Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 5, No. 2, pp. 211–225, May 2009

DECIDING NON-REALIZABILITY OF ORIENTED MATROIDS BY SEMIDEFINITE PROGRAMMING

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Abstract: The concept of oriented matroid is a combinatorial abstraction of many geometric objects such as hyperplane arrangements. The problem to decide whether an oriented matroid has a geometric realization or not is called the realizability problem. This is a fundamental problem in the oriented matroid theory, and many important issues in combinatorial geometry such as stretchability of pseudoline arrangements [4] can be reduced to this problem. The realizability problem is known to be NP-hard [12] and there are many realizability certificates and non-realizability certificates based on sufficient conditions which can be checked efficiently. However, they cannot decide the realizability of all oriented matroids. Therefore new certificates are needed to determine the realizability of those that cannot be decided by existing methods. In this paper, we propose a new certificate for non-realizability of oriented matroids based on semidefinite programming relaxation of Grassmann-Plücker relations, and apply our method to oriented matroids with 8 elements and rank 4, and 9 elements and rank 3.

Key words: *oriented matroids, realizability problem, semidefinite programming*

Mathematics Subject Classification: *05B35, 90C22*

1 Introduction

The concept of oriented matroid is a combinatorial abstraction of hyperplane arrangements, vector configurations, point configurations and digraphs, and it provides a unified combinatorial setting to treat these objects. A fundamental problem to decide whether a given oriented matroid has a geometric realization (for example, hyperplane arrangements or equivalently, vector configurations, point configurations) or not is called the realizability problem and has been studied for a long time. The realizability problem shows a gap between the abstract combinatorial model and the geometric objects, and many important problems in combinatorial geometry can be reduced to the realizability problem of oriented matroids. For example, the realizability problem is known to be equivalent to that of specifying the gap between pseudohyperplane arrangements and hyperplane arrangements [4]. The realizability problem also has practical applications such as robust geometric computation as seen in [1].

By Mnëv's universality theorem [12], the realizability problem is polynomially equivalent to the decision problem for the existential theory of the reals, which is known to be NP-hard. Therefore it is almost impossible to solve the realizability problem completely and efficiently. On the other hand, some oriented matroids can be decided to be realizable or non-realizable by simple certificates. For example, a solvability sequence method [2], a reduction sequence method [17] and a realizability certificate using polynomial optimization

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and generalized mutation graphs [13, 15] were proposed as realizability certificates, and non-Euclidean property [7], a biquadratic final polynomial method (BFP) [3] and non-HK* property [8] as non-realizability certificates.

Since Finschi and Fukuda developed an enumeration algorithm of oriented matroids [6], and made a database of oriented matroids [5] recently, the existing certificates were applied to OM(8,4) and OM(9,3), the rank-4 oriented matroids with 8 elements and the rank-3 oriented matroids with 9 elements respectively [8, 13, 14, 15]. They are minimal cases having non-realizable ones with respect to the number of elements for each rank. In these studies, oriented matroids in OM(8,4) and OM(9,3) including non-uniform ones were classified with respect to the realizability using the above certificates and the database of oriented matroids. They determined 172183 of 181472 oriented matroids in OM(8,4) to be realizable and 3968 to be non-realizable as in Figure 1, and 452231 of 461053 oriented matroids in OM(9,3) to be realizable and 274 to be non-realizable as in Figure 2. However, the remaining oriented matroids are still unknown to be realizable or non-realizable. Therefore new realizability or non-realizability certificates are needed to know the realizability of the remaining oriented matroids.

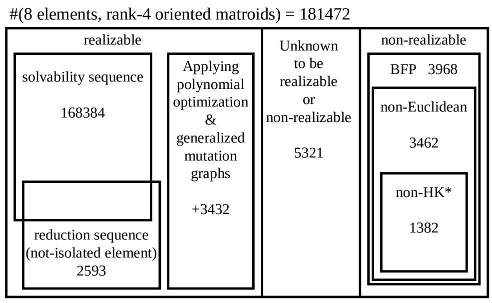

Figure 1: A classification of OM(8,4) w.r.t. certificates [8, 13, 14, 15]

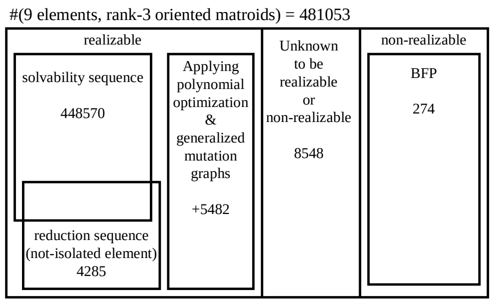

Figure 2: A classification of OM(9,3) w.r.t. certificates [8, 13, 14, 15]

In this paper, we propose a new certificate for non-realizability of oriented matroids based on semidefinite programming (SDP). First, we focus attention on a realizability certificate using polynomial optimization [13, 15]. In [13, 15], Nakayama, Moriyama and Fukuda gave realizations of oriented matroids by solving the following system, which describes the conditions for realizability of an oriented matroid $(\{1,...,n\},\chi)$:

$$\mathrm{sign}(\det(v_{i_1},v_{i_2},...,v_{i_r}))=\chi(i_1,i_2,...,i_r) \text{ for all } 1\le i_1<i_2<...<i_r\le n,$$

where $\chi : E^r \to \{+1,-1,0\}$ is a chirotope which defines the oriented matroid and $v_1,...,v_n \in \mathbb{R}^r$, as a polynomial optimization problem. Polynomial optimization problems (POPs) are optimization problems whose objective functions are polynomial and constraints are polynomial inequalities and equalities. Recently, Lasserre [10] and Parrilo [16] developed algorithms to solve POPs by SDP relaxation and there appeared solvers that solve POPs efficiently such as SparsePOP [19, 23]. Nakayama, Moriyama and Fukuda succeeded to give realizations of new oriented matroids by solving polynomial systems using SparsePOP as follows [13, 15]. They eliminated equality constraints from the above system as much as possible and solved them using SparsePOP. A solution of the SDP sometimes gives a solution of the original POP because most of the constraints of the POP are inequalities and they can be satisfied even if there is a relaxation gap. Therefore they certified realizability of an oriented matroid by computing an approximate solution of the above system by SDP relaxation and by checking if it is also a solution of the original POP. However, non-realizability of an oriented matroid cannot be certified by this approach even if this method fails to give a realization of the oriented matroid.

