Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 5, No. 2, pp. 225–234, May 2009

DISCRETE L-CONVEX FUNCTION MINIMIZATION BASED ON CONTINUOUS RELAXATION

Satoko Moriguchi and Nobuyuki Tsuchimura

Abstract: We consider the problem of minimizing a nonlinear discrete function with L$^\natural$-convexity proposed in the theory of discrete convex analysis. For this problem, a steepest descent algorithm and a steepest descent scaling algorithm are known. In this paper, we propose a continuous relaxation approach which first minimizes the continuous variable version in order to find a good initial solution of the steepest descent algorithm. For discrete L$^\natural$-convex functions, we give a proximity theorem showing that a discrete global minimizer exists in a neighborhood of a continuous global minimizer. This proximity theorem affords a theoretical guarantee for the efficiency of the proposed algorithm.

Key words: *discrete optimization, discrete convex function, submodular function, algorithm*

Mathematics Subject Classification: *52A41, 90C27*

1 Introduction

In recent research towards a unified framework of discrete convex analysis [12], the concept of L$^\natural$-convex functions was proposed as a generalization of the Lovász extension of submodular set functions [9]. The concept of M$^\natural$-convex functions was also proposed as an extension of that of valuations on matroids invented by Dress and Wenzel [2]. These two concepts of discrete convexity are conjugate to each other, and a Fenchel-type duality theorem holds for L$^\natural$- and M$^\natural$-convex/concave functions [12]. Applications of L$^\natural$-/M$^\natural$-convexity can be found in mathematical economics with indivisible commodities [1,14,15], system analysis by mixed polynomial matrices [11], etc. These two discrete convexities play central roles in the theory of discrete convex analysis [12] and provide a nice framework of nonlinear combinatorial optimization; global optimality is guaranteed by local optimality and descent algorithms work for minimization. Steepest descent algorithms, which terminate in pseudo-polynomial time, and steepest descent scaling algorithms, which terminate in polynomial time with the aid of a scaling technique, are also known. The proximity theorems on a scaled local optimum for L$^\natural$-convexity and M$^\natural$-convexity guarantee the efficiency of scaling algorithms.

The objective of this paper is to show that we can minimize an L$^\natural$-convex function more efficiently, in the case where the continuous variable version which can be minimized tractably is available. Rescent progress of continuous optimization enables us today to solve convex minimization problems in a practical time [16]. We propose a continuous relaxation approach which first minimizes the continuous variable version, i.e., a continuous convex function, in order to find a good initial solution of a steepest descent algorithm. In general, for discrete function minimization, we can say that the rounded continuous

Copyright © 2008 Yokohama Publishers **http://www.ybook.co.jp**

relaxation solution is almost certainly nonoptimal and may be very far away from the optimal integer solution. For separable convex optimization problems, proximity results between the continuous and integral optimal solutions were obtained [4, 5]. In this paper, for the discrete L$^\natural$-convex function minimization problem, which is a nonseparable optimization problem, we give a proximity theorem showing that a discrete global minimizer exists in a neighborhood of a continuous global minimizer. On the basis of our new proximity, we can minimize a discrete L$^\natural$-convex function efficiently by using continuous relaxation. In order to compare the performance of our new continuous relaxation approach with those of the previously proposed algorithms, we make numerical experiments with randomly generated test problems. It is observed from numerical results that our new approach, when applicable, is much faster than the previously proposed algorithms.

2 Preliminaries

Let $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a function. The effective domain, the epigraph and the set of minimizers of g are given by

$$\begin{aligned} \mathrm{dom}\, g &= \{p \in \mathbf{R}^n \mid g(p) < +\infty\}, \\ \mathrm{epi}\, g &= \{(p, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq g(p)\}, \\ \arg\min g &= \{p \in \mathbf{R}^n \mid g(p) \leq g(q)\ (\forall q \in \mathbf{R}^n)\}, \end{aligned}$$

respectively. For a function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$, we use the notation

$$\begin{aligned} \mathrm{dom}_{\mathbf{Z}}\, g &= \{p \in \mathbf{Z}^n \mid g(p) < +\infty\}, \\ \arg\min\nolimits_{\mathbf{Z}} g &= \{p \in \mathbf{Z}^n \mid g(p) \leq g(q)\ (\forall q \in \mathbf{Z}^n)\} \end{aligned}$$

for the effective domain and the set of minimizers of g, respectively.

A convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be proper if $\mathrm{dom}\, g \neq \emptyset$, and closed if $\mathrm{epi}\, g$ is a closed set. For a closed proper convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, $\arg\min g \neq \emptyset$ if $\mathrm{dom}\, g$ is bounded.

For vectors $p, q \in \mathbf{R}^n$, we write $p \vee q$ and $p \wedge q$ for their componentwise maximum and minimum. We write $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{Z}^n$. Functions defined on integer points are said to be discrete functions. A discrete function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is called L-convex [12] if it satisfies

(SBF[Z]) $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)\quad (p, q \in \mathbf{Z}^n)$,
(TRF[Z]) $\exists r \in \mathbf{R}$ such that $g(p+\mathbf{1}) = g(p)+r\quad (p \in \mathbf{Z}^n)$,

where it is understood that the inequality (SBF[**Z**]) is satisfied if $g(p)$ or $g(q)$ is equal to $+\infty$.

A function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is called L$^\natural$-convex [3, 12] if there exists an L-convex function $\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ such that

$$g(p_1, \ldots, p_n) = \tilde{g}(0, p_1, \ldots, p_n) \tag{2.1}$$

for each $(p_1, \ldots, p_n) \in \mathbf{Z}^n$. It turns out that L$^\natural$-convexity can be characterized by a kind of generalized submodularity:

(SBF$^\natural$[Z]) $g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1}))\quad (0 \leq \alpha \in \mathbf{Z}, p, q \in \mathbf{Z}^n)$, which is called translation submodularity.

The concepts of L-/L$^\natural$-convexity can also be defined for functions in real variables through an appropriate adaptation of the conditions (SBF[**Z**]) and (TRF[**Z**]). Namely, we call a function $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ L-convex [12] if $\bar{g}$ is convex and satisfies

(SBF[R]) $\bar{g}(p) + \bar{g}(q) \geq \bar{g}(p \vee q) + \bar{g}(p \wedge q) \quad (p, q \in \mathbf{R}^n)$,
(TRF[R]) $\exists r \in \mathbf{R}$ such that $\bar{g}(p+\mathbf{1}) = \bar{g}(p)+r \quad (p \in \mathbf{R}^n)$.

L$^\natural$-convex functions are defined as the restriction of L-convex functions, as in (2.1), and are characterized by

(SBF$^\natural$[R]) $\bar{g}(p) + \bar{g}(q) \geq \bar{g}((p - \alpha\mathbf{1}) \vee q) + \bar{g}(p \wedge (q + \alpha\mathbf{1})) \quad (0 \leq \alpha \in \mathbf{R}, p, q \in \mathbf{R}^n)$.

Throughout the paper, we assume that a continuous L$^\natural$-convex function is a closed proper convex function. It is known that a closed proper L$^\natural$-convex function is continuous on the effective domain [13].

Minimization of a continuous L$^\natural$-convex function is tractable with a firm theoretical basis provided by convex analysis. For minimization of a discrete L$^\natural$-convex function, we have the following optimality criterion, which shows that global minimality is characterized by local minimality. The characteristic vector of a subset $X \subseteq \{1, 2, \ldots, n\}$ is denoted by

$$\chi_X(i) = \begin{cases} 1 & (i \in X), \\ 0 & (i \in \{1, 2, \ldots, n\} \setminus X). \end{cases}$$

Theorem 2.1 (Theorem 7.14 in [12]). *Let $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete L$^\natural$-convex function. For $p \in \mathrm{dom}_{\mathbf{Z}}\, g$, $g(p) \leq g(q)$ $(q \in \mathbf{Z}^n)$ if and only if*

$$g(p) \leq g(p \pm \chi_X) \quad (X \subseteq \{1, 2, \ldots, n\}). \tag{2.2}$$

3 Proposed Algorithm

For discrete L$^\natural$-convex function minimization, our continuous relaxation approach and proximity theorems between the discrete minimizer and the relaxation solution are given in Section 3.1. Section 3.2 is devoted to the proofs of proximity theorems.

