mathrm{k_3y_2} \geq -\mathrm{k_2k_3}, \\ (1 - k_2(k_3 - 1))y_1 - (k_3 - 1)y_2 \geq -k_2k_3\}. \end{aligned} \tag{2.11}$$

The integer widths of $\mathcal{R}$ along e_1 and e_2 are:

$$\begin{aligned} W_I(e_2, \mathcal{R}) &= \left\lfloor \frac{k_1 - 1}{k_1} \right\rfloor - \left\lceil \frac{1 + k_2}{k_1(1 - k_2k_3 + k_2) + k_3 - 1} \right\rceil + 1 = 1, \\ W_I(e_1, \mathcal{R}) &= \left\lfloor 1 + \frac{k_1(1 + k_2)}{k_1(k_2k_3 - k_2 - 1) - k_3 + 1} \right\rfloor - \left\lceil \max\{2 - k_1, -1 - \frac{k_1 + k_3(k_1 - 1)}{k_1(k_2k_3 - 1)}\} \right\rceil \\ &\quad +1 \\ &= 3. \end{aligned}$$

The integral optimal solution $[1, 0]^T$ of (2.10) is readily available by branching on e_2 first. This shows that Mehrotra-Li method generates a better branching scheme by considering the geometry of the relaxed feasible set. In the following two sections we give particular instances of (2.1) to numerically illustrate the value of above constructed example.

3 Feasible Numerical Examples

3.1 A Numerical Example Comparing Euclidean and Ellipsoidal Norm Reduced Basis Reformulations

A particular instance of (2.1) is given by letting $k_1 = 3$, $k_2 = 11$, $k_3 = 180$:

$$\begin{array}{lll} \min & \sum_{i=1}^{4} x_i & \\ s.t. & x_1 - 5904x_3 - 5937x_4 & = -23445178 \\ & x_2 + 5725x_3 + 5757x_4 & = 22734359 \\ & x_1, x_2, x_3, x_4 \in \mathbb{Z}_+. & \end{array} \tag{3.1}$$

This problem has a unique integral solution $[2, 0, 3959, 12]^T$. A short particular solution to the equality constraint is $v = [2, -1, 1980, 1980]^T$.

From the HNF computation of the coefficient matrix in (3.1), a kernel basis and its corresponding adjoint lattice basis are given by $Z = \begin{bmatrix} 5904 & 5937 \\ -5725 & -5757 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Z^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Reformulation Using Euclidean Norm

The LLL-reduced basis obtained from Z is $Z_{LLL} = \begin{bmatrix} 33 & -3 \\ -32 & 3 \\ -1 & 180 \\ 1 & -179 \end{bmatrix}$, which is obtained by post-multiplying Z with $V = \begin{bmatrix} -1 & 180 \\ 1 & -179 \end{bmatrix}$. Using the transformation $x = Z_{LLL}y + v$, (3.1) is reformulated as:

$$\begin{array}{lll} \min & y_1 + y_2 & \\ s.t. & 33y_1 - 3y_2 & \geq -2 \\ & -32y_1 + 3y_2 & \geq 1 \\ & -y_1 + 180y_2 & \geq -1980 \\ & y_1 - 179y_2 & \geq -1980 \\ & y_1, y_2 \in \mathbb{Z}. & \end{array} \tag{3.2}$$

The optimal solution of linear programming relaxation of (3.2) is $[-1, -10.3333]^T$. Denote by $\mathcal{P}$, $\mathcal{Q}$ the relaxed feasible sets of (3.1) and (3.2), respectively. Then $W_I(e_2, \mathcal{Q}) = \lfloor 11.067 \rfloor - \lceil -10.333 \rceil + 1 = 22$, and $W_I(e_1, \mathcal{Q}) = \lfloor 1.0063 \rfloor - \lceil -1 \rceil + 1 = 3$. This shows that it requires at least 22 subproblems if branching on e_2 first. We also used CPLEX 9.0 MIP solver with default options for comparison. It correctly solved (3.1) using 19 nodes.