On the other hand, if one proves infeasibility of the SDP system which is obtained as an SDP relaxation of the original POP in some way, infeasibility of the original POP follows. Therefore one can prove non-realizability of the oriented matroid by certifying infeasibility of the SDP system. In this paper, we investigate this approach with another system, which we call Grassmann-Plücker system:

$$\begin{cases}\sum_{s=1}^{r+1}(-1)^s[i_1...i_{r-1}j_s][j_1...j_{s-1}j_{s+1}...j_{r+1}]=0 \\ \qquad\qquad\qquad\text{for all } 1\le i_1<...<i_{r-1}\le n, 1\le j_1<...<j_{r+1}\le n, \\ \mathrm{sign}([i_1i_2...i_r])=\chi(i_1,i_2,...,i_r) \text{ for all } 1\le i_1<i_2<\cdots<i_r\le n,\end{cases}$$

where all occurring brackets are variables. This system is the same system as that of BFP [3]. We choose this system for the following reasons (a non-realizability certificate based on SDP relaxation of the formulation in [15] is studied in [11]). First of all, the Grassmann-Plücker system has a rich algebraic structure. As demonstrating in Section 4, we can compute minimal equality constraints of this system efficiently using this structure. Secondly, BFP seems to be a very powerful certificate [14] and an SDP relaxation of this system may produce a more powerful non-realizability certificate. BFP is a method that proves non-realizability of oriented matroids as follows. First, we choose 3-term Grassmann-Plücker relations supposing that all occurring brackets and indices are sorted to be positive.

$$\begin{cases}[\tau,a,b][\tau,c,d]-[\tau,a,c][\tau,b,d]+[\tau,a,d][\tau,b,c]=0, \\ [\tau,a,b]>0,[\tau,c,d]>0,[\tau,a,c]>0,[\tau,b,c]>0,[\tau,a,d]>0,[\tau,b,c]>0 \\ \text{for all } a,b,c,d\in E,\tau\in E^{r-2}.\end{cases}$$

(For simplicity, we explain the uniform case). Then we make an LP system by relaxing conditions by $[\tau,a,b][\tau,c,d]<[\tau,a,c][\tau,b,d]$ and $[\tau,a,d][\tau,b,c]<[\tau,a,c][\tau,b,d]$, and by taking the logarithm of both sides of these inequalities and prove infeasibility of this LP system. However, this special LP relaxation seems difficult to be extended to an SDP relaxation. Hence in this paper we consider an SDP relaxation which is not an obvious extension of this special LP relaxation of BFP. Taking the problem size into account, we investigate the case of relaxing Grassmann-Plücker system with relaxation order 1. This is the first attempt to use SDP for a non-realizability certificate of oriented matroids.

We apply our method to oriented matroids in OM(8,4) and OM(9,3) which are not decided to be realizable by the existing methods using Finschi and Fukuda's database of oriented matroids [5] and the SDP solver SeDuMi [18, 20]. Our method with relaxation order 1 decides non-realizability of 440 oriented matroids in OM(8,4) and none of oriented matroids in OM(9,3). Consequently we find that it cannot find new non-realizable oriented matroids, but finds non-realizable oriented matroids which are not decided by non-Euclidean property or non-HK* property.

Our main result. Relation between our method with relaxation order 1 and the existing methods is as in Figure 3.

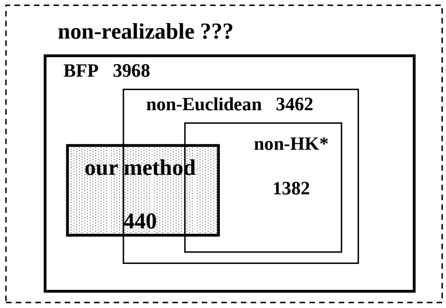

Figure 3: Comparison between our method and the existing methods (8 elements and rank 4)

Although our method with relaxation order 1 is weaker than BFP for OM(8,4) and OM(9,3), this method with higher relaxation order may be available as a more powerful method. Actually, observing dual solutions, we notice that in many cases, one can prove non-realizability by much smaller number of constraints (This remarkable property is never seen in [11]). Furthermore, if a given oriented matroid has some symmetries, one can apply techniques used in [9] to reduce problem size.

2 Preliminaries on Oriented Matroids

In this section, we explain the concept of oriented matroids and the realizability problem briefly. For more details, see [1].

Consider a vector configuration $V = (v_i)_{i=1}^n \in \mathbb{R}^{r\times n}$. Then, it satisfies Grassmann-Plücker relations:

$$[i_1...i_r][j_1...j_r] - \sum_{s=1}^{r} [j_s i_2...i_r][j_1...j_{s-1} i_1 j_{s+1}...j_r] = 0,$$

where $[i_1, ..., i_r] := \det(v_{i_1}, ..., v_{i_r})$. Abstracting values of the $r \times r$ minors to sign patterns, we obtain a chirotope χ that satisfies the following axioms.

Definition 2.1. Let E be a finite set and $r \geq 1$ an integer. A chirotope of rank r on E is a mapping $\chi : E^r \to \{+1, -1, 0\}$ which satisfies the following properties.

