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QUASIEFFICIENT SOLUTIONS OF MULTICRITERIA LOCATION PROBLEMS WITH RECTILINEAR NORM IN $\mathbb{R}^n$

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Abstract: A multicriteria location problem with rectilinear norm in $\mathbb{R}^n$ and quasiefficient solutions of the problem are considered. In applications, for example, the problem is important and applicable to the development of new products. Algorithms to find all quasiefficient solutions of the problem in $\mathbb{R}^2$ and $\mathbb{R}^3$ are known. However, the algorithms to find all quasiefficient solutions of the problem with $n \leq 3$ cannot be applied straightforwardly to the problem with $n > 3$. We give some properties of quasiefficient solutions of the problem. Based on the obtained properties and known algorithms to find all efficient solutions of the problem, we give a procedure to find all quasiefficient solutions of the problem. Furthermore, we give a numerical example.

Key words: *location problem, multicriteria problem, rectilinear norm, quasiefficiency*

Mathematics Subject Classification: *90B85*

1 Introduction

In a general continuous location model, finitely many points called demand points, modeling existing factories or customers, are given. The aim of decision makers is to decide the location of a new facility. In many situations, decision makers usually have conflicting objectives, e.g., on the priorities to be put on the demand points. Then the problem is naturally formulated as a multicriteria location problem. Distances between the facility to be located and demand points are considered as the typical objectives. Furthermore, rectilinear norm is considered as one of typical distance measures. In multicriteria location problems, one of main interests is to find efficient or quasiefficient solutions, which will be defined later and are very important concepts of solutions for general multicriteria optimization. In terminology for the efficiency, efficient solutions are also called Pareto optimal solutions, and quasiefficient solutions are also called weak Pareto optimal solutions or weak efficient solutions. For a comprehensive overview, see [11] and references therein. In this article, we consider quasiefficient solutions of multicriteria location problems with rectilinear norm. Note that sets of all quasiefficient solutions of multicriteria location problems are different according to distance measures which are used in the problems. Because decision making depends on the sets, these differences are important for applications. For example, Euclidean norm and rectilinear norm are very different in the sense of the difference of the sets.

In [8], an application to the development of new products is considered for an artificial data set by using efficient solutions of multicriteria location problems with block norm in $\mathbb{R}^2$. Block norm is a class of norms containing rectilinear norm as a special case. In [5], a public opinion survey on home education is considered for an actual data set, and new learning contents which are near required learning contents as much as possible are determined by using efficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^2$ and $\mathbb{R}^3$, where $\mathbb{R}^2$ and $\mathbb{R}^3$ correspond to $\mathbb{R}^n$ when $n = 2$ and $n = 3$, respectively. The limitations of $n = 2$ and $n = 3$ are due to the results of the quantification method of type III for the results of the questionnaire on home education. Because learning contents can be considered as products, it is a kind of the development of new products. Within our knowledge, it is the first application of location problems to the development of new products for an actual data set. The results in [5] suggest that rectilinear norm is suitable for measuring the differences between human preferences which are represented as points in $\mathbb{R}^n$, and that efficient and quasiefficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^n (n > 3)$ are important and applicable to the development of new products.

Algorithms to find all efficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^n (n > 3)$ were proposed, respectively, in [1], [4] and [6]. All quasiefficient solutions of multicriteria location problems with block norm in $\mathbb{R}^2$ can be determined by using an algorithm proposed in [3], and all quasiefficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^3$ can be determined by using an algorithm proposed in [7]. However, the algorithms to find all quasiefficient solutions of the problems with $n \leq 3$ cannot be applied straightforwardly to those with $n > 3$.

In this article, our main interest is to find *all* quasiefficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^n$. Of course, our results are available to the case $n = 1, 2$ and 3. However, it is trivial when $n = 1$, and one had better use algorithms in [3] and [7] when $n = 2$ and 3, respectively, because of the efficiency of algorithms. Our results are meaningful in the case $n > 3$.