Reformulation Using Ellipsoidal Norm

We now illustrate the performance when ellipsoidal-norm is used in this example. We find an analytic center by solving the optimization problem:

$$\begin{array}{lll} \min & -\sum_{i=1}^{4} \ln(x_i) & \\ s.t. & x_1 - 5904x_3 - 5937x_4 & = -23445178 \\ & x_2 + 5725x_3 + 5757x_4 & = 22734359 \\ & x_1, x_2, x_3, x_4 \in \mathbb{R}_+. & \end{array} \tag{3.3}$$

A near optimal solution of (3.3) is given by $w = [0.7498, 0.7271, 2998.742, 966.92]^T$ (compare this with the approximate center $[0.5, 1, 3059.5, 906.5]^T$ given by (2.9)).

An ellipsoid that approximates $\mathcal{P}$ is given by $\mathcal{E}(w, \mathcal{P}) := \{x \in \mathbb{R}^4 | (\mathrm{x}-\mathrm{w})^{\mathrm{T}}\mathrm{Q}(\mathrm{x}-\mathrm{w}) \le 1\}$, where $Q^{1/2} = \mathrm{diag}\{1.334, 1.375, 0.000333, 0.00103\}$. Then we have $Q^{1/2}Z_{LLL} = \begin{bmatrix} 44.012 & -4.001 \\ -44.012 & 4.126 \\ -0.000333 & 0.060 \\ 0.00103 & -0.185 \end{bmatrix}$. The LLL-reduced $Q^{1/2}Z_{LLL}$ basis is given by $Q^{1/2}Z_{LLL} = \begin{bmatrix} 0 & -4.001 \\ 1.375 & 4.126 \\ 0.66 & 0.060 \\ -2.035 & -0.185 \end{bmatrix}$, obtained by using the unimodular transformation $V = \begin{bmatrix} 1 & 0 \\ 11 & 1 \end{bmatrix}$. The corresponding reduced integral kernel basis, denoted by $Z_{Q,LLL}$ is given by $Z_{Q,LLL} = \begin{bmatrix} 0 & -3 \\ 1 & 3 \\ 1979 & 180 \\ -1968 & -179 \end{bmatrix}$.

Therefore, we can reformulate (3.2) as

$$\begin{array}{lll} \min & -10y_1 + y_2 & \\ s.t. & -3y_2 & \ge -2 \\ & y_1 + 3y_2 & \ge 1 \\ & 1979y_1 + 180y_2 & \ge -1980 \\ & -1968y_1 - 179y_2 & \ge -1980 \\ & y_1, y_2 \in \mathbb{Z}, & \end{array} \tag{3.4}$$

Now branching on e_2 and e_1 gives a better branching scheme. In fact, the integral optimal solution is $[1, 0]^T$ and the relaxed optimal solution is $[1.00490, -0.000163]^T$. Denote by $\mathcal{R}$ the feasible set of (3.4). Then $W_I(e_2, \mathcal{R}) = \lfloor \frac{2}{3} \rfloor - \lceil -0.0021 \rceil + 1 = 1$, and $W_I(e_1, \mathcal{R}) = \lfloor 1.0063 \rfloor - \lceil -1 \rceil + 1 = 3$. Only one subproblem is needed when branching on e_2 first.

We can also generate the branching directions in the original space by using the columns of an adjoint lattice basis introduced by Mehrotra and Li [9]. Given an integral kernel lattice basis Z of A, its adjoint basis is given by an integral matrix Z^* satisfying $Z^T Z^* = I$. In the above numerical example, the adjoint lattice basis is given by $Z^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

According to Proposition A.1 (Proposition 4.1 in [9]) we run the LLL basis reduction algorithm on the adjoint lattice basis Z^* with respect to the projected ellipsoidal norm

$$\begin{aligned} P &= Q^{-1/2} P_{AQ^{-1/2}} Q^{-1/2} = Q^{-1} - Q^{-1}A^T(AQ^{-1}A^T)^{-1}AQ^{-1} \\ &= Z(Z^TQZ)^{-1}Z^T. \end{aligned} \tag{3.5}$$

The LLL-reduced adjoint lattice basis, denoted by $Z^*_{Q,LLL}$, is given by

$$Z^*_{Q,LLL} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1968 & -1789 \\ -1979 & -1799 \end{bmatrix}. \tag{3.6}$$