1. χ is not identically zero.
2. $\chi(i_{\sigma(1)}, ..., i_{\sigma(r)}) = \mathrm{sgn}(\sigma)\chi(i_1, ..., i_r)$ for all $i_1, ..., i_r$ and every permutation σ.
3. For all $i_1, ..., i_r, j_1, ..., j_r \in E$ such that

$$\chi(j_s, i_2, ..., i_r) \cdot \chi(j_1, ..., j_{s-1}, i_1, j_{s+1}, ..., j_r) \geq 0$$

for $s = 1, ..., r$, we have

$$\chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) \geq 0.$$

We define an oriented matroid as a pair of a finite set E and a chirotope $\chi : E^r \to \{+1, -1, 0\}$, and it is called a uniform oriented matroid if $\chi(i_1, ..., i_r) \neq 0$ for all $1 \leq i_1 < ... < i_r \leq n$. If $|E| = n$, we call the pair (E, χ) rank-r oriented matroid with n elements.

By definition, oriented matroids can be regarded as an abstraction of vector configurations. If an oriented matroid has a corresponding vector configuration, it is called a realizable oriented matroid. The precise definition of a realizable oriented matroid is stated as follows.

Definition 2.2. (The realizability of oriented matroids)
Let $M = (E, \chi)$ be a rank-r oriented matroid with n elements. If there exists a vector configuration $V = (v_i)_{i=1}^n \in \mathbb{R}^{r \times n}$ such that

$$\mathrm{sign}(\det(v_{i_1}, v_{i_2}, ..., v_{i_r})) = \chi(i_1, i_2, ..., i_r)$$

for all $i_1, ..., i_r \in E$, M is said to be realizable, otherwise non-realizable.

3 Formulating Non-realizability Certificates by SDP

In this section, we formulate non-realizability certificates of oriented matroids as an infeasibility certificate of SDP. From now on, we will use the following notations throughout the paper.

- $A \bullet B := \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices.
  $S_+^n := \{X \mid X \text{ is an } n \times n \text{ positive semidefinite symmetric matrix}\}$.
- $\Lambda(n, r) := \{[i_1 ... i_r] \mid 1 \leq i_1 < ... < i_r \leq n\}$.
- $[i_{\sigma(1)} i_{\sigma(2)} ... i_{\sigma(r)}] := \mathrm{sgn}(\sigma)[i_1 i_2 ... i_r]$, where $[i_1 i_2 ... i_r] \in \Lambda(n, r)$ and σ is a permutation on $\{1, ..., r\}$.
- $[i_1 ... i_r] := 0$, where there exist $k, l \in \{1, ..., r\}$ $(k \neq l)$ that satisfy $i_k = i_l$.
- $K[x_1, ..., x_n]$: The polynomial ring over a field K in n variables $x_1, ..., x_n$.
- $T(\{x_1, ..., x_n\}, d) := \{x_1^{\alpha_1} ... x_n^{\alpha_n} \mid \alpha_1 + ... + \alpha_n \leq d, \alpha_1, ..., \alpha_n \in \mathbb{Z}_{\geq 0}\}$

Let $M = (E, \chi)$ be an oriented matroid of rank r where $|E| = n$. Suppose that M is realizable. Then there exists a vector configuration $V = (v_i)_{i=1}^n \in \mathbb{R}^{r \times n}$ such that for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$\begin{cases} [i_1 ... i_r][j_1 ... j_r] - \sum_{k=1}^r [j_k i_2 ... i_r][j_1 ... j_{k-1} i_1 j_{k+1} ... j_r] = 0, \\ \qquad\qquad\qquad\qquad \text{(Grassmann-Plücker relations)} \\ \mathrm{sign}([i_1 ... i_r]) \quad = \quad \chi(i_1, ..., i_r) \\ \qquad\qquad\qquad\qquad \text{(Conditions of chirotope)} \end{cases}$$

where $[i_1...i_r] := \det(v_{i_1}, ..., v_{i_r})$. Then, we rewrite the above system as follows. For $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$\begin{cases} x_{(i_1,...,i_r),(j_1,...,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 ... i_r),(j_1...j_{k-1} i_1 j_{k+1}...j_r)} = 0, \\ \mathrm{sign}(x_{(i_1,...,i_r),(j_1,...,j_r)}) = \chi(i_1,...,i_r) \cdot \chi(j_1,...,j_r), \\ \mathrm{sign}(x_{(i_1,...,i_r)}) = \chi(i_1,...,i_r), \\ x_{(i_1,...,i_r),(j_1,...,j_r)} = [i_1...i_r][j_1...j_r], \\ x_{(i_1,...,i_r)} = [i_1...i_r] \end{cases}$$

where $x_{(i_{\sigma(1)},...,i_{\sigma(r)}),(j_{\tau(1)},...,j_{\tau(r)})}$ denotes $\mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) \cdot x_{(i_1,...,i_r),(j_1,...,j_r)}$ for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$, and σ and τ are permutations on $\{1, ..., r\}$.

Here we consider a vector u_1 given by listing 1 and elements of $\Lambda(n, r)$ in lexicographic order:

$$u_1 := (1, [1, 2, ..., r], [1, 2, ..., r-1, r+1], ..., [n-r, n-r+1, ..., n])^T.$$

Then, under the constraints $x_{(i_1,...,i_r),(j_1,...,j_r)} = [i_1...i_r][j_1...j_r]$ and $x_{(i_1,...,i_r)} = [i_1...i_r]$ for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$X := \begin{pmatrix} 1 & x_{(1,2,...,r)} & \cdots & x_{(n-r,n-r+1,...,n)} \\ x_{(1,2,...,r)} & x_{(1,2,...r),(1,2,...,r)} & \cdots & x_{(1,2,...,r),(n-r,...,n)} \\ x_{(1,2,...,r-1,r+1)} & x_{(1,2,...,r-1,r+1),(1,2,...,r)} & \cdots & x_{(1,2,...,r-1,r+1),(n-r,...,n)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(n-r,...,n)} & x_{(n-r,...,n),(1,2,...,r)} & \cdots & x_{(n-r,...,n),(n-r,...,n)} \end{pmatrix}$$

$(= u_1 u_1^T)$ should be a positive semidefinite symmetric matrix. Therefore if M is realizable, the following system (SDP A) is feasible: for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$\begin{cases} x_{(i_1,...,i_r),(j_1,...,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 ... i_r),(j_1...j_{k-1} i_1 j_{k+1}...j_r)} = 0, \\ \mathrm{sign}(x_{(i_1,...,i_r),(j_1,...,j_r)}) = \chi(i_1,...,i_r) \cdot \chi(j_1,...,j_r), \\ \mathrm{sign}(x_{(i_1,...,i_r)}) = \chi(i_1,...,i_r), \\ X \text{ is a positive semidefinite symmetric matrix,} \end{cases} \qquad \text{(SDP A)}$$

where $x_{(i_{\sigma(1)},...,i_{\sigma(r)}),(j_{\tau(1)},...,j_{\tau(r)})}$ denotes $\mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) \cdot x_{(i_1,...,i_r),(j_1,...,j_r)}$ for all $i_1, ..., i_r, j_1, ..., j_r$ for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$, and σ and τ are permutations on $\{1, ..., r\}$.

