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A VARIATIONAL APPROACH TO THE INVERSION OF SOME COMPACT OPERATORS

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Abstract: Inverse problems of Fourier synthesis can be regularized by constraining the resolution of the reconstructed object. One may speak of *regularization by mollification*. This regularization principle has been shown to behave nicely in practice, and more recently to give rise to interesting Tikhonov-like theorems. In this paper, we propose and analyse an extension of the regularization by mollification to a wider class of ill-posed problems.

Key words: *variational methods, ill-posed problems, regularization*

Mathematics Subject Classification: *35A15, 47A52*

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1 Introduction

Let us consider an ill-posed operator equation of the form $Tf = g$, in which T is some linear mapping from a normed space F into a normed space G and g is the data, which is assumed to be an approximation of the image by T of the original unknown object f_0. In order to deal with ill-posedness of such problems, it is customary to infer regularized solutions via an optimization problem of the form

$$(\mathscr{E}) \quad \left| \begin{array}{rl} \text{Minimize} & \mathscr{F}(f) := \dfrac{1}{2}\left\| g - Tf \right\|_G^2 + \alpha \mathscr{H}(f) \\ \text{s.t.} & f \in F, \end{array} \right.$$

in which α is a positive parameter and $\mathscr{H}$ is a convex functional generically called an *regularizer*.

The abundant literature devoted to the choice of a particular regularizer and, for this regularizer, of a particular value of α, is symptomatic of a fundamental difficulty in the interpretation of the above scheme. Focusing primarily on the variational definition of the regularized solution leaves the new objective unclear: to some extent, the new unknown (which is no longer f_0) remains unspecified.

In [7], Lannes *et al.* proposed an alternative scheme for problems of *Fourier synthesis*. They addressed problems in which

$$F = L^2(V), \quad G = L^2(W) \quad \text{and} \quad T = 1_W U.$$

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Here, V and W are bounded subsets of R^d, $L^2(V)$ denotes the subspace of $L^2(\mathrm{R}^d)$ of the functions having their support in V, 1_W denotes both the indicator function of W and the operator $(g \mapsto 1_W g)$, and U denotes the Fourier operator:

$$(1_W g)(\xi) = 1_W(\xi)g(\xi) \quad \text{and} \quad Uf(\xi) = \hat{f}(\xi) = \int e^{-2i\pi\langle\xi,x\rangle} f(x)\,\mathrm{d}x.$$

The operator T is called a *truncated Fourier operator*. For convenience, the Fourier transform of a function ϕ will also be denoted by $\hat{\phi}$. The regularization scheme of Lannes *et al.* can be outlined as follows:

Step 1 Define the *object to be reconstructed* (or *target object*) as $\phi_\beta * f_0$, where $\{\phi_\beta\}_{\beta>0}$ is an *approximation of unity*.

Step 2 Replace the original data g (the approximate truncated Fourier transform of f_0) by *regularized data*: $\hat{\phi}_\beta g$. The reason for this is that, if g is an approximation of the Fourier transform of f_0, $\hat{\phi}_\beta g$ will be an approximation of the Fourier transform of $\phi_\beta * f_0$.

Step 3 Finally, define the *reconstructed object* as the solution of the following optimization problem:

$$(\mathscr{P}) \quad \left| \begin{array}{ll} \text{Minimize} & \dfrac{1}{2}\left\|\hat{\phi}_\beta g - T_W f\right\|^2_{L^2(W)} + \dfrac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}\right\|^2_{L^2(\mathrm{R}^d)} \\ \text{s.t.} & f \in L^2(V_1), \end{array} \right.$$

in which V_1 is a compact set containing V. Ideally, V_1 should be chosen so as to contain the support of $\phi_\beta * f_0$ for every β in the interval of interest (*e.g.* $\beta \in (0,1]$). In practice, if ϕ_β has unbounded support, it is sufficient to ensure that most of the *energy* of $\phi_\beta * f_0$ is contained in V_1 for β in the range under consideration.

In a recent paper, Alibaud *et al.* [1] proved a theorem on the behavior of the regularized solution as β tends to zero (see Theorem 2.1 below). This theorem is the counterpart, for the regularization principle under consideration, of Tikhonov's convergence theorem, in which the *homogeneous* parameter α is in force.