3.1 Algorithms and Proximity Theorems

The local characterization of global minimality for discrete L$^\natural$-convex functions (Theorem 2.1) naturally leads to the following steepest descent algorithm [12, Sec. 10.3.1].

Steepest descent algorithm for an L$^\natural$-convex function: SD(g, p)
Input: a discrete L$^\natural$-convex function g and $p \in \mathrm{dom}_{\mathbf{Z}}\, g$
Output: a minimizer of g
S1: Find $\varepsilon \in \{1, -1\}$ and $X \subseteq \{1, 2, \ldots, n\}$ that minimize $g(p + \varepsilon\chi_X)$.
S2: If $g(p) \leq g(p + \varepsilon\chi_X)$, then return p (p is a minimizer of g).
S3: Set $p := p + \varepsilon\chi_X$ and go to S1.

Step S1, i.e., the verification of (2.2), amounts to minimizing a pair of submodular set functions which can be done in polynomial time [6,17,18]. If the effective domain is bounded, the number of iterations is $\mathrm{O}(\hat{K}_\infty)$ where

$$\hat{K}_\infty = \max\{\|p - q\|_\infty \mid p, q \in \mathrm{dom}_{\mathbf{Z}}\, g\}$$

[7]. We assume that the minimizer of a submodular set function can be computed in $\mathrm{O}(S)$ function evaluations. Then, the steepest descent algorithm finds a minimizer of g with $\mathrm{O}(S\hat{K}_\infty)$ function evaluations. Furthermore, the steepest descent algorithm, which is a pseudo-polynomial time algorithm, can be made more efficient with the aid of a scaling technique. The resulting steepest descent scaling algorithm [12, Sec. 10.3.2] terminates in polynomial time. This is guaranteed by the proximity theorem (Theorem 7.18 in [12]) on a scaled local optimum for L-convexity.

Now, we propose a continuous relaxation approach which is the steepest descent algorithm starting with a continuous relaxation solution as the initial solution. We assume that a continuous L$^\natural$-convex function $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ with

$$g(p) = \bar{g}(p) \quad (p \in \mathbf{Z}^n) \tag{3.1}$$

and $\arg\min \bar{g} \neq \emptyset$ is known. Note that the existence of a continuous L$^\natural$-convex function $\bar{g}$ with (3.1) is guaranteed by the convex extensibility of a discrete L$^\natural$-convex function (Theorem 7.20 in [12]). If a continuous L$^\natural$-convex function $\bar{g}$ which can be minimized tractably is available, our continuous relaxation approach minimizes g efficiently.

Continuous relaxation algorithm for an L$^\natural$-convex function: $\mathrm{RELAX}(g, \bar{g})$
Input: a discrete L$^\natural$-convex function g and a continuous L$^\natural$-convex function $\bar{g}$ with (3.1)
Output: a minimizer of g
S1: Find $\bar{p} \in \arg\min \bar{g}$.
S2: Round off $\bar{p}$ to obtain $p \in \mathrm{dom}_{\mathbf{Z}}\, g$.
S3: Return $\mathrm{SD}(g, p)$.

We have to take care of the proximity between $\bar{p}$ and a minimizer of g before we can assert that this approach is efficient. First, we obtain the following "proximity theorem," showing that a continuous relaxation solution of a discrete L$^\natural$-convex function minimization problem exists in a neighborhood of the integer minimizer.