Therefore, the first branching direction is $u = [0, 0, -1968, -1979]^T$. Only one subproblem is generated by adding constraint

$$1968x_3 + 1979x_4 = 7815060.$$

The integral optimal solution $[2, 0, 3959, 12]^T$ is obtained readily by adding another constraint

$$1789x_3 + 1799x_4 = 7104239.$$

3.2 A Numerical Example Showing CPLEX Failure

A harder example is given by setting $k_1 = 11$, $k_2 = 41$, $k_3 = 600$.

$$\begin{array}{lll} \min & \sum_{i=1}^4 x_i & \\ s.t. & x_1 - 270138x_3 - 270589x_4 & = -13301884190 \\ & x_2 + 269539x_3 + 269989x_4 & = 13272388799 \\ & x_1, x_2, x_3, x_4 \in \mathbb{Z}_+. & \end{array} \tag{3.7}$$

The only optimal integral solution is $[10, 0, 49199, 42]^T$ and a short particular solution is $[10, -1, 24600, 24600]^T$. The reformulated problem using Euclidean-norm reduced lattice basis is given as

$$\begin{array}{lll} \min & y_1 + y_2 & \\ s.t. & 451y_1 - 11y_2 & \geq -10 \\ & -450y_1 + 11y_2 & \geq 1 \\ & -y_1 + 600y_2 & \geq -24600 \\ & y_1 - 599y_2 & \geq -24600 \\ & y_1, y_2 \in \mathbb{Z}. & \end{array} \tag{3.8}$$

Denote by $\mathcal{Q}$ the relaxed feasible set of Problem (3.8). Then $W_I(e_2, \mathcal{Q}) = \lfloor 41.07 \rfloor - \lceil -41.0017 \rceil + 1 = 83$, and $W_I(e_1, \mathcal{Q}) = \lfloor 1.0017 \rfloor - \lceil -1.0222 \rceil + 1 = 3$.

Using the ellipsoidal-norm reduced basis we obtain the following transformed problem:

$$\begin{array}{lll} \min & -40y_1 + y_2 & \\ s.t. & -11y_2 & \geq -10 \\ & y_1 + 11y_2 & \geq 1 \\ & 24599y_1 + 600y_2 & \geq -24600 \\ & -24558y_1 - 599y_2 & \geq -24600 \\ & y_1, y_2 \in \mathbb{Z}. & \end{array} \tag{3.9}$$

Let $\mathcal{R}$ be its relaxed feasible set. The analytic center solution to (2.8) for this example is $w = [4.52, 4.51, 25597.88, 23603.79]^T$ (compare with the approximated center $[4.5, 5, 37199.5, 12021.5]^T$

obtained from (2.9)). Then $W_I(e_2, \mathcal{R}) = \lfloor \frac{10}{11} \rfloor - \lceil -1.558E-4 \rceil + 1 = 1$, and $W_I(e_1, \mathcal{R}) = \lfloor 1.00172 \rfloor - \lceil -1.022214502 \rceil + 1 = 3$. The CPLEX 9.0 MIP [6] solver produced the following output output for Problem (3.7). It experienced numerical difficulties and reached an incorrect conclusion while exploring more than 500 nodes.

```
Reduced MIP has 2 rows, 4 columns, and 6 nonzeros. Presolve time = -0.00
sec. MIP emphasis: balance optimality and feasibility Root relaxation
solution time =   -0.00 sec.

        Nodes                                           Cuts/
   Node  Left     Objective  IInf  Best Integer     Best Node    ItCnt     Gap

      0     0    49166.9778     2                  49166.9778        2
                 49166.9800     2                   Fract:  1        3
    100     1    49167.1562     2                  49167.1559        4
    200     1    49167.3272     2                  49167.3266        4
    300     1    49167.4982     2                  49167.4973        4
    400     1    49167.6692     2                  49167.6681        4
    500     1    49167.8403     1                  49167.8388        4
*   596     0                   0    49168.0044    49168.0027        4    0.00%
Warning: integer solution contains unscaled infeasibilities.

Maximum unscaled integer infeasibility = 0.00443538. Integer optimal with
unscaled infeasibilities: Objective = 4.9168004443e+04 Solution time =
0.05 sec. Iterations = 4  Nodes = 596
```

4 Infeasible Problems

One may question the value of using ellipsoidal norm for infeasible integer programs. In this section we modify the right-hand side of Problem (2.1) to generate infeasible problems and show that similar performance improvements are possible when ellipsoidal norm is considered in the reduced lattice basis computation.