In section 2, we formulate a multicriteria location problem, and give some properties of quasiefficient solutions of the problem and a procedure to find all quasiefficient solutions of the problem. In section 3, we give a numerical example. Finally, we give some conclusions in section 4.

2 Formulation and Quasiefficient Solutions

In this section, we formulate a multicriteria location problem, and give some properties of quasiefficient solutions of the problem and a procedure to find all quasiefficient solutions of the problem.

Given demand points in $\mathbb{R}^n$, a problem to locate a new facility in $\mathbb{R}^n$ is called a single facility location problem. The problem is usually formulated as a minimization problem with an objective function involving distances between the facility and demand points. It is assumed that m demand points $\boldsymbol{d}_i \equiv (d_i^1, d_i^2, \cdots, d_i^n)^T \in \mathbb{R}^n$, $i \in I \equiv \{1, 2, \cdots, m\}$ and rectilinear norm $\|\cdot\|_1$ defined on $\mathbb{R}^n$ are given. Let $\boldsymbol{x} \equiv (x^1, x^2, \cdots, x^n)^T \in \mathbb{R}^n$ be the variable location of the facility. We put $J \equiv \{1, 2, \cdots, n\}$ and $D \equiv \{\boldsymbol{d}_i \colon i \in I\}$. Our main problem is a multicriteria location problem formulated as follows:

$$\text{(P)} \qquad \min_{\boldsymbol{x} \in \mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}) \equiv (\|\boldsymbol{x} - \boldsymbol{d}_1\|_1, \|\boldsymbol{x} - \boldsymbol{d}_2\|_1, \cdots, \|\boldsymbol{x} - \boldsymbol{d}_m\|_1)^T.$$

(P) is a problem to find an efficient or quasiefficient solution. A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is called *an efficient solution of* (P) if there is no $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$ and $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{f}(\boldsymbol{x}_0)$, and $\boldsymbol{x}_0$ is called *a quasiefficient solution of* (P) if there is no $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{f}(\boldsymbol{x}_0)$, where $\boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{f}(\boldsymbol{x}_0)$ ($\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$) means that $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1 < \|\boldsymbol{x}_0 - \boldsymbol{d}_i\|_1$ ($\|\boldsymbol{x} - \boldsymbol{d}_i\|_1 \leq \|\boldsymbol{x}_0 - \boldsymbol{d}_i\|_1$, respectively) for all $i \in I$. Let $E(D)$ and $QE(D)$ be the set of all efficient solutions of (P) and the set of all quasiefficient solutions of (P), respectively. By the above definition, it can be seen that $D \subset E(D) \subset QE(D)$. Our aim is to find $QE(D)$. In order to characterize quasiefficient solutions of (P), we also consider a minisum location problem formulated as follows:

$$(\mathrm{P}_{\boldsymbol{\lambda}}) \qquad \min_{\boldsymbol{x}\in\mathbb{R}^n} g(\boldsymbol{x}) \equiv \sum_{i=1}^{m} \lambda^i \|\boldsymbol{x} - \boldsymbol{d}_i\|_1$$

where each λ^i, $i \in I$ is a non-negative weight associated with $\boldsymbol{d}_i$, and not all λ^i's are zero. We put $\boldsymbol{\lambda} \equiv (\lambda^1, \lambda^2, \cdots, \lambda^m)^T$.

First, recall the following relationships among efficient and quasiefficient solutions of (P) and optimal solutions of $(\mathrm{P}_{\boldsymbol{\lambda}})$.