Theorem 3.1. *Let $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete L$^\natural$-convex function and $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a continuous L$^\natural$-convex function with $\arg\min \bar{g} \neq \emptyset$. We assume that*

$$g(p) = \bar{g}(p) \quad (p \in \mathbf{Z}^n).$$

Then, for any $p^* \in \arg\min_{\mathbf{Z}} g$, there exists some $\bar{p} \in \arg\min \bar{g}$ such that

$$p^* - n\mathbf{1} \leq \bar{p} \leq p^* + n\mathbf{1}.$$

The proof of Theorem 3.1 is given later in Section 3.2.

What is really needed for the proposed algorithm is a kind of reverse direction of Theorem 3.1, that is, a theorem that shows a minimizer of a discrete L$^\natural$-convex function g exists in a neighborhood of the continuous relaxation solution. As the main theorem of this paper we obtain the following proximity theorem. We assume now the boundedness of the effective domain of $\bar{g}$.

Theorem 3.2. *Let $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete L$^\natural$-convex function and $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a continuous L$^\natural$-convex function with $\arg\min \bar{g} \neq \emptyset$. We assume that*

$$g(p) = \bar{g}(p) \quad (p \in \mathbf{Z}^n),$$

and $\mathrm{dom}\, \bar{g}$ is bounded. For any $\bar{p} \in \arg\min \bar{g}$, there exists some $p^* \in \arg\min_{\mathbf{Z}} g$ such that

$$\bar{p} - n\mathbf{1} \leq p^* \leq \bar{p} + n\mathbf{1}.$$

The proof of Theorem 3.2 is also given later in Section 3.2.

Theorem 3.2 guarantees that our continuous relaxation approach is efficient if the relaxation solution can be found fast. In order to find the relaxation solution $\bar{p}$ in step S1, we can utilize continuous convex minimization algorithms for $\bar{g}$ since a continuous L$^\natural$-convex function is convex by the definition. The number of iterations in step S3 is $\mathrm{O}(n)$ from Theorem 3.2, while, in the steepest descent algorithm starting with an arbitrary solution in the effective domain, it is $\mathrm{O}(\hat{K}_\infty)$. Thus the continuous relaxation algorithm, denoting by T an upper bound on the number of function evaluations to find a relaxation solution, finds minimizer of g with $\mathrm{O}(nS+T)$ function evaluations.

3.2 Proofs

We give proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. For an integer $s \geq 2$, we define $g_s : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ as

$$g_s(p) := \bar{g}(\frac{p}{s}) \quad (p \in \mathbf{Z}^n).$$

We have

$$g(p) = g_s(sp) \quad (p \in \mathbf{Z}^n). \tag{3.2}$$

For all $p, q \in \mathbf{Z}^n$ and $0 \leq \alpha \in \mathbf{Z}$, we have

$$\begin{aligned} g_s(p) + g_s(q) &= \bar{g}(\frac{p}{s}) + \bar{g}(\frac{q}{s}) \\ &\geq \bar{g}((\frac{p}{s} - \frac{\alpha}{s}\mathbf{1}) \vee \frac{q}{s}) + \bar{g}(\frac{p}{s} \wedge (\frac{q}{s} + \frac{\alpha}{s}\mathbf{1})) \\ &= \bar{g}(\frac{(p-\alpha\mathbf{1}) \vee q}{s}) + \bar{g}(\frac{p \wedge (q+\alpha\mathbf{1})}{s}) \\ &= g_s((p-\alpha\mathbf{1}) \vee q) + g_s(p \wedge (q+\alpha\mathbf{1})), \end{aligned}$$

where the inequality is by translation submodularity (SBF$^\natural$[$\mathbf{R}$]). This means discrete L$^\natural$-convexity of g_s.