By eliminating the constraints $\mathrm{sign}(x_{(i_1,...,i_r)}) = \chi(i_1, ..., i_r)$ and variables $x_{(i_1,...,i_r)}$ for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$, we obtain the following system (SDP B). For $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$\begin{cases} x_{(i_1,...,i_r),(j_1,...,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 ... i_r),(j_1...j_{k-1} i_1 j_{k+1}...j_r)} = 0, \\ \mathrm{sign}(x_{(i_1,...,i_r),(j_1,...,j_r)}) = \chi(i_1,...,i_r) \cdot \chi(j_1,...,j_r), \\ X' \text{ is a positive semidefinite symmetric matrix} \end{cases} \qquad \text{(SDP B)}$$

where

$$X' := \begin{pmatrix} x_{(1,2,...r),(1,2,...,r)} & \cdots & x_{(1,2,...,r),(n-r,n-r+1,...,n)} \\ x_{(1,2,...,r-1,r+1),(1,2,...,r)} & \cdots & x_{(1,2,...,r-1,r+1),(n-r,n-r+1,...,n)} \\ \vdots & \ddots & \vdots \\ x_{(n-r,n-r+1,...,n),(1,2,...,r)} & \cdots & x_{(n-r,n-r+1,...,n),(n-r,n-r+1,...,n)} \end{pmatrix}.$$

Proposition 3.1. (SDP A) *is feasible if and only if* (SDP B) *is feasible.*

Before proving this proposition, we explain some terminologies. A polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ is said to be square-free if for every monomials $x_1^{\alpha_1} ... x_n^{\alpha_n}$ in f, $\alpha_1, ..., \alpha_n$ are equal to 0 or 1. A matrix X is called a strictly feasible solution of an SDP system if X is a feasible solution of the SDP system and is positive definite.

Proof of Proposition 3.1. Sufficiency is trivial and we prove necessity. Let $\hat{X}$ be a feasible solution of (SDP B). Taking into account that all Grassmann-Plücker relations are square-free, we obtain a strictly feasible solution $\bar{X} := \hat{X} + I$, where I is a unit matrix whose size is equal to that of $\hat{X}$. Then, we see that

$$\begin{pmatrix} 1 & \chi(1, 2, ..., r) \cdot \epsilon & \cdots & \chi(n-r, ..., n) \cdot \epsilon \\ \chi(1, 2, ..., r) \cdot \epsilon & & & \\ \vdots & & \bar{X} & \\ \chi(n-r, , ..., n) \cdot \epsilon & & & \end{pmatrix}$$

is a solution of (SDP A) for sufficiently small $\epsilon > 0$. □

If X is a feasible solution of (SDP B), αX is also a solution of (SDP B) for all $\alpha > 0$. Therefore the following system is feasible if and only if (SDP B) is feasible: for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$\begin{cases} x_{(i_1,...,i_r),(j_1,...,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 ... i_r),(j_1 ... j_{k-1} i_1 j_{k+1} ... j_r)} = 0, & \\ x_{(i_1,...,i_r),(j_1,...,j_r)} - 1 \geq 0 & \text{if } \chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) = +, \\ x_{(i_1,...,i_r),(j_1,...,j_r)} = 0 & \text{if } \chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) = 0, \\ x_{(i_1,...,i_r),(j_1,...,j_r)} + 1 \geq 0 & \text{if } \chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) = -, \\ X' \text{ is a positive semidefinite symmetric matrix,} & \end{cases}$$

We transform the above system to a standard form of SDP by adding slack variables. For $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$,

$$\begin{cases} x_{(i_1,...,i_r),(j_1,...,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 ... i_r),(j_1 ... j_{k-1} i_1 j_{k+1} ... j_r)} = 0, & \\ x_{(i_1,...,i_r),(j_1,...,j_r)} - 1 = y_{(i_1,...,i_r),(j_1,...,j_r)} & \text{if } \chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) = +1, \\ x_{(i_1,...,i_r),(j_1,...,j_r)} = 0 & \text{if } \chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) = 0, \\ x_{(i_1,...,i_r),(j_1,...,j_r)} + 1 = -y_{(i_1,...,i_r),(j_1,...,j_r)} & \text{if } \chi(i_1, ..., i_r) \cdot \chi(j_1, ..., j_r) = -1 \\ y_{(i_1,...,i_r),(j_1,...,j_r)} \geq 0, & \\ X' \text{ is a positive semidefinite symmetric matrix,} & \end{cases}$$

where $x_{(i_{\sigma(1)},...,i_{\sigma(r)}),(j_{\tau(1)},...,j_{\tau(r)})}$ denotes $\text{sgn}(\sigma) \cdot \text{sgn}(\tau) \cdot x_{(i_1,...,i_r),(j_1,...,j_r)}$ for $1 \leq i_1 < ... < i_r \leq n$ and $1 \leq j_1 < ... < j_r \leq n$, and σ and τ are permutations on $\{1, ..., r\}$.

Then, if one proves that the above SDP system has no feasible solution, M is decided to be non-realizable.