Theorem 2.1 ([9]). *A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is an efficient solution of (P) if and only if $\boldsymbol{x}_0$ is an optimal solution of $(P_{\boldsymbol{\lambda}})$ for some $\boldsymbol{\lambda} > \boldsymbol{0}$, where $\boldsymbol{\lambda} > \boldsymbol{0}$ means that $\lambda^i > 0$ for all $i \in I$.*

Theorem 2.2 ([10]). *A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is a quasiefficient solution of (P) if and only if $\boldsymbol{x}_0$ is an optimal solution of $(P_{\boldsymbol{\lambda}})$ for some $\boldsymbol{\lambda} \geq \boldsymbol{0}$ with $\boldsymbol{\lambda} \neq \boldsymbol{0}$.*

Corollary 2.3 ([10]). *We put $\mathcal{D} \equiv \{D' \subset D : D' \neq \emptyset\}$. For each $D' = \{\boldsymbol{d}_{i_1}, \boldsymbol{d}_{i_2}, \cdots, \boldsymbol{d}_{i_k}\} \in \mathcal{D}$, let $E(D')$ be the set of all efficient solutions of the following multicriteria location problem:*

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} (\|\boldsymbol{x} - \boldsymbol{d}_{i_1}\|_1, \|\boldsymbol{x} - \boldsymbol{d}_{i_2}\|_1, \cdots, \|\boldsymbol{x} - \boldsymbol{d}_{i_k}\|_1)^T. \qquad (2.1)$$

Then

$$QE(D) = \bigcup_{D' \in \mathcal{D}} E(D').$$

Next, we give some properties of quasiefficient solutions of (P) and a procedure to find all quasiefficient solutions of (P).

Since the objective function of $(\mathrm{P}_{\boldsymbol{\lambda}})$, g, can be rewritten as

$$g(\boldsymbol{x}) = \sum_{i=1}^{m} \lambda^i \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 = \sum_{i=1}^{m} \lambda^i \sum_{j=1}^{n} |x^j - d_i^j| = \sum_{j=1}^{n}\sum_{i=1}^{m} \lambda^i |x^j - d_i^j|,$$

$(\mathrm{P}_{\boldsymbol{\lambda}})$ reduces to n independent one-dimensional problems. Namely, $\boldsymbol{x}^* \equiv (x^{1*}, x^{2*}, \cdots x^{n*})^T$ is an optimal solution of $(\mathrm{P}_{\boldsymbol{\lambda}})$ if and only if each x^{j*}, $j \in J$ is an optimal solution of the following one-dimensional problem:

$$(\mathrm{P}_j) \qquad \min_{x\in\mathbb{R}} g_j(x) \equiv \sum_{i=1}^{m} \lambda^i |x - d_i^j|.$$

These one-dimensional problems can be solved by using an algorithm in [2].

Lemma 2.4 ([6]). *For $j \in J$ and any fixed $\boldsymbol{\lambda} > \boldsymbol{0}$, $\min\{d_i^j : i \in I\} \leq x^* \leq \max\{d_i^j : i \in I\}$, where x^* is any optimal solution of (P_j).*

Theorem 2.5. *We put*

$$B \equiv \{(x^1, x^2, \cdots, x^n)^T \in \mathbb{R}^n : \min\{d_i^j : i \in I\} \leq x^j \leq \max\{d_i^j : i \in I\}, j \in J\}.$$

Then any quasiefficient solution of (P) belongs to B. Namely, $QE(D) \subset B$.

Proof. From Theorem 2.1 and Lemma 2.4, $E(D') \subset B$ for any $D' \in \mathcal{D}$. Therefore, $QE(D) \subset B$ from Corollary 2.3. □

In the case $n = 1$, $E(D) = QE(D) = B$ from definitions of the efficiency and the quasiefficiency. In the case $n = 2$, it is known that $QE(D) = B$ (see [3]). In the case $n \geq 3$, the following example shows that $QE(D) \neq B$ in general.

Example. In $\mathbb{R}^n$, $n \geq 3$, we put $\boldsymbol{d}_i = \boldsymbol{e}_i$, $i \in J$, where $\{\boldsymbol{e}_i\colon\ i \in J\}$ is the canonical basis of $\mathbb{R}^n$, that is, for each $\boldsymbol{e}_i$, $i \in J$, the ith component is one and the others are zero. In this case, $B = [0,1]^n$, where $[0,1] \equiv \{x \in \mathbb{R}\colon\ 0 \leq x \leq 1\}$. We put $\boldsymbol{x}_0 = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})^T \in B$. From Theorem 2.2, $\boldsymbol{x}_0 \in QE(D)$ if and only if $\boldsymbol{x}_0$ is an optimal solution of $(\mathrm{P}_{\boldsymbol{\lambda}})$ for some $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \cdots, \lambda^n)^T$ such that $\boldsymbol{\lambda} \geq \boldsymbol{0}$ and $\boldsymbol{\lambda} \neq \boldsymbol{0}$. The latter condition holds if and only if $\boldsymbol{\lambda}$ satisfies