Let p^* be a minimizer of g. Optimality criterion for g, i.e., (2.2), yields

$$g_s(sp^*) \leq g_s(sp^* \pm s\chi_X) \ (X \subseteq \{1, 2, \ldots, n\})$$

from (3.2). By applying L-proximity theorem on a scaled local optimum (Theorem 7.18 (2) in [12]) to g_s and sp^*, there exists $p_s \in \arg\min_{\mathbf{Z}} g_s$ with

$$sp^* - (s-1)n\mathbf{1} \leq p_s \leq sp^* + (s-1)n\mathbf{1}. \tag{3.3}$$

Dividing all parts of (3.3) by s shows

$$p^* - n\mathbf{1} \leq p^* - \frac{s-1}{s}n\mathbf{1} \leq \frac{p_s}{s} \leq p^* + \frac{s-1}{s}n\mathbf{1} \leq p^* + n\mathbf{1}.$$

Put $K := \{p \in \mathbf{R}^n \mid p^* - n\mathbf{1} \leq p \leq p^* + n\mathbf{1}\}$. Since K is compact, every sequence in K has a convergent subsequence, the limit point of which belongs to K. For $k \in \mathbf{Z}$ with $k \geq 1$, we suppose $s_k = 2^k$, $p_{s_k} \in \arg\min_{\mathbf{Z}} g_{s_k}$ and $\frac{p_{s_k}}{s_k} \in K$. From the sequence $\{\frac{p_{s_k}}{s_k}\}$, we take a convergent subsequence $\{\frac{p_{s_{k_i}}}{s_{k_i}}\}$ and put $\lim_{i\to\infty} \frac{p_{s_{k_i}}}{s_{k_i}} = p' \in K$. Continuity of $\bar{g}$

implies $\lim_{i\to\infty} \bar{g}(\frac{p_{s_{k_i}}}{s_{k_i}}) = \bar{g}(\lim_{i\to\infty} \frac{p_{s_{k_i}}}{s_{k_i}}) = \bar{g}(p')$. Note that $\{\bar{g}(\frac{p_{s_{k_i}}}{s_{k_i}})\}$ is a monotonically decreasing sequence $(\bar{g}(\frac{p_{s_{k_1}}}{s_{k_1}}) \geq \bar{g}(\frac{p_{s_{k_2}}}{s_{k_2}}) \geq \cdots \geq \bar{g}(\frac{p_{s_{k_i}}}{s_{k_i}}) \geq \cdots)$ and

$$\bar{g}(p') \leq \bar{g}(\frac{p_{2^{k_i}}}{2^{k_i}}) = \min g_{2^{k_i}} \quad (i \in \mathbf{Z}, i \geq 1). \tag{3.4}$$

Now, we prove $\bar{g}(p') = \min \bar{g}$, i.e., $p' \in \arg\min \bar{g}$, by contradiction. Assume that $\bar{g}(p') > \min \bar{g}$ and put $\varepsilon_0 := \bar{g}(p') - \min \bar{g} > 0$. We fix $\bar{p} \in \arg\min \bar{g}$ arbitrarily. For any number $\delta > 0$, there exist a number $N \in \{k_i \mid i = 1, 2, \ldots\}$ and a sequence $\{b_k\}$ with $b_0 = \lfloor \bar{p} \rfloor$ and $b_k \in \{0,1\}^n$ for $k = 1, 2, \ldots, N$ such that $|\bar{p} - q| < \delta$ where $q := \sum_{k=0}^{N} \frac{b_k}{2^k}$. Note that $2^N q \in \mathbf{Z}$. Continuity of $\bar{g}$ gives

$$\forall \varepsilon' > 0, \exists \delta_{\varepsilon'} > 0 : |x - y| < \delta_{\varepsilon'} \Rightarrow |\bar{g}(x) - \bar{g}(y)| < \varepsilon'. \tag{3.5}$$

Now, we suppose that $x = \bar{p}$ and $\varepsilon' = \frac{\varepsilon_0}{2}$ in (3.5) and choose as above $y = q$ such that $|\bar{p} - q| < \delta_{\varepsilon'}$. Then we have

$$\min g_{2^N} \leq \bar{g}(q) < \min \bar{g} + \frac{\varepsilon_0}{2} < \min \bar{g} + \varepsilon_0 = \bar{g}(p'),$$

which contradicts (3.4). This proves $\bar{g}(p') = \min \bar{g}$. □

Proof of Theorem 3.2. To use Theorem 3.1 in the reverse direction, we consider the case where $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ has a unique minimizer $\bar{p}$. Then, the fact that the condition

$$p^* - n\mathbf{1} \leq \bar{p} \leq p^* + n\mathbf{1}$$

holds for all $p^* \in \arg\min_{\mathbf{Z}} g$ is immediate from Theorem 3.1. In particular, there exists $p^* \in \arg\min_{\mathbf{Z}} g$ satisfying this condition.