4.1 A General Infeasible Example

Let $\mathcal{P} = \{x \in \mathbb{R}^4_+ | \mathrm{Ax} = \mathrm{b}\}$, where the coefficient matrix A is given in (2.2), and

$$b = \begin{bmatrix} k_1k_2^2k_3 - (k_2k_3 - 1)(2k_1k_2k_3 - k_1k_2 - 2k_1) + (k_1 - 1) \\ 2k_3(k_1k_2 - 1)(k_2k_3 - k_2 - 1) - k_1(2k_2k_3 - k_2 - 2) - 1 \end{bmatrix}. \tag{4.1}$$

We consider the following integer feasibility problem (IFP):

$$\text{Does there exist a feasible integer solution in } \mathcal{P}? \tag{4.2}$$

A particular short solution to $Ax = b$ is given by $v = [k_1 - 1, -1, k_2k_3, k_2k_3 - k_2 - 2]^T$. Using the transformation $x = Z_{LLL}y + v$, the set $\mathcal{P}$ is transformed to:

$$\begin{aligned} \mathcal{Q} = \{y \in \mathbb{R}^2 \mid \mathrm{k_1k_2y_1 - k_1y_2 \geq -(k_1 - 1), -(k_1k_2 - 1)y_1 + k_1y_2 \geq 1}, \\ -y_1 + k_3y_2 \geq -k_2k_3,\ y_1 - (k_3 - 1)y_2 \geq -k_2k_3 + k_2 + 2\}. \end{aligned} \tag{4.3}$$

The IFP problem (4.2) is equivalent to the following feasibility problem:

$$\text{Does there exist a feasible integer solution in } \mathcal{Q}? \tag{4.4}$$

One can verify that $\mathcal{Q}$ does not contain any feasible integer solution with the conditions given in (2.4). The integer widths of $\mathcal{Q}$ along e_2, e_1 are given as

$$\begin{aligned} W_I(e_2,\mathcal{Q}) &= \left\lfloor \frac{(k_2k_3-k_2-2)(k_1k_2-1)-1}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor \\ &\quad - \left\lceil \max\left\{ k_2(2-k_1)+\frac{k_1-1}{k_1}, -k_2-\frac{(k_1k_2+k_1-1)}{k_1k_2k_3-k_1)} \right\} \right\rceil + 1 \\ &= \left\lfloor k_2 - \frac{k_1k_2-1}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor - \max\{k_2(2-k_1)+1, -k_2\} + 1 \\ &= k_2-1+\min\{k_2(k_1-2)-1, k_2\}+1, \end{aligned}$$

which is $2k_1-1$ (if $k_1=3$) or $2k_2$ (if $k_1>3$), and

$$\begin{aligned} W_I(e_1,\mathcal{Q}) &= \left\lfloor \frac{k_1(k_2k_3-k_2-2)-(k_3-1)}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor \\ &\quad - \left\lceil \max\left\{ 2-k_1, -\frac{k_3(k_1k_2+k_1-1)}{k_1k_2k_3-k_1} \right\} \right\rceil + 1 \\ &= \left\lfloor 1-\frac{k_1}{(k_3-1)(k_1k_2-1)-k_1} \right\rfloor + \min\{k_1-2, 1\}+1 = 2. \end{aligned}$$

Using ellipsoidal norm reduced lattice basis the transformed feasible set is given by

$$\begin{aligned} \mathcal{R} = \{y \in \mathbb{R}^2 \mid -\mathrm{k_1y_2} \geq -(\mathrm{k_1}-1), \mathrm{y_1}+\mathrm{k_1y_2} \geq 1, (\mathrm{k_2k_3}-1)\mathrm{y_1}+\mathrm{k_3y_2} \geq -\mathrm{k_2k_3}, \\ (1-k_2(k_3-1))y_1-(k_3-1)y_2 \geq -k_2k_3+k_2+2\}. \end{aligned}$$