The relaxation explained above corresponds with the SDP relaxation of a polynomial system with relaxation order 1 which was introduced in [10]. When we construct an SDP relaxation with relaxation order d, we make equalities and inequalities system as follows.

$$\begin{cases} ([i_1...i_r][j_1...j_r] - \sum_{k=1}^{r} [j_k i_2...i_r][j_1...j_{k-1} i_1 j_{k+1}...j_r])m \\ \qquad\qquad\qquad =: \sum_{m' \in T(\Lambda(n,r),2d)} c_{m'}^{(i_1,...,i_r,j_1,...,j_r,m)} m' = 0 \\ \text{for } 1 \le i_1 < ... < i_r \le n,\ 1 \le j_1 < ... < j_r \le n, \text{and } m \in T(\Lambda(n,r), 2d-2), \\ \text{sign}([k_{1,1}...k_{1,r}]...[k_{l,1}...k_{l,r}]) = \chi(k_{1,1},...,k_{1,r}) \cdot \ldots \cdot \chi(k_{l,1},...,k_{l,r}), \\ [k_{1,1}...k_{1,r}]...[k_{l,1}...k_{l,r}] = y_{[k_{1,1}...k_{1,r}]...[k_{l,1}...k_{l,r}]} \\ \quad \text{for } 1 \le k_{1,1} < ... < k_{1,r} \le n,\ ...,\ 1 \le k_{2d,1} < ... < k_{2d,r} \le n,\ \text{and } l \le 2d. \end{cases}$$

Then, we consider a vector u_d given by listing all elements of $T(\Lambda(n,r), d)$ by lexicographic order:

$$u_d := (1, [1,2,...,r], ..., [n-r,...,n], [1,2,..,r]^2, ..., [n-r,...,n]^d)^T.$$

Under the constraints $[k_{1,1}...k_{1,r}]...[k_{l,1}...k_{l,r}] = y_{[k_{1,1},...,k_{1,r}]...[k_{l,1},...,k_{l,r}]}$ for $1 \le k_{1,1} < ... < k_{1,r} \le n,\ ...,\ 1 \le k_{l,1} < ... < k_{l,r} \le n$, a matrix $Y = (y_{l,m})_{l,m \in T(\Lambda(n,r),2d)}$ is equal to $u_d u_d^T$ and should be positive semidefinite. Therefore if M is realizable, the following SDP system is feasible.

$$\begin{cases} \sum_{m' \in T(\Lambda(n,r),2d)} c_{m'}^{(i_1,...,i_r,j_1,...,j_r,m)} y_{m'} = 0 \\ \text{for } 1 \le i_1 < ... < i_r \le n,\ 1 \le j_1 < ... < j_r \le n, \text{and } m \in T(\Lambda(n,r), 2d-2), \\ \text{sign}(y_{[k_{1,1}...k_{1,r}]...[k_{l,1}...k_{l,r}]}) = \chi(k_{1,1},...,k_{1,r}) \cdot \ldots \cdot \chi(k_{l,1},...,k_{l,r}) \\ \quad \text{for } 1 \le k_{1,1} < ... < k_{1,r} \le n,\ ...,\ 1 \le k_{2d,1} < ... < k_{2d,r} \le n, \text{and } l \le 2d, \\ Y = (y_{l,m})_{l,m \in T(\Lambda(n,r),2d)} \text{ is a positive semidefinite symmetric matrix.} \end{cases}$$

Hence, if we can prove infeasibility of the above SDP system, M is proved to be non-realizable.

If we use the higher order relaxation, we will obtain a more powerful certificate. However, the problem size will become very large. Therefore in this paper, we focus on the SDP relaxation with relaxation order 1.

4 Eliminating Redundancy of Constraints

In the previous section, we formulated a non-realizability certificate by SDP. However, the problem size becomes very large even if the number of elements and the rank of a given oriented matroid are small. About 2000 variables and 3000 constraints are needed for OM(8,4), and about 3000 variables and 4500 constraints are needed for OM(9,3). Therefore it is very important to know which constraints are really needed. In this section, we discuss this issue. To treat the case of higher order relaxation together, we investigate redundancy of the original polynomial system, not that of SDP system.

First, we consider inequality constraints. Although discovering redundant inequalities is sometimes achieved easily (for example, $y > 0$ is redundant for the system $x - y + z = 0, x > 0, y > 0, z > 0$), in general it is as hard task as the problem to decide emptiness of semialgebraic sets. On the other hand, redundancy of equations can be discussed in a relatively simple way by considering generating sets of the corresponding ideals. Therefore we discuss redundancy of equalities in this section. To investigate redundancy of Grassmann-Plücker relations, we explain Grassmann-Plücker ideal well known in invariant theory [21, 22]. We will explain terminologies in invariant theory such as *K-algebra homomorphism, K-vector space, total degree* and *homogeneous polynomial* when we use these terminologies.

Definition 4.1. ([21, 22])
For a field K, let $f : K[\Lambda(n,r)] \to K[x_{11},...,x_{1r},x_{21},...,x_{nr}]$ be the K-algebra homomorphism that takes $[i_1...i_r]$ to $\det\begin{pmatrix} x_{i_1 1} & \cdots & x_{i_r 1} \\ \vdots & \ddots & \vdots \\ x_{i_1 r} & \cdots & x_{i_r r}\end{pmatrix}$. Then, $I_{n,r} := ker(f)$ is called Grassmann-Plücker ideal.

A map $\phi : K[\Lambda(n,r)] \to K[x_{11},...,x_{1r},x_{21},...,x_{nr}]$ is called a K-algebra homomorphism if it satisfies $\phi(a+b) = \phi(a)+\phi(b)$, $\phi(sab) = s\phi(a)\phi(b)$ for all $a,b \in K[\Lambda(n,r)]$ and $s \in K$. Thus, f described above is the map that substitutes $[i_1...i_r]$ with $\det\begin{pmatrix} x_{i_1 1} & \cdots & x_{i_r 1} \\ \vdots & \ddots & \vdots \\ x_{i_1 r} & \cdots & x_{i_r r}\end{pmatrix}$ and Grassmann-Plücker ideal $I_{n,r}$ is the set of polynomial identities which consist of the $r \times r$ minors of $\begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1r} & \cdots & x_{nr}\end{pmatrix}$.