We consider a perturbation of $\bar{g}$ so that we can use this fact. We arbitrarily fix a minimizer $\bar{p} \in \arg\min \bar{g}$. For any number $\varepsilon > 0$, we define functions $\bar{g}_\varepsilon : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and $g_\varepsilon : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ as

$$\bar{g}_\varepsilon(p) := \bar{g}(p) + \sum_{i=1}^{n} \varepsilon(p(i) - \bar{p}(i))^2 \quad (p \in \mathbf{R}^n)$$

and

$$g_\varepsilon(p) := \bar{g}_\varepsilon(p) \quad (p \in \mathbf{Z}^n).$$

The functions $\bar{g}_\varepsilon$ and g_ε are L$^\natural$-convex by Theorem 7.11(1) in [12] and $\bar{g}_\varepsilon$ has a unique minimizer $\bar{p}$. Now, we explain that we can fix sufficiently small ε such that $p^*_\varepsilon \in \arg\min_{\mathbf{Z}} g_\varepsilon$ is also a minimizer of g. Recall that we assume now the boundedness of the effective domain of $\bar{g}$ and put $\tilde{K}_\infty = \max\{\|p - q\|_\infty \mid p, q \in \mathrm{dom}\, \bar{g}\}$. Assume that $g(p') = \min\{g(p) \mid p \in \mathrm{dom}_{\mathbf{Z}}\, g \setminus \arg\min_{\mathbf{Z}} g\}$. We fix $\varepsilon < \{g(p') - \min g\}/\{n\tilde{K}_\infty^2\}$. From $p^*_\varepsilon \in \arg\min_{\mathbf{Z}} g_\varepsilon$ and the definition of g_ε, for $p \in \arg\min_{\mathbf{Z}} g$, we have

$$g(p^*_\varepsilon) \leq g(p) + \varepsilon \sum_{i=1}^{n} \{(p(i) - \bar{p}(i))^2 - (p^*_\varepsilon(i) - \bar{p}(i))^2\},$$

where the last term is nonnegative because of $p \in \arg\min_{\mathbf{Z}} g$. Hence, it holds that

$$\begin{aligned} g(p^*_\varepsilon) &\leq g(p) + \frac{g(p') - \min g}{n\tilde{K}_\infty^2} \sum_{i=1}^{n} \{(p(i) - \bar{p}(i))^2 - (p^*_\varepsilon(i) - \bar{p}(i))^2\} \\ &< g(p) + g(p') - \min g = g(p'). \end{aligned}$$

This means $p_\varepsilon^* \in \arg\min_{\mathbf{Z}} g$.

Now, we apply the fact in the case of a unique continuous minimizer to show that there exists $p_\varepsilon^* \in \arg\min_{\mathbf{Z}} g$ such that $p_\varepsilon^* - n\mathbf{1} \leq \bar{p} \leq p_\varepsilon^* + n\mathbf{1}$. □

4 Numerical Experiments

We here mainly compare the performance of our new continuous relaxation approach with those of the previously proposed algorithms. We observe from numerical experiments that our approach is much faster than the previous algorithms.

We implemented three algorithms for minimization of a discrete $\mathrm{L}^\natural$-convex function shown in Table 1 in the C language to compare the performance of these algorithms.