One can show that

$$\begin{aligned} W_I(e_2,\mathcal{R}) &= \left\lfloor \frac{k_1-1}{k_1} \right\rfloor - \left\lceil \frac{1}{k_1(k_2k_3-k_2-1)-(k_3-1)} \right\rceil + 1 = 0, \\ W_I(e_1,\mathcal{R}) &= \left\lfloor 1-\frac{k_1}{k_1(k_2k_3-k_2-1)-(k_3-1)} \right\rfloor - \left\lceil \max\{2-k_1, -1-\frac{k_1+k_3(k_1-1)}{k_1(k_2k_3-1)}\} \right\rceil \\ &\quad +1 = 2. \end{aligned}$$

We can readily detect the infeasibility of this problem by branching on e_2 first.

4.2 Numerical Illustration of the Infeasible Example

We now give a numerical example for the problem constructed in Section 4.1. We convert our example in Section 3.1 with a new b computed from for this purpose. The relaxed feasible set is given as

$$\mathcal{P} = \{x \in \mathbb{R}^4_+ | \mathrm{x_1} - 5904\mathrm{x_3} - 5937\mathrm{x_4} = -23367997,\ \mathrm{x_2} + 5725\mathrm{x_3} + 5757\mathrm{x_4} = 22659518\},$$

and the transformed relaxed feasible set is

$$\begin{aligned} \mathcal{Q} = \{y \in \mathbb{R}^2 | 33\mathrm{y_1} - 3\mathrm{y_2} \geq -2,\ -32\mathrm{y_1} + 3\mathrm{y_2} \geq 1, \\ -y_1 + 180y_2 \geq -1980,\ y_1 - 179y_2 \geq -1967\}, \end{aligned}$$

where $W_I(e_2, \mathcal{Q}) = \lfloor 10.99 \rfloor - \lceil -10.33 \rceil + 1 = 21$, $W_I(e_1, \mathcal{Q}) = \lfloor 0.99 \rfloor - \lceil -1 \rceil + 1 = 2$.

Following the method in Mehrotra and Li [9], we obtain the following transformed set:

$$\mathcal{R} = \{y \in \mathrm{IR}^2 | -3\mathrm{y}_2 \geq -2,\ \mathrm{y}_1 + 3\mathrm{y}_2 \geq 1,\\ 1979y_1 + 180y_2 \geq -1980,\ -1968y_1 - 179y_2 \geq -1967\},$$

where $W_I(e_2, \mathcal{R}) = \lfloor 2/3 \rfloor - \lceil 0.000175 \rceil + 1 = 0$, and $W_I(e_1, \mathcal{R}) = \lfloor 0.99476 \rfloor - \lceil -1 \rceil + 1 = 2$. CPLEX 9.0 MIP solver with default options detected the integer infeasibility of $\mathcal{P}$ using 116 nodes. The relaxed feasible region of the infeasible version of the example in Section 3.2 obtained by changing the right hand side according to (4.1) is given by:

$$\mathcal{P} = \{x \in \mathrm{IR}^4_+ | \mathrm{x}_1 - 270138\mathrm{x}_3 - 270589\mathrm{x}_4 = -13290248863,\\ x_2 + 269539x_3 + 269989x_4 = 13260779272\}.$$

A short particular solution is $[10, -1, 24600, 24557]^T$, and the transformed feasible set is

$$\mathcal{Q} = \{y \in \mathrm{IR}^2 | 451\mathrm{y}_1 - 11\mathrm{y}_2 \geq -10,\ -450\mathrm{y}_1 + 11\mathrm{y}_2 \geq 1,\\ -y_1 + 600y_2 \geq -24600,\ y_1 - 599y_2 \geq -24557\}.$$

Then $W_I(e_2, \mathcal{Q}) = \lfloor 40.998 \rfloor - \lceil -41.0017 \rceil + 1 = 82$, and $W_I(e_1, \mathcal{Q}) = \lfloor 0.99996 \rfloor - \lceil -1.0222 \rceil + 1 = 2$.