$$-\lambda^j + \sum_{i \neq j} \lambda^i = 0, \qquad j \in J \tag{2.2}$$

and

$$\boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{\lambda} \neq \boldsymbol{0}. \tag{2.3}$$

From (2.2), we have

$$\sum_{i=1}^{n} \lambda^i = 0.$$

Since $\boldsymbol{\lambda} \geq \boldsymbol{0}$, we have $\lambda^i = 0, i \in J$. Thus, there does not exist $\boldsymbol{\lambda}$ which satisfies (2.2) and (2.3) simultaneously. Therefore, $\boldsymbol{x}_0 \notin QE(D)$.

Theorem 2.6. *Assume that* $m \geq n+1$. *Then*

$$QE(D) = \bigcup_{\{i_1, i_2, \cdots, i_{n+1}\} \subset I} QE(\{\boldsymbol{d}_{i_k} : k \in J_1\})$$

where $J_1 \equiv \{1, 2, \cdots, n+1\}$ *and each* $QE(\{\boldsymbol{d}_{i_k} : k \in J_1\})$ *is the set of all quasiefficient solutions of (2.1) when* $D' = \{\boldsymbol{d}_{i_k} : k \in J_1\}$ *in Corollary 2.3.*

Proof. For $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{x}_0 \in QE(D)$ if and only if

$$\bigcap_{i=1}^{m} \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 < \|\boldsymbol{x}_0 - \boldsymbol{d}_i\|_1\} = \emptyset.$$

Since each $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1, i \in I$ is a convex function in $\boldsymbol{x} \in \mathbb{R}^n$, the above intersection is empty if and only if there exist $i_1, i_2, \cdots, i_{n+1} \in I$ such that

$$\bigcap_{k=1}^{n+1} \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{d}_{i_k}\|_1 < \|\boldsymbol{x}_0 - \boldsymbol{d}_{i_k}\|_1\} = \emptyset$$

(by Corollary 21.6.1 in [12]), and this is equivalent to $\boldsymbol{x}_0 \in QE(\{\boldsymbol{d}_{i_k} : k \in J_1\})$. Thus, the result follows. □

In the case $m \leq n+1$, $QE(D)$ can be determined by using Corollary 2.3, where $E(D')$ in Corollary 2.3 can be determined by using an algorithm in [6] (or [4] or [1]). In the case $m > n+1$, by Theorem 2.6, it is essential to determine $QE(D')$ for $D' \subset D$ which has $n+1$ elements. As a direct consequence of the results of this section, we have the following pseudoalgorithm to determine $QE(D)$ in the case $m \geq n+1$.

Input: $\boldsymbol{d}_i \in \mathbb{R}^n, i \in I$: demand points.

Output: $QE(D)$.

Steps:

1. Compute $QE(\{\boldsymbol{d}_{i_k} : k \in J_1\})$ for all $i_1, i_2, \cdots, i_{n+1} \in I$.
2. Compute $QE(D) = \bigcup_{\{i_1,i_2,\cdots,i_{n+1}\}\subset I} QE(\{\boldsymbol{d}_{i_k} : k \in J_1\})$.
3. END.

Now, we consider the computational time of the above procedure for fixed n. For each $i_1, i_2, \cdots, i_{n+1} \in I$, determining $E(D')$ in Corollary 2.3 by using an algorithm in [6] (or [4] or [1]) and determining $QE(\{\boldsymbol{d}_{i_k} : k \in J_1\})$ by using Corollary 2.3 require $O(1)$ computational time. Therefore, determining $QE(D)$ by using the above procedure requires $O(m^{n+1})$ computational time.