Theorem 4.2. (The Second Fundamental Theorem of Invariant Theory [21])
Grassmann-Plücker ideal $I_{n,r}$ is generated by Grassmann-Plücker polynomials:

$$\sum_{s=1}^{r+1} (-1)^s [i_1...i_{r-1}j_s][j_1...j_{s-1}j_{s+1}...j_{r+1}],$$

for $1 \le i_1 < ... < i_{r-1} \le n, 1 \le j_1 < ... < j_{r+1} \le n$.

By this theorem, all polynomial equations which consist of the generic $r \times r$ minors are generated by Grassmann-Plücker relations. However, not all Grassmann-Plücker relations are needed. We will show an example of redundant relations.

$$\begin{aligned}
&[123][456] - [124][356] + [125][346] - [126][345] = 0 \quad \cdots (1)\\
&[123][456] - [124][356] + [134][256] - [156][234] = 0 \quad \cdots (2)\\
&[125][346] - [126][345] - [134][256] + [156][234] = 0 \quad \cdots (3)
\end{aligned}$$

Because (1)−(2)=(3), the above relations have redundancy.

To obtain a set of independent relations, we consider a minimal generating set of $I_{n,r}$. In general, computing a minimal generating set of ideals is a very hard task, but it is not too difficult in the case of $I_{n,r}$ because it has the generating set which consists of quadratic *homogeneous* polynomials. Before demonstrating how to compute a minimal generating set efficiently, we explain some definitions. The total degree of a monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ is defined to be $\alpha_1 + \cdots + \alpha_n$ and the total degree of a polynomial is a maximum total degree of monomials which appear in the polynomial. A polynomial is said to be homogeneous if all monomials which appear in the polynomial have the same total degree. In addition, we use the terminology *K-vector space spanned by a set $S \subset K[\Lambda(n,r)]$* for the following vector space:

$$\{a_1s_1 + \cdots + a_Ms_M \mid a_1,...,a_M \in K, s_1,...,s_M \in S, M \in \mathbb{N}\}.$$

Using these terminologies, we explain an algorithm to compute a minimal generating set of $I_{n,r}$ efficiently.

Proposition 4.3. *For a field K, let $h, g_1, g_2, ..., g_m \in K[x_1, ..., x_n]$ be homogeneous polynomials whose total degrees are d. Then, $h \in \langle g_1, g_2, ..., g_m\rangle := \{b_1g_1 + \cdots + b_mg_m \mid b_1, ..., b_m \in K[x_1, ..., x_n]\}$ if and only if there exist $a_1, ..., a_m \in K$ such that $h = a_1g_1 + a_2g_2 + \cdots + a_mg_m$.*

Proof. Sufficiency is trivial and we prove necessity. If $h \in \langle g_1, g_2, ..., g_m\rangle$, we obtain $h - (b_1g_1 + b_2g_2 + \cdots + b_mg_m) = 0$ for $b_1, b_2, \cdots, b_m \in K[x_1, ..., x_n]$. Then, we obtain the proposition by looking at the part of total degree d. □

By this proposition, a minimal generating set of $I_{n,r}$ is equal to a basis of the K-vector space spanned by Grassmann-Plücker polynomials, denoted by $V_{n,r}$. It can be computed efficiently using Gaussian Elimination. Furthermore, $V_{n,r}$ can be decomposed to $V_{n,r} = \bigoplus_A V_{n,r}^{(A)}$, where $\bigoplus$ denotes a direct sum, A runs over $\{k_1, ..., k_r, l_1, ..., l_r\}$ that satisfies $[k_1...k_r], [l_1...l_r] \in \Lambda(n, r)$, and $V_{n,r}^{(\{k_1,...,k_r,l_1,...,l_r\})}$ is the K-vector space spanned by Grassmann-Plücker polynomials:

$$\sum_{s=1}^{r+1} (-1)^s [i_1...i_{r-1}j_s][j_1...j_{s-1}j_s...j_{r+1}]$$

that satisfy $\{i_1, ..., i_{r-1}, j_1, ..., j_{r+1}\} = \{k_1, ..., k_r, l_1, ..., l_r\}$. It is because $V_{n,r}^{(A)}$ and $V_{n,r}^{(A')}$ do not include common monomials if $A \neq A'$. Therefore one can find a minimal generating set of $I_{n,r}$ more efficiently by computing bases of each $V_{n,r}^{(A)}$ separately. Further observe, as $V_{n,r}^{(\sigma\cdot A)} = \sigma \cdot V_{n,r}^{(A)}$, where σ is a permutation on $\{1, 2, ..., n\}$, we only need to know bases of $V_{n,r}^{(\{1,2,...,r+2\})}, ..., V_{n,r}^{(\{1,2,...,\min\{n,2r\}\})}$.

Algorithm 4.4. (Computing a minimal generating set of $I_{n,r}$)

1. For $m = r + 2, r + 3, ..., \min\{n, 2r\}$ do
   $T_m := \{\sum_{s=1}^{r+1} (-1)^s [i_1...i_{r-1}j_s][j_1...j_{s-1}j_{s+1}...j_{r+1}] \mid \{i_1, ..., i_{r+1}, j_1, ..., j_{r-1}\} = \{1, ..., m\}, 1 \leq i_1 < i_2 < ... < i_{r+1} \leq n, 1 \leq j_1 < j_2 < ... < j_{r-1} \leq n\}$ and compute a basis B_m of the K-vector space spanned by T_m using Gaussian Elimination.
2. For $m = r + 2, r + 3, ..., \min\{n, 2r\}$ do
$$U_m := \bigcup_{1\leq k_1<...<k_m\leq n} \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} \cdot T_m.$$
3. Output $\bigcup_{m=r+1}^{\min\{n,2r\}} U_m$.