Using the ellipsoidal norm reduced lattice basis we obtain the following transformed problem:

$$\mathcal{R} = \{y \in \mathrm{IR}^2 | -11\mathrm{y}_2 \geq -10,\ \mathrm{y}_1 + 11\mathrm{y}_2 \geq 1,\ 24599\mathrm{y}_1 + 600\mathrm{y}_2 \geq -24600,\\ -24558y_1 - 599y_2 \geq -24557\}.$$

Then $W_I(e_2, \mathcal{R}) = \lfloor \frac{10}{11} \rfloor - \lceil 3.71E-6 \rceil + 1 = 0$, $W_I(e_1, \mathcal{R}) = \lfloor 0.99996 \rfloor - \lceil -1.02221 \rceil + 1 = 2$. For comparison, CPLEX 9.0 MIP solver with default options detected the infeasibility of this problem using 2886 nodes.

5 Concluding Remarks

In this paper we have constructed four variable examples illustrating an integer programming reformulation technique which uses an LLL-reduced basis in ellipsoidal norm. We have demonstrated the value of using the ellipsoidal norm over Euclidean norm proposed previously by showing that in these examples the branching nodes in the ellipsoidal norm based reformulation is $O(1)$, while the number of nodes in the Euclidean norm based reformulations grows with the magnitude of problem data. We have also observed the failure of a state of the art commercial package in solving four variable integer programs constructed using the technique described in this paper.

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A Appendix

General Notation

The superscript ^{T} represents transpose of a vector or a matrix. For $x \in \mathbb{R}^n$, $\|x\|$ represents the l_2 (Euclidean) norm, and $\|x\|_Q$ represents the ellipsoidal norm $\sqrt{x^TQx}$. e is a vector of all ones. $Q^{1/2}$ represents the square-root of a positive definite matrix, and $Q^{-1/2}$ represents the inverse of $Q^{1/2}$. $\lfloor\alpha\rceil$ represents an integer closest to α. $\lfloor\alpha\rfloor$ denotes the largest integer less than or equal to α, and $\lceil\alpha\rceil$ represents the smallest integer greater than or equal to α. $\lfloor x\rfloor$ and $\lceil x\rceil$ represents integral vectors obtained by rounding each component of a real vector x as described above. I represents an Identity matrix of an appropriate size.

Feasibility Integer Programs

The pure feasibility integer linear programming problem (FILP) is to

$$\text{find } \{x \in \mathbb{Z}^n_+ \mid Ax = a\}, \tag{A.1}$$

or to show that no such solution exists. Here $A \in \mathbb{Z}^{m\times n}$, $a \in \mathbb{Z}^m$, and A is assumed to have full row rank. Let $\mathcal{P} := \{x \mid Ax = a, x \geq 0\}$.

Width of a Convex Set

The width of a convex set $\mathcal{C}$ along an integral vector u is defined as

$$\mathcal{W}(u,\mathcal{C}) := \max_{x\in\mathcal{C}} u^Tx - \min_{x\in\mathcal{C}} u^Tx,$$

and its integer width is defined as

$$\mathcal{W}_I(u,\mathcal{C}) := \lfloor\max_{x\in\mathcal{C}} u^Tx\rfloor - \lceil\min_{x\in\mathcal{C}} u^Tx\rceil + 1.$$

Kernel and Adjoint Lattices

Given $B = [b_1,\ldots,b_k]$, $n \geq k$, $\mathcal{L}(B) := \{x \in \mathbb{R}^n \mid \mathrm{x} = \sum_{\mathrm{i}=1}^{\mathrm{k}} \mathbb{Z}\mathrm{b_i}\}$, is the lattice generated by column vectors b_i, $i = 1,\ldots,k$. A lattice is called integral if all vectors in $\mathcal{L}(B)$ are integer vectors. An integral lattice has an associated unique integral kernel lattice $\mathcal{K}(B) := \{u \in \mathbb{Z}^n \mid u^Tb = 0 \text{ for all } b \in \mathcal{L}(B)\}$. The lattice $\mathcal{K}(A^T)$ is represented by Λ. The existence of Λ is well known. A lattice $\mathcal{K}^*(A^T)$ is called an adjoint lattice of A if for any basis Z of Λ there exist a basis Z^* of $\mathcal{K}^*(A^T)$ such that

$$Z^TZ^* = I. \tag{A.2}$$

An adjoint lattice is integral if all its elements are integral. We only consider integral adjoint lattices.