Remark. For $\boldsymbol{x}_0 \equiv (x_0^1, x_0^2, \cdots, x_0^n)^T \in \mathbb{R}^n$, $\boldsymbol{x}_0$ is called *an intersection point* if $x_0^j \in \{d_i^j : i \in I\}, j \in J$. It has been mentioned before that $QE(D)$ can be determined by using an algorithm proposed in [7] when $n = 3$. The idea of its algorithm is to trace quasiefficient adjacent intersection points from some initial point. When $n > 3$, its framework is available if it can be checked that a given intersection point is quasiefficient or not. In [7], when $n = 3$, it is checked that a given intersection point $\boldsymbol{x}_0 \in \mathbb{R}^3$ is quasiefficient or not by using the concept of the summary diagram. Roughly speaking, the summary diagram represents locations of demand points from $\boldsymbol{x}_0$. It means that some demand point is contained in a cone $\boldsymbol{x}_0 + \mathcal{C}(\{\alpha\boldsymbol{e}_1, \beta\boldsymbol{e}_2, \gamma\boldsymbol{e}_3\})$ or not, where $\alpha, \beta, \gamma \in \{-1, +1\}$ and $\mathcal{C}(\{\alpha\boldsymbol{e}_1, \beta\boldsymbol{e}_2, \gamma\boldsymbol{e}_3\}) \equiv \{\mu^1\alpha\boldsymbol{e}_1 + \mu^2\beta\boldsymbol{e}_2 + \mu^3\gamma\boldsymbol{e}_3 : \mu^1 \geq 0, \mu^2 \geq 0, \mu^3 \geq 0\}$. In order to check that a given intersection point is quasiefficient or not, it seems to be useless to extend the summary diagram in the case $n = 3$ to that in the case $n > 3$. It is due to the difficulty of the extension of the necessary and sufficient condition for a given intersection point to be quasiefficient in [7]. Therefore, our approach is to use Corollary 2.3 and Theorem 2.6 and algorithms in [6] (or [4] or [1]), and is not to use the summary diagram. It is the main difference in approaches between this article and [7].

3 Numerical Example

In this section, we give a numerical example.

Consider the following multicriteria location problem:

$$\min_{\boldsymbol{x}\in\mathbb{R}^4} (\|\boldsymbol{x}-\boldsymbol{d}_1\|_1, \|\boldsymbol{x}-\boldsymbol{d}_2\|_1, \|\boldsymbol{x}-\boldsymbol{d}_3\|_1, \|\boldsymbol{x}-\boldsymbol{d}_4\|_1, \|\boldsymbol{x}-\boldsymbol{d}_5\|_1)^T$$

where $\boldsymbol{d}_1 = (3,0,4,1)^T$, $\boldsymbol{d}_2 = (4,2,0,2)^T$, $\boldsymbol{d}_3 = (2,1,3,3)^T$, $\boldsymbol{d}_4 = (0,4,5,4)^T$ and $\boldsymbol{d}_5 = (1,5,2,5)^T$. Then we have $QE(D)$ illustrated in the following figures.

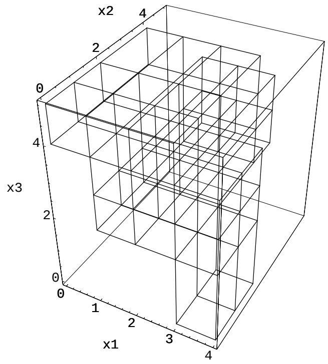

Figure 1. $QE(D)(1)$.

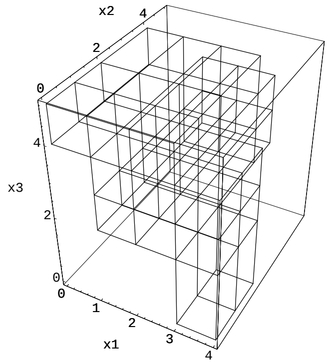

Figure 2. $QE(D)(\eta)$ with $1 < \eta < 2$.