Using the above algorithm, we obtain systems whose numbers of constraints are about 2000 for OM(8,4), and 4000 for OM(9,3). We notice that the above algorithm preserves the symmetry of the Grassmann-Plücker system. It will be desirable when we apply techniques used in [9].

Example. (Computing a minimal generating set of $I_{6,3}$)
By the Second Fundamental Theorem of Invariant Theory, $I_{6,3}$ is generated by $p_{i_1i_2,j_1j_2j_3j_4} := [i_1i_2j_1][j_2j_3j_4] - [i_1i_2j_2][j_1j_3j_4] + [i_1i_2j_3][j_1j_2j_4] - [i_1i_2j_4][j_1j_2j_3]$,
for $1 \leq i_1 < i_2 \leq 6$ and $1 \leq j_1 < j_2 < j_3 < j_4 \leq 6$. To obtain a minimal generating set of $I_{6,3}$, we compute a basis of the k-vector space $V_{6,3}$ spanned by the above polynomials.

First, look at the monomials in $p_{i_1 i_2, j_1 j_2 j_3 j_4}$. We notice that sets of indices which appear in each monomial are all $\{i_1, i_2, j_1, j_2, j_3, j_4\}$. Then, we classify the above polynomials into the following seven groups by index sets.

$$\begin{array}{lll}
\{1,2,3,4,5,6\} & : \{p_{12,3456}, p_{13,2456}, p_{14,2345}, p_{15,2346}, ..., p_{56,1234}\} & (=: G_{123456}) \\
\{1,2,3,4,5\} & : \{p_{12,1345}, p_{12,2345}, p_{13,1245}, p_{13,2345}, ..., p_{45,1235}\} & (=: G_{12345}) \\
\{1,2,3,4,6\} & : \{p_{12,1346}, p_{12,2346}, p_{13,1246}, p_{13,2346}, ..., p_{46,1236}\} & (=: G_{12346}) \\
\{1,2,3,5,6\} & : \{p_{12,1356}, p_{12,2356}, p_{13,1256}, p_{13,2356}, ..., p_{56,1236}\} & (=: G_{12356}) \\
\{1,2,4,5,6\} & : \{p_{12,1456}, p_{12,2456}, p_{14,1256}, p_{14,2456}, ..., p_{56,1246}\} & (=: G_{12456}) \\
\{1,3,4,5,6\} & : \{p_{13,1456}, p_{13,3456}, p_{14,1356}, p_{14,3456}, ..., p_{56,1346}\} & (=: G_{13456}) \\
\{2,3,4,5,6\} & : \{p_{23,2456}, p_{23,3456}, p_{24,2356}, p_{24,3456}, ..., p_{45,1235}\} & (=: G_{23456})
\end{array}$$

We see that the elements of G_{123456} cannot be written as a linear combination of the elements of $G_{12345}, G_{12346}, ..., G_{23456}$. Therefore we can compute a maximal linear independent set of the whole system by merging maximal linear independent sets of each group. Thus, we consider linear dependency of each group separately. First, G_{123456} has the following linear dependencies:

$$\begin{array}{ll}
p_{12,3456} + p_{34,1256} + p_{56,1234} = 0, & p_{13,2456} + p_{25,1346} + p_{46,1235} = 0, \\
p_{14,2356} + p_{26,1345} + p_{35,1246} = 0, & p_{15,2346} + p_{24,1356} + p_{36,1245} = 0, \\
p_{16,2345} + p_{23,1456} + p_{45,1236} = 0. &
\end{array}$$

Eliminating these dependencies, we obtain a maximal linearly independent set:

$$\begin{aligned}
G'_{123456} = \{&p_{12,3456}, p_{13,2456}, p_{14,2356}, p_{15,2346}, p_{16,2345}, \\
&p_{34,1256}, p_{25,1346}, p_{26,1345}, p_{24,1356}, p_{23,1456}\}.
\end{aligned}$$

On the other hand, G_{12345} has no linear dependency. Next, let us focus on the following relation.

$$\begin{aligned}
p_{12,1346} &= [123][146] - [124][136] + [126][134] \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix} \cdot ([123][145] - [124][135] + [125][134]) \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix} \cdot p_{12,1345},
\end{aligned}$$

where the symmetry group S_6 acts on $K[[123],[124],...,[456]]$ by the linear extension of the following relations.

$$\sigma \cdot [i_1 j_1 k_1] \cdots [i_m j_m k_m] = [\sigma(i_1)\sigma(j_1)\sigma(k_1)] \cdots [\sigma(i_m)\sigma(j_m)\sigma(k_m)]$$

for $i_1, .., i_m, j_1, ..., j_m, k_1, ..., k_m \in \{1,2,3,4,5,6\}$ and $\sigma \in S_6$.
In the same way,

$$\begin{aligned}
G_{12346} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix} \cdot G_{12345}, \\
G_{12356} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & 4 \end{pmatrix} \cdot G_{12345}, \\
&\vdots \\
G_{23456} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} \cdot G_{12345},
\end{aligned}$$

where the symmetry group S_6 acts on the power set of $K[[123],[124],...,[456]]$ by

$$\sigma \cdot \{p_1,...,p_m\} = \{\sigma \cdot p_1,...,\sigma \cdot p_m\}$$

for $p_1,...,p_m \in K[[123],[124],...,[456]]$ and $\sigma \in S_6$.
It follows that $G_{12346}, G_{12356}, G_{12456}, G_{13456}, G_{23456}$ have no linear dependency, too. (Recall that G_{12345} has no linear dependency.) Therefore we conclude that $G'_{123456} \cup G_{12345} \cup G_{12346} \cup ... \cup G_{23456}$ is a minimal generating set of $I_{6,3}$.

Remark. Proposition 4.3 also implies that adding redundant equations whose total degrees are two to Grassmann-Plücker systems never strengthens SDP relaxations because these redundant equations can be written as linear combinations of Grassmann-Plücker relations. Similarly, eliminating redundant equations from Grassmann-Plücker systems does not weaken SDP relaxations if one considers the case of relaxation order 1.