Lenstra, Lenstra, and Lovász Reduced Basis

Let $\hat{B} = [\hat{b}_1,\ldots,\hat{b}_k]$ be the orthogonal basis vectors computed by using the Gram-Schmidt orthogonalization procedure as follows:

$$\hat{b}_i = b_i - \sum_{j=1}^{i-1}\Gamma_{j,i}\hat{b}_j, i = 1,\ldots,k, \tag{A.3}$$

where

$$\Gamma_{j,i} = b_i^TE\hat{b}_j/\|\hat{b}_j\|_E^2, \text{ and } \hat{b}_1 = b_1. \tag{A.4}$$

Here E is a positive semi-definite matrix, which is defined later depending on the context. It is assumed that $\|\cdot\|_E \neq 0$ for the vectors of interest.

Definition A.1 *A basis $b_1, \ldots, b_k$ of a lattice $\mathcal{L}$ is called an* LLL-reduced basis, *for $\delta \in (\frac{1}{4}, 1)$, if it has the following two properties:*

C1. (Size Reduced) $|\Gamma_{j,i}| \leq 1/2$ *for* $1 \leq j < i \leq k$.

C2. (2-Reduced) $\|\hat{b}_{i+1}\|_E^2 \geq (\delta - \Gamma_{i,i+1}^2)\|\hat{b}_i\|_E^2$, $|\Gamma_{i,i+1}| \leq 1/2$ *for* $i = 1, \ldots, k-1$.

A.1 The Branching Hyperplane Problem

Aardal *et al.* [1, 2] have given a particular reformulation technique that reformulates FILP in a full dimensional space. This technique is shown to be computationally useful for the market split problems [3], and knapsack problems [4]. Aardal *et al.* [1, 2] transform (FILP) to an equivalent feasibility problem:

$$\text{find } \{y \in \mathbb{Z}^k \mid y \in \mathcal{Y} := v + Z^T y \geq 0\}, \tag{A.5}$$

where $Z \in \mathbb{Z}^{n \times k}$ is a basis of $\mathcal{K}(A^T)$, and $v \in \mathbb{Z}^n$ is an integral solution satisfying $Ax = a$. Aardal *et al.* [1, 2, 3] and Aardal and Lenstra [4] use an LLL reduced Z and a short v in their reformulation.

The following result from Mehrotra and Li [9] shows that computation of the width of $\mathcal{P}$ and $\mathcal{Y}$ are equivalent.

Theorem A.1 *Let $u \in \Lambda^*$, and $u \neq 0$, then there exists a $p \in \mathbb{Z}^k$ ($p \neq 0$) such that*

$$\mathcal{W}(u, \mathcal{P}) = \mathcal{W}(p, \mathcal{Y}). \tag{A.6}$$

Furthermore, for $p \in \mathbb{Z}^k$ there exists a $u \in \Lambda^*$ such that (A.6) also holds. In particular, $u = Z^* p$, where Z^* is a basis of Λ^* satisfying $Z^T Z^* = I$.

This results in the following corollary.

Corollary A.1 $\min_{u \in \Lambda^* \setminus 0} \mathcal{W}(u, \mathcal{P}) = \min_{p \in \mathbb{Z}^k \setminus 0} \mathcal{W}(p, \mathcal{Y})$.

A.2 The Branching Hyperplane Problem in Ellipdsoidal Norm

The branching hyperplane problem using the ellipsoidal norm is formulated as follows. Let $\mathcal{E}(w, Q) := \{x \in \mathrm{I\!R}^n \mid (x - w)^T Q (x - w) \leq 1, Ax = a\}$. If w is taken as the log-barrier analytic center, which is the solution of:

$$\min\{-\sum_{i=1}^{n} \ln x_i \mid Ax = a\}.$$

It is well known that $\mathcal{E}(w, Q)$ gives an n-approximation of $\mathcal{P}$ [10]. In particular,

$$\mathcal{E}(w, \nabla^2 \rho(w, \mathcal{P})) \subseteq \mathcal{P} \subseteq \mathcal{E}(w, \nabla^2 \rho(w, \mathcal{P})/n).$$