The present paper is an attempt to extend this result to a wider context. It is motivated by the desire to address ill-posed operator equations of the general form $Tf = g$ in which T has no explicit connexion with any truncated Fourier operator. In such an extended context, what should indeed become Step 2 ? We shall see that it makes sense to replace the data g by *regularized data* $\Phi_\beta g$, in which the linear operator Φ_β itself results from the minimization of some functional.

The paper is organized as follows. In Section 2, we give an overview of Fourier synthesis and we give a detailed account of our extended regularization scheme. Then, in section 3, we address the definition and computation of Φ_β and we give examples. Finally, in section 4 we show that the convergence result of Alibaud *et al.* [1] can be extended to our generalized setting.

2 A General Regularization Scheme

2.1 Overview of Fourier Synthesis

We call *Fourier synthesis* the generic problem of recovering a function f_0 from a partial and approximate knowledge of its Fourier transform. Whenever *partial* is understood as *limited to a bounded domain*, we speak of *Fourier extrapolation*.

Important milestones in the history of Fourier Synthesis are Landau's paper [5] on Fourier sampling theory and the eigenvalue analysis of truncated Fourier operators, and later on the paper by Lannes *et al.* [7], devoted to the regularization of the *Fourier extrapolation problem*. In the latter paper, the authors considered the following abstract problem:

> Let V and W be bounded subsets of R^d, where W is assumed to have non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on W.

Recall that T_W is injective, since Fourier transforms of compactly supported functions are analytic and W is assumed to have nonempty interior. Recall also that T_W is compact, as a Hilbert-Schmidt operator.

Ill-posedness of the Fourier extrapolation problem [7, 1] led Lannes *et al.* [7] to propose a regularization principle which, in essence, consists in constraining the resolution of the object to be inferred. The problem of identifying f_0 is replaced by that of recovering a *limited resolution* version of it, namely, $\phi_\beta * f_0$, in which

$$\phi_\beta(x) := \frac{1}{\beta^d}\,\phi\left(\frac{x}{\beta}\right), \quad \text{with} \quad \phi \in L^1(\mathrm{R}^d) \quad \text{and} \quad \int_{\mathrm{R}^d} \phi(x)\,\mathrm{d}x = 1. \tag{2.1}$$

We shall refer to $\phi_\beta \in L^1(\mathrm{R}^d)$ as an *apodized point spread function*, and to the parametrized family $\{\phi_\beta\}_{\beta>0}$ as an *approximation of unity*. It is then reasonable to define the reconstructed object as the solution to Problem $(\mathscr{P})$.

The above regularization scheme clearly refers to mollification theory, and we may then speak of *regularization by mollification*. The parameter β, which can be regarded as the inverse of a *cutoff frequency*, appears as a *regularization parameter* for the inversion of T_W. In [1], Alibaud *et al.* considered the behavior of the solution to Problem $(\mathscr{P})$ as this parameter tends to zero. They proved the following result:

Theorem 2.1. *Consider Problem $(\mathscr{P})$ above, in which ϕ_β is as in Equation* (2.1). *Let T_W^+ denote the Moore-Penrose pseudo-inverse of $T_W : L^2(V_1) \to L^2(W)$.*

I. *Let $\alpha > 0$ and $\beta > 0$ be fixed. Then $(\mathscr{P})$ has a unique solution f_β. Moreover, f_β depends continuously on $g \in L^2(W)$.*

II. *Assume that $\hat{\phi}(\xi) \neq 1$ for all $\xi \in \mathrm{R}^d \setminus \{0\}$, and that there exist positive numbers K and s such that $|1 - \hat{\phi}(\xi)| \sim_{\xi\to 0} K \,\|\,\xi\,\|^s$. If $g \in \operatorname{ran} T_W$ is such that its analytic extension $\tilde{g} = UT_W^+ g$ satisfies*

$$\int_{\mathrm{R}^d} \|\,\xi\,\|^{2s}\,|\,\tilde{g}(\xi)\,|^2\,\mathrm{d}\xi < \infty,$$

then f_β converges to $T_W^+ g$ strongly, in $L^2(V_1)$, as $\beta \downarrow 0$.