5 Proving Infeasibility of SDP Systems

In Section 3, we formulated a non-realizability certificate by SDP, but we cannot discern infeasibility of the SDP system by the interior point method directly because of numerical errors. In this section, we provide a way to resolve this issue.

To discern infeasibility of SDP system

$$(a) \quad A_i \bullet X = b_i \ (i = 1, 2, ..., m), X \in S^n_+,$$

we consider the following SDP system.

$$(b) \quad \begin{array}{ll} \text{Primal SDP:} & \begin{array}{l} \min s \\ \begin{pmatrix} A_i & 0 \\ 0 & b_i \end{pmatrix} \bullet \begin{pmatrix} X & 0 \\ 0 & s \end{pmatrix} = b_i \ (i = 1, 2, ..., m) \\ \begin{pmatrix} X & 0 \\ 0 & s \end{pmatrix} \in S^{n+1}_+. \end{array} \\ \text{Dual SDP:} & \begin{array}{l} \max \sum_{i=1}^m b_i z_i \\ Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \sum_{i=1}^m \begin{pmatrix} A_i & 0 \\ 0 & b_i \end{pmatrix} z_i \in S^{n+1}_+. \end{array} \end{array}$$

Note that this SDP system has a trivial primal solution $X = 0, s = 1$ and a trivial dual solution $(Y, z_1, ..., z_m) = (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0, ..., 0)$, and (a) is feasible if and only if the primal optimal value of (b) is 0. Therefore if one certifies the primal optimal value of (b) is larger than 0, one can certify that (a) is infeasible. However, an exact primal optimal value cannot be obtained efficiently, and we search a dual solution of (b) proving that the primal optimal value of (b) is larger than 0 as follows. We apply the interior point method for SDP to (b), and obtain an approximately dual feasible solution (z^*, Y^*), at which objective function may take large value and obtain an exact dual feasible solution (z^{**}, Y^{**}) of (b), which is sufficiently close to (z^*, Y^*). We compute (z^{**}, Y^{**}) by substituting z^*'s entries that are nearly 0 with exact 0 and putting Y^{**} by $Y^{**} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \sum_{i=1}^m \begin{pmatrix} A_i & 0 \\ 0 & b_i \end{pmatrix} z_i^{**}$ (Of course, this way does not always produce a dual feasible solution, but really works well in our particular case in which z^{**} is often very sparse. See also next section). If (z^{**}, Y^{**}) is really dual feasible and the object function's value at (z^{**}, Y^{**}) is larger than 0, the primal optimal value of (b) is larger than 0 by weak duality. In other words, one can decide that (a) is infeasible.

6 Experimental Results

Using SeDuMi [18, 20] and Finschi and Fukuda's database of oriented matroids [5], we apply our method to 3968 oriented matroids in OM(8,4) decided to be non-realizable by non-HK* property [8], non-Euclidean property [7] or BFP [3] and 5321 oriented matroids that could not be decided to be realizable or non-realizable by the above methods, heuristics [13], a solvability sequence [2], a reduction sequence (not-isolated point) [17] or a realizability certificate using POP and generalized mutation graphs [13, 15]. Our method with relaxation order 1 decides non-realizability of 440 oriented matroids of the 3968 ones and no oriented matroids of the 5321 ones in OM(8,4). In the same way, we apply our method to 8822 oriented matroids in OM(9,3) which were not decided to be realizable by the existing methods, but none of those is decided to be non-realizable by our method.

As a result, the number of oriented matroids decided to be non-realizable by our method with relaxation order 1 is less than that of oriented matroids decided to be non-realizable by BFP, but found non-realizable oriented matroids which were not decided to be non-realizable by non-Euclidean property or non-HK* property (See Figure 3). IC(8,4,000018) in Finschi and Fukuda's database [5] is an example of an oriented matroid which is decided to be non-realizable by our method, but cannot be decided to be non-realizable by non-Euclidean property or non-HK* property. On the other hand, we observe the following property. In many cases, while we solve an SDP with about 2000 constraints, the dual solutions that prove non-realizability consist of many zero-entries and few non-zero entries, 8-48. This property, which is never seen in [11], implies that we often need a very small number of constraints to prove non-realizability and may be able to apply our method with higher relaxation order.

7 Conclusion and Future Works

In this paper, we proposed a new non-realizability certificate of oriented matroids by using SDP relaxation of Grassmann-Plücker relations. We have showed that our method with relaxation order 1 is weaker than the existing method BFP for OM(8,4) and OM(9,3), but found non-realizable rank-4 oriented matroids with 8 elements which were not decided to be non-realizable by non-Euclidean property or non-HK* property.

We have considered SDP relaxations of Grassmann-Plücker relations with relaxation order 1, but we expect that we can obtain a more powerful certificate with higher relaxation order by reducing the problem sizes. To reduce the problem sizes, the following issues remain as future works.

1. Developing techniques to predict Grassmann-Plücker relations unnecessary to prove non-realizability.
   By experiments, we have observed that the number of Grassmann-Pücker relations necessary to prove non-realizability is often very small. Therefore if predicting some of unnecessary relations, one can reduce the problem size largely.
2. Exploiting symmetry and reducing the sizes of SDP.
   Some oriented matroids have non-trivial symmetries and they induce the same sym-

metries to the polynomial systems which describe the conditions of the realizability. Therefore reducing sizes of SDP using a technique introduced in [9] may be hopeful.

Acknowledgements

The authors would like to thank Professor Komei Fukuda for suggesting to study a new non-realizability certificate by semidefinite programming. They are also thankful to Dr. Hiroki Nakayama for providing his data of realizable oriented matroids and non-realizable ones. They would also like to express their appreciation to anonymous referees for many valuable suggestions to improve earlier drafts of this manuscript.

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Manuscript received 29 February 2008
revised 21 October 2008
accepted for publication 24 October 2008

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