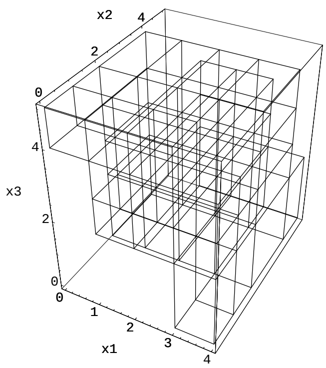

Figure 3. $QE(D)(2)$.

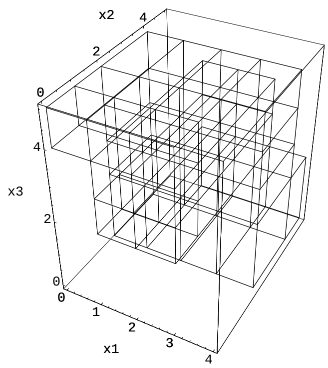

Figure 4. $QE(D)(\eta)$ with $2 < \eta < 3$.

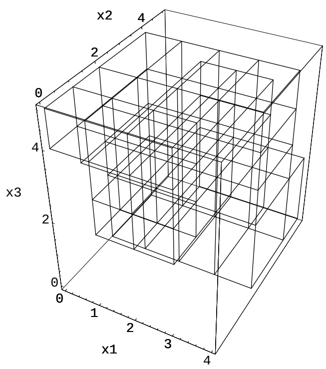

Figure 5. $QE(D)(3)$.

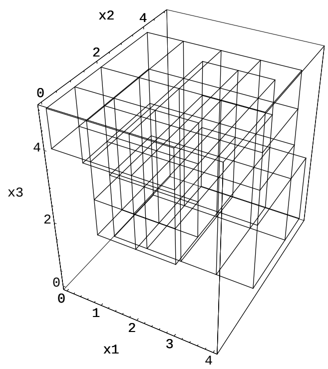

Figure 6. $QE(D)(\eta)$ with $3 < \eta < 4$.

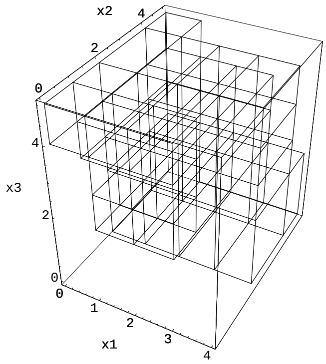

Figure 7. $QE(D)(4)$.

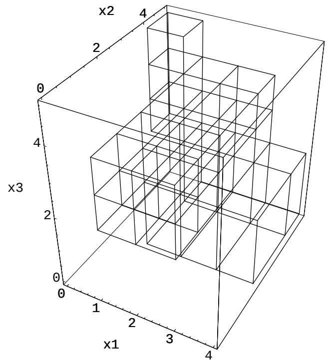

Figure 8. $QE(D)(\eta)$ with $4 < \eta < 5$.

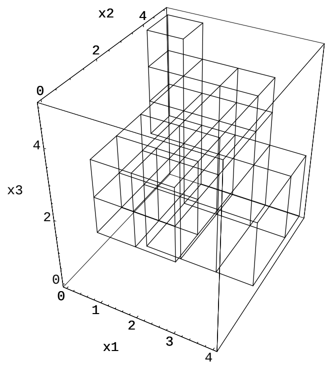

Figure 9. $QE(D)(5)$.