Notice that T_W^+ is nothing but T_W^{-1}, the inverse of $T_W : L^2(V_1) \to \operatorname{ran} T_W$. Furthermore, in the second part of the theorem, the condition on g can be rewritten as

$$g \in T_W\left(L^2(V_1) \cap H^s(\mathrm{R}^d)\right),$$

in which $H^s(\mathrm{R}^d)$ denotes as usual the Sobolev space

$$H^s(\mathrm{R}^d) := \left\{ f \in L^2(\mathrm{R}^d) \;\middle|\; \int \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi < \infty \right\},$$

endowed with the inner product

$$\langle f_1, f_2 \rangle_s := \int \left(1 + \|\xi\|^2\right)^s \hat{f}_1(\xi)\overline{\hat{f}_2(\xi)}\,\mathrm{d}\xi \tag{2.2}$$

and the corresponding norm

$$\|f\|_s^2 := \int \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi.$$

2.2 Extension

One of the key points in the above regularization scheme is that the *data* corresponding to the target object is easily computed from the original data: if g is an approximation of Uf_0 on W, $\hat{\phi}_\beta g$ is an approximation of $U(\phi_\beta * f_0)$. This corresponds to the existence of an operator Φ_β such that $\Phi_\beta U = UC_\beta$, where C_β denotes the convolution operator $(f \mapsto \phi_\beta * f)$ on $L^2(\mathrm{R}^d)$ (and is understood as the identity for $\beta = 0$). This nice aspect of (truncated) Fourier operators is shared with a few other operators, such as convolution operators or the classical Radon operators. The latter are important in practice [10, 8]. Nevertheless, there are relevant applications in which it is not possible to find such a Φ_β. This motivates the generalization outlined in the introduction, which we now describe in detail.

Throughout, we shall assume the following:

> STANDING ASSUMPTIONS: V is a bounded domain in R^d containing the support of the original unknown object f_0; G is an infinite dimensional separable Hilbert space; $T\colon L^2(\mathrm{R}^d) \to G$ is a continuous injective linear operator (modelling the data aquisition process) whose restriction to $L^2(V)$ is compact.

The restriction of an operator T to a subspace E will be subsequently denoted by $T\lfloor_E$, or merely by T whenever no confusion is to be feared.

Let $\phi_\beta * f_0$ be the new *target object*, where $\{\phi_\beta\}_{\beta>0}$ is an approximation of unity. The target object may have an unbounded support in R^d. Nevertheless we choose here to reconstruct the object in $L^2(V_1)$ for some compact set $V_1 \supset V$: on the one hand, working with unbounded supports makes little sense in practice; on the other hand, it is reasonable to choose V_1 in such a way that it contains most of the target object's *energy*. For example, one may fix a small parameter $\varepsilon > 0$, define $\mathrm{supp}_\varepsilon\,\phi$ as a ball B such that $\int_{B^c} |\phi| \leq \varepsilon$, and then choose V_1 to be the closure of $V + \mathrm{supp}_\varepsilon\,\phi$.

Finally, define the reconstructed object as the solution to

$$(\mathscr{P}_\beta) \quad \left|\; \begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\left\|\Phi_\beta g - Tf\right\|_G^2 + \dfrac{\alpha}{2}\left\|(I - C_\beta)f\right\|_{L^2(\mathrm{R}^d)}^2 \\ \text{s.t.} & f \in L^2(V_1), \end{array}\right.$$

in which I is the identity and $\Phi_\beta\colon G \to G$ is a solution to the following optimization problem:

$$(\mathscr{Q}_\beta) \quad \left|\; \begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\left\|TC_\beta - XT\lfloor_E\right\|_{L(E,G)}^2 \\ \text{s.t.} & X \in L(G),\;\; X = 0 \text{ on } (\mathrm{ran}\,T\lfloor_E)^\perp. \end{array}\right.$$

Here, E is some subspace of $L^2(\mathrm{R}^d)$, $L(E,G)$ denotes as usual the space of continuous linear mappings from E to G and $L(G) := L(G,G)$. We emphasize that the Hilbert space structure of E may not be the one inherited from that of $L^2(\mathrm{R}^d)$.

Remark 2.2. The assumption made on T in the above extension is not minimal: the reader may check that continuity of T on well chosen subspaces of $L^2(\mathrm{R}^d)$ is sufficient. Our stronger assumption is made for the sake of clarity.

The following sections will demonstrate the relevance of our extended regularization principle.

3 Regularizing the Data

In this section, we give a few basic results concerning Problem $(\mathscr{Q}_\beta)$. Let E be a subspace of $L^2(V_1)$, a particular instance of which being considered in Section 4.