Table 1: Algorithms we implemented for $\mathrm{L}^\natural$-convex function minimization.

symbol	algorithm
SD	steepest descent algorithm [12, Sec. 10.3.1]
SCALING	steepest descent scaling algorithm [12, Sec. 10.3.2]
RELAX	our new continuous relaxation approach

We use the following libraries:

- 'L-BFGS' by J. Nocedal* with its C++ wrapper by T. Kudo†, which is an implementation of quasi-Newton method for continuous function optimization [8]. As the routine requires the gradient of the objective function, we calculate a finite-difference approximation by calling the function evaluation oracle $n+1$ times. We use this only in RELAX (our new continuous relaxation approach).
- 'SFM8' by S. Iwata, which is an implementation of Iwata–Fleischer–Fujishige [6]. This minimizes a submodular function with $\mathrm{O}(n^5 \log_2 M)$ function evaluations, where M is the maximum absolute value of the submodular function.
- 'SIMD-oriented Fast Mersenne Twister' developed by M. Saito and M. Matsumoto‡, which generates pseudorandom numbers. We make use of this to generate test problems.

As test problems, we consider the following function:

$$g(p) = \sum_{i=1}^{n} h_i(p(i)) + \sum_{1 \leq i < j \leq n} h_{ij}(p(i) - p(j)) \quad (p \in \mathbf{Z}^n),$$

where $h_i(z) = a_i(z - c_i)^2 + b_i(z - c_i)$ and $h_{ij}(z) = a_{ij}z^2 + b_{ij}z$ are univariate functions. In our continuous relaxation approach, we use

$$\bar{g}(p) = \sum_{i=1}^{n} h_i(p(i)) + \sum_{1 \leq i < j \leq n} h_{ij}(p(i) - p(j)) \quad (p \in \mathbf{R}^n).$$

*http://www.ece.northwestern.edu/~nocedal/lbfgs.html
†http://chasen.org/~taku/software/misc/lbfgs/
‡http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/SFMT/

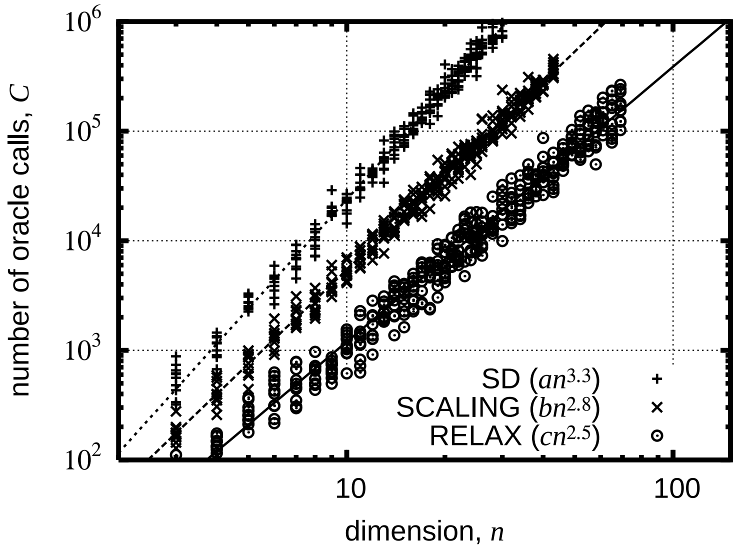

Figure 1: The number of oracle calls for L$^\natural$-convex function minimization.

Table 2: Observed computational complexity for L$^\natural$-convex function minimization.

algorithm	SD	SCALING	RELAX
oracle calls C	$n^{3.3}$	$n^{2.8}$	$n^{2.5}$

For each n, we generate ten test problems with randomly chosen integer variables $1 \leq a_i, a_{ij} \leq n$, $-n^2 \leq b_i, c_i, b_{ij} \leq n^2$. For each problem, we randomly choose an initial discrete solution p_0 satisfying $-10n \leq p_0(i) \leq 10n$.

Our computational environment is the following: HP dx5150 SF/CT, AMD Athlon 64 3200+ processor (2.0GHz, 512KB L2 cache), 4GB memory, Vine Linux 4.1 (kernel 2.6.16), gcc 3.3.6.