In Figures 1–9, $QE(D)(\eta) \equiv \{\boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x} \in QE(D), x^4 = \eta\}$ for $\eta \in \mathbb{R}$, where $\boldsymbol{x} = (x^1, x^2, x^3, x^4)^T \in \mathbb{R}^4$. In this case, $QE(D)(\eta) = \emptyset$ for $\eta < 1$ or $\eta > 5$, and $QE(D) = QE(D)(1) \cup QE(D)(\eta^1) \cup QE(D)(2) \cup QE(D)(\eta^2) \cup QE(D)(3) \cup QE(D)(\eta^3) \cup QE(D)(4) \cup QE(D)(\eta^4) \cup QE(D)(5)$ with $1 < \eta^1 < 2, 2 < \eta^2 < 3, 3 < \eta^3 < 4, 4 < \eta^4 < 5$. Each $QE(D)(\eta)$ in Figures 1–9 is the union of all drawn line segments, which connect two adjacent intersection points, and the set of all points surrounded by such line segments in $\{\boldsymbol{x} \in \mathbb{R}^4 : x^4 = \eta\}$, where "a point is surrounded by such line segments" means that the point is a (relative) interior point of a rectangle constituted by four drawn line segments or a (relative) interior point of a rectangular parallelepiped constituted by twelve drawn line segments.

4 Conclusions

We dealt with a multicriteria location problem (P) with rectilinear norm in $\mathbb{R}^n$. Our main interest was to find the set $QE(D)$ of all quasiefficient solutions of (P). First, as Theorem 2.5 and 2.6, we gave properties of quasiefficient solutions of (P). Next, based on Theorem 2.6, we gave a procedure to compute $QE(D)$.

For multicriteria location problems (P) with rectilinear norm in $\mathbb{R}^n$, it is known that all efficient solutions for $n = 2, 3$ and $n > 3$ can be determined by using algorithms proposed, respectively, in [1], [4] and [6], and that all quasiefficient solutions for $n = 2$ and 3 can be determined by using algorithms proposed in [3] and [7], respectively. A procedure to find $QE(D)$, which we proposed, is meaningful for $n > 3$ and requires $O(m^{n+1})$ computational time for fixed n. If n is large, it is a hard task to find $QE(D)$ by using the proposed procedure. Thus, future research could be conducted to construct a more efficient algorithm. On the other hand, in practical applications to the development of new products, the results in [5] suggest that, in some cases, efficient and quasiefficient solutions of multicriteria location problems (P) with rectilinear norm in $\mathbb{R}^n$ are effective and important, where n is not so large. Thus, the proposed procedure is useful and important for such cases.

References

[1] L.G. Chalmet, R.L. Francis and A. Kolen, Finding efficient solutions for rectilinear distance location problems efficiently, *Eur. J. Oper. Res.* 6 (1981) 117–124.

[2] Z. Drezner and G.O. Wesolowsky, The asymmetric distance location problem, *Trans. Sci.* 23 (1989) 201–207.

[3] M. Kon, Efficient solutions for multicriteria location problems under the block norm, *Math. Japon.* 47 (1998) 295–303.

[4] M. Kon, Efficient solutions of multicriteria location problems with rectilinear norm in $\boldsymbol{R}^3$, *Sci. Math. Jpn.* 54 (2001) 289–299.

[5] M. Kon, Public opinion survey on home education: application of location problems with rectilinear norm, *Sci. Math. Jpn.* 58 (2003) 99–111.

[6] M. Kon, Efficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^n$, *Bull. Fac. Sci. Tech. Hirosaki Univ.* 7 (2004) 21–30.

[7] M. Kon, Quasiefficient solutions of multicriteria location problems with rectilinear norm in $\mathbb{R}^3$, *J. Oper. Res. Soc. Japan* 50 (2007) 264–275.

[8] M. Kon and S. Kushimoto, Efficient solutions for multicriteria location problems under the block norm II: application to the development of new products, *Sci. Math.* 1 (1998) 133–140.

[9] M. Kon and S. Kushimoto, On efficient solutions of multicriteria location problems with the block norm, *Sci. Math.* 2 (1999) 245–254.

[10] T.J. Lowe, J.-F. Thisse, J.E. Ward and R.E. Wendell, On efficient solutions to multiple objective mathematical programs, *Manage. Sci.* 30 (1984) 1346–1349.

[11] S. Nickel and J. Puerto, *Location Theory*, Springer-Verlag, Berlin, 2005.

[12] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, N. J., 1970.

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