Thoughout this section, the restriction $T\lfloor_E$ of T to E will be denoted by T. In particular, T^+ will denote the peudo-inverse of T with respect to the Hilbert space structure of E, and for every continuous linear mapping Y from E to G, $\|Y\|$ will stand for $\|Y\|_{L(E,G)}$.

The functional to be minimized in $(\mathscr{Q}_\beta)$ is obviously convex, as the post-composition of an affine function by a norm. It should be noticed that, in most cases of interest, $(\mathscr{Q}_\beta)$ is an ill-posed optimization problem, for the function to be minimized then fails to be inf-compact.

As a matter of fact, consider the (translated) functional $X \mapsto \|XT\|$, in which T is compact and, in accordance with the rest of the paper, X belongs to a set of bounded operators vanishing on $(\operatorname{ran} T)^\perp$. Let us show that, in the case where $\operatorname{ran} T$ is not of finite dimension, the level sets of the latter function are unbounded. Let K denote the closure of the image by T of the closed unit ball of F. The set K is a compact subset of G. Let $(g_k)_{k\in\mathrm{N}}$ be a Hilbert basis of G and let v_k denote the element of maximum norm in $K \cap \operatorname{vect}\{g_k\}$, whose existence is ensured by the compactness of K. Then, $v_{k_r} \neq 0$ for an infinity of $k_r \in \mathrm{N}$ and necessarily $\|v_{k_r}\| \to 0$ as $r \to \infty$, for otherwise there would exist $\lambda > 0$ such that $\lambda g_{k_r} \in K$ for all $r \in \mathrm{N}$, and since $\|\lambda g_p - \lambda g_q\| = \sqrt{2}\lambda$ for all $p, q \in \mathrm{N}$, the sequence $(\lambda g_{k_r})_{r\in\mathrm{N}}$ cannot have any accumulation point, in contradiction with the compactness of K. Therefore, $\|v_{k_r}\| \to 0$ as $r \to \infty$ and we can define a sequence $(X_r)_{r\in\mathrm{N}} \subset L(G)$ in such a way that:

$$X_r v_{k_r} = 1 \quad \text{and} \quad X_r|_{\{g_{k_r}\}^\perp} = 0,$$

so that $\|X_r T\| = 1$ for all $r \in \mathrm{N}$ and

$$\|X_r\| = \frac{1}{\|v_{k_r}\|} \underset{r\to\infty}{\longrightarrow} \infty.$$

Moreover, in the case where T is not injective, uniqueness of a solution also fails.

However, Problem $(\mathscr{Q}_\beta)$ turns out to have explicit solutions, under reasonable assumptions. This is the purpose of Proposition 3.1 below. Moreover, in practice, the computation of a solution may be performed by means of a proximal iterative procedure. The latter is known to introduce numerical well-posedness in ill-posed optimization problems.

Proposition 3.1. *If $TC_\beta T^+$ belongs to $L(\mathscr{D}(T^+), G)$, then $TC_\beta T^+$ can be extended to a continuous operator on G which is the unique solution to Problem $(\mathscr{Q}_\beta)$.*

Proof. Recall that $\mathscr{D}(T^+) = \operatorname{ran} T + (\operatorname{ran} T)^\perp$ is a dense subset of G, and that, since T is assumed to be injective, T^+T is the identity. Since $TC_\beta T^+$ is assumed to be bounded, it admits a unique continuous linear extension to G; this extension is denoted likewise and still satisfies :

$$\|TC_\beta T^+T - TC_\beta\| = 0.$$

This proves that $TC_\beta T^+$ is a solution to $(\mathscr{Q}_\beta)$. Moreover, if Φ is another minimizer, $\Phi T = TC_\beta$, which yields $\Phi TT^+ = TC_\beta T^+$; since $TT^+ \colon \mathscr{D}(T^+) \to G$ is the orthogonal projection onto the closure of $\operatorname{ran} T$, the latter equality implies that Φ and $TC_\beta T^+$ coincide on $\operatorname{ran} T$. Since $\Phi((\operatorname{ran} T)^\perp)$ is constrained to be $\{0\}$, we finally obtain that $\Phi = TC_\beta T^+$. □

Remark 3.2. In this paper, we are mostly interested in the case where E is a dense subspace of $L^2(V_1)$. In this case, $T(E)$ is dense in $T(L^2(V_1))$ so that the domain $\mathscr{D}(T^+_{\lfloor E})$ is dense in the domain $\mathscr{D}(T^+_{\lfloor L^2(V_1)})$. Moreover, since T is injective, the pseudoinverses $T^+_{\lfloor E}$ and $T^+_{\lfloor L^2(V_1)}$ coincide on $\mathscr{D}(T^+_{\lfloor E}) = T(E) + T(E)^\perp$. Problem $(\mathscr{Q}_\beta)$ is then equivalent to

$$(\mathscr{Q}'_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\left\|TC_\beta - XT_{\lfloor L^2(V_1)}\right\|^2_{L(L^2(V_1),G)} \\ \text{s.t.} & X \in L(G), \ \ X = 0 \text{ on } (\operatorname{ran} T_{\lfloor L^2(V_1)})^\perp. \end{array}\right.$$

The reason for this is that, in the use of Proposition 3.1, $TCT^+_{\lfloor E}$ is bounded if and only if $TCT^+_{\lfloor L^2(V_1)}$ is bounded.

Remark 3.3. Note that, due to the injectivity of T, the minimum value of Problem $(\mathscr{Q}_\beta)$ is equal to zero. If T were not injective, one could still prove that $TC_\beta T^+$ is a solution to $(\mathscr{Q}_\beta)$, but uniqueness would clearly fail.

Notice that, since T^+ is defined on $\operatorname{ran} T \oplus \ker T^\star \subset G$, the boundedness of $TC_\beta T^+$ is equivalent to the existence of a positive constant K_β such that

$$\forall g \in \mathscr{D}(T^+), \quad \|TC_\beta T^+ g\| \leq K_\beta \|g\|.$$

Since every g in $\mathscr{D}(T^+)$ can written as $g = g_1 + g_2$ with $g_1 \in \operatorname{ran} T$ and $g_2 \in (\operatorname{ran} T)^\perp$, the above condition is equivalent to

$$\forall (g_1, g_2) \in \operatorname{ran} T \times (\operatorname{ran} T)^\perp \quad \|TC_\beta T^+ g_1\| \leq K_\beta \sqrt{\|g_1\|^2 + \|g_2\|^2}.$$

Since the latter inequality is true in particular when $g_2 = 0$ and since TT^+ coincides with the identity on $\operatorname{ran} T$, an equivalent condition is

$$\forall g_1 \in \operatorname{ran} T, \quad \|TC_\beta T^+ g_1\| \leq K_\beta \|TT^+ g_1\|.$$

Finally, $T^+(\operatorname{ran} T) = E$, so that the boundedness of $TC_\beta T^+$ is equivalent to the existence of a positive constant K_β such that

$$\forall f \in E, \quad \|TC_\beta f\| \leq K_\beta \|Tf\|.$$

We now describe a class of operators T such that TCT^+ is bounded, where C is some convolution operator (that is, $Cf = \phi * f$ for some convolution kernel $\phi \in L^1(\mathrm{R}^d)$). We assume here that $E = L^2(V_1)$, endowed with the standard inner product.

Recall that the integral operator of kernel $\alpha \colon \mathrm{R}^d \times \mathrm{R}^d \to \mathrm{C}$ is defined by

$$Tf(x) = \int \alpha(x,y) f(y)\,\mathrm{d}y,$$

the domain of T being the space of measurable functions for which the above integral is well-defined for almost all $x \in \mathrm{R}^d$.

Proposition 3.4. *Let T be the integral operator of kernel α. Assume that*

(i) $\int_{\mathrm{R}^d\times\mathrm{R}^d} |\alpha(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y < \infty$ *(that is, T is Hilbert-Schmidt on $L^2(\mathrm{R}^d)$);*

(ii) *there exists a function k such that for all $x, y, z \in \mathrm{R}^d$, $\alpha(x, y+z) = \alpha(x,y)k(x,z)$;*

(iii) *there exists a positive constant M_ϕ, depending on ϕ only, such that*

$$\forall x \in \mathrm{R}^d, \quad \left|\int_{\mathrm{R}^d} \phi(z)k(x,z)\,\mathrm{d}z\right| < M_\phi.$$

Then T is well-defined on $\operatorname{ran} C_{\lfloor L^2(V_1)}$ and TCT^+ is bounded on its domain.

Proof. Let f be any function in $L^2(V_1)$. Then,

$$\begin{aligned}
\int_{\mathrm{R}^d} \left|\alpha(x,y)Cf(y)\right|\,\mathrm{d}y &= \int_{\mathrm{R}^d} \left|\alpha(x,y)\int_{\mathrm{R}^d}\phi(z)f(y-z)\,\mathrm{d}z\right|\,\mathrm{d}y \\
&\le \int_{\mathrm{R}^d\times\mathrm{R}^d} \left|\alpha(x,y)\phi(z)f(y-z)\right|\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_{\mathrm{R}^d\times\mathrm{R}^d} \left|\alpha(x,y'+z)\phi(z)f(y')\right|\,\mathrm{d}z\,\mathrm{d}y' \\
&= \int_{\mathrm{R}^d\times\mathrm{R}^d} \left|\alpha(x,y')k(x,z)\phi(z)f(y')\right|\,\mathrm{d}z\,\mathrm{d}y' \\
&= \int_{\mathrm{R}^d} \left|\alpha(x,y')f(y')\right|\left(\int_{\mathrm{R}^d}\left|k(x,z)\phi(z)\right|\right)\mathrm{d}y' \\
&\le M_\phi \int_{\mathrm{R}^d} \left|\alpha(x,y')f(y')\right|\,\mathrm{d}y'.
\end{aligned}$$

This proves that T is well-defined on $\operatorname{ran} C$. Now, it is easy to check that

$$TCf(x) = \int_{\mathrm{R}^d\times\mathrm{R}^d} \phi(z)k(x,z)\alpha(x,y')f(y')\,\mathrm{d}y'\,\mathrm{d}z.$$

Consequently,

$$\begin{aligned}
\|TCf\|^2_{L^2(\mathrm{R}^d)} &= \int_{\mathrm{R}^d}\left|\int_{\mathrm{R}^d}\phi(z)k(x,z)\,\mathrm{d}z\right|^2 |Tf(x)|^2\,\mathrm{d}x, \\
&\le {M_\phi}^2 \int_{\mathrm{R}^d} |Tf(x)|^2\,\mathrm{d}x, \\
&\le {M_\phi}^2 \|Tf\|^2_{L^2(\mathrm{R}^d)},
\end{aligned}$$

whence the desired conclusion. □

4 Regularizing the Pseudo-inverse

In this section, we set:

$$E := L^2(V_1) \cap H^s(\mathrm{R}^d),$$

and we endow E with the inner product of $H^s(\mathrm{R}^d)$ (defined in Equation (2.2)). We consider the instances of $(\mathscr{P}_\beta)$ and $(\mathscr{Q}_\beta)$ corresponding to the above choice of E, and we study the asymptotic behavior of $(\mathscr{P}_\beta)$ as $\beta \downarrow 0$.

Clearly, E is dense in $L^2(V_1)$. As expressed in Remark 3.2, the solution Φ_β obtained with this choice is the same as the solution to $(\mathscr{Q}'_\beta)$.

Throughout this section, T^+ denotes the pseudo-inverse of T regarded as an operator from $L^2(V_1)$ into G. Notice that, since $T\colon L^2(\mathrm{R}^d) \to G$ is assumed to be injective, T^+ is nothing but the inverse of

$$T\colon L^2(V_1) \to T\big(L^2(V_1)\big).$$

Notice also that, whenever $g \in T(L^2(V))$, T^+g is also the inverse image of g by $T\colon L^2(V) \to T(L^2(V))$.

We emphasize that the operator norm to be minimized in $(\mathscr{Q}_\beta)$ is $\|\cdot\|_{L(E,G)}$. This choice, which comes from the particular class of functions ϕ considered here, is justified by the proofs of the technical lemmas we are about to state.

We shall prove that, in some sense, C_β converges to the identity and Φ_β converges to the projection onto the closure of $\operatorname{ran} T\lfloor_E$ as $\beta \downarrow 0$. Notice that this projection is a solution to the following limit problem:

$$(\mathscr{Q}_0)\quad \left|\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\big\| T\lfloor_E - XT\lfloor_E \big\|^2_{L(E,G)} \\ \text{s.t.} & X \in L(G),\ \ X = 0 \text{ on } (\operatorname{ran} T\lfloor_E)^\perp. \end{array}\right.$$

As for Problem $(\mathscr{P}_\beta)$, the corresponding limit problem reads:

$$\left|\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\big\| TT^+