All the algorithms implemented here provide an optimal solution under the assumption that an oracle for computing L$^\natural$-convex function values is available. We measure the number of oracle calls for each problem. Our numerical result is summarized in Figure 1 which shows the relationship between the number of oracle calls C and dimension n for L$^\natural$-convex function minimization. In all the algorithms the relationship is linear in $\log C$ and $\log n$, which implies $C = \mathrm{O}(n^l)$ for some l. This result is displayed in Table 2.

By numerical experiments with randomly generated test problems, we can conclude that our continuous relaxation approach is faster than the previously proposed algorithms.

Acknowledgement

The authors thank Hiroshi Hirai, Satoru Iwata, Kazuo Murota, and an anonymous referee for stimulating comments. This work is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

[1] V. Danilov, G. Koshevoy and K. Murota, Discrete convexity and equilibria in economies with indivisible goods and money, *Math. Social Sci.* 41 (2001) 251–273.

[2] A.W.M. Dress and W. Wenzel, Valuated matroids, *Adv. Math.* 93 (1992) 214–250.

[3] S. Fujishige and K. Murota, Notes on L-/M-convex functions and the separation theorems, *Math. Program.* 88 (2000) 129–146.

[4] D.S. Hochbaum and J.G. Shanthikumar, Convex separable optimization is not much harder than linear optimization, *J. ACM* 37 (1990) 843–862.

[5] D.S. Hochbaum, Lower and upper bounds for the allocation problem and other nonlinear optimization problems, *Math. Oper. Res.* 19 (1994) 390–409.

[6] S. Iwata, L. Fleischer and S. Fujishige, A combinatorial strongly polynomial algorithm for minimizing submodular functions, *J. ACM* 48 (2001) 761–777.

[7] V. Kolmogorov and A. Shioura, New algorithms for the dual of the convex cost network flow problem with application to computer vision, preprint, 2007.

[8] D.C. Liu and J. Nocedal, On the limited memory method for large scale optimization, *Math. Program. B* 45 (1989) 503–528.

[9] L. Lovász, Submodular functions and convexity, in *Mathematical Programming—The State of the Art*, A. Bachem, M. Grötschel and B. Korte (eds.), Springer, Berlin, 1983, pp. 235–257.

[10] S. Moriguchi, K. Murota and A. Shioura, Scaling algorithms for M-convex function minimization, *IEICE Transactions on Fundamentals* E85-A (2002) 922–929.

[11] K. Murota, *Matrices and Matroids for Systems Analysis*, Springer, Berlin, 2000.

[12] K. Murota, *Discrete Convex Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[13] K. Murota and A. Shioura, Note on the continuity of M-convex and L-convex functions in continuous variables, METR 2008-12, Department of Mathematical Informatics, the University of Tokyo, 2008.

[14] K. Murota and A. Tamura, New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities, *Discrete Appl. Math.* 131 (2003) 495–512.

[15] K. Murota and A. Tamura, Application of M-convex submodular flow problem to mathematical economics, *Japan J. Indust. Appl. Math.* 20 (2003) 257–277.

[16] Y. Nesterov and A.S. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[17] J.B. Orlin, A faster strongly polynomial time algorithm for submodular function minimization, in *Integer Programming and Combinatorial Optimization*, 12th International IPCO Conference, Ithaca, NY, USA, June 25-27, 2007, Proceedings, Lecture Notes in Computer Science 4513, M. Fischetti and D. Williamson (eds.), Springer, Berlin, 2007, pp. 240–251.

[18] A. Schrijver, A combinatorial algorithm minimizing submodular functions in strongly polynomial time, *J. Combin. Theory Ser. B* 80 (2000) 346–355.

Manuscript received 24 March 2008
revised 12 June 2008
accepted for publication 17 June 2008

Satoko Moriguchi
Department of Information Systems Architecture, School of Industrial Technology
Advanced Institute of Industrial Technology
E-mail address: satoko@aiit.ac.jp

Nobuyuki Tsuchimura
School of Science and Technology, Kwansei Gakuin University
E-mail address: tutimura@kwansei.ac.jp