The branching hyperplane finding problem for the ellipsoid $\mathcal{E}(w, Q)$ is to solve the minimization problem:

$$\min_{u \in \Lambda^* \setminus 0} \mathcal{W}(u, \mathcal{E}(w, Q)), \quad \text{or equivalently,} \quad \min_{u \in \Lambda^* \setminus 0} \mathcal{W}(u, \mathcal{E}(0, Q)), \tag{A.7}$$

where $\mathcal{E}(0, Q) = \{x \in \mathrm{I\!R}^n \mid \|x\|_Q \leq 1, Ax = 0\}$. Since for any $u \in \mathrm{I\!R}^n$, $\min_{x \in \mathcal{E}(0,Q)} u^T x = -\max_{x \in \mathcal{E}(0,Q)} u^T x$, we have the following result.

Proposition A.1 *The width of the ellipsoid $\mathcal{E}(0, Q)$ along $u \in \mathbb{Z}^n$ is*

$$\mathcal{W}(u, \mathcal{E}(0, Q)) = 2\|Q^{-1/2}u\|_{P_{AQ^{-1/2}}}, \tag{A.8}$$

where $P_{AQ^{-1/2}} = I - Q^{-1/2}A^T(AQ^{-1}A^T)^{-1}AQ^{-1/2}$, *or* $P_{AQ^{-1/2}} = Q^{1/2}Z(Z^TQZ)^{-1}Z^TQ^{1/2}$ *is an orthogonal projection matrix. In particular, if $u \in \Lambda^*$, then*

$$\frac{1}{2}\min_{u\in\Lambda^*\setminus 0} \mathcal{W}(u, \mathcal{E}(0, Q)) = \min_{p\in\mathbb{Z}^k\setminus 0} \|Q^{-1/2}Z^*p\|_{P_{AQ^{-1/2}}} \tag{A.9}$$

$$= \min_{p\in\mathbb{Z}^k\setminus 0} \sqrt{p^T(Z^TQZ)^{-1}p} \tag{A.10}$$

$$= \min_{p\in\mathbb{Z}^k\setminus 0} \|Q^{1/2}Z(Z^TQZ)^{-1}p\|. \tag{A.11}$$

The following corollary establishes a relationship between branching in the original problem and its full dimensional reformulation.

Corollary A.2 *Consider the polyhedron $\mathcal{Y} = \{y \mid Z^Ty + v \geq 0\}$, where Z is a basis for Λ, and $v \in \mathbb{Z}^n$ satisfies $Av = a$. Then,*

$$\mathcal{W}(e_k, \mathcal{Y}) = \mathcal{W}(Z_k^*, \mathcal{P}),$$

where $Z_k^* = Z^*e_k$ is the $k-th$ column of Z^*.

If Z is LLL-reduced in $\|\cdot\|_Q$ norm and Z^* is the corresponding adjoint lattice, then Mehrotra and Li [9] show that $P_{AQ^{-1/2}}Z^*$ is 2-reduced in the reverse order. Similarly, if $P_{AQ^{-1/2}}Z^*$ is LLL-reduced then the corresponding $Q^{1/2}Z$ is 2-reduced in the reverse order.

Proposition A.2 *Let Z, Z^* be bases of Λ and Λ^* satisfying $Z^TZ^* = I$. Let us consider the problem obtained by adding a constraint $u^Tx = \alpha$ to the set $Ax = b$, where $u = Z_k^*$. Then $\tilde{Z} = [Z_1, \ldots, Z_{k-1}]$ is an integral basis for $\mathcal{K}(\tilde{A})$ where $\tilde{A} := \begin{bmatrix} A \\ u^T \end{bmatrix}$. Furthermore, $\tilde{Z}^* = \left[Z_1^*, \ldots, Z_{k-1}^*\right]$ satisfies $\tilde{Z}^T\tilde{Z}^* = I$.*

From Proposition A.2 and Corollary A.2 Mehrotra and Li [9] conclude that branching on the coordinates $e_k, \ldots, e_1$ in $\mathcal{Y}$, is equivalent to branching on the vectors $Z_k^*, \ldots, Z_1^*$ of the adjoint lattice basis satisfying $Z^TZ^* = I$. Consequently they suggest an alternative reformulation of $FILP$ in (A.5) where Z is such that it is LLL-reduced under $\|\cdot\|_Q$ norm instead of l-2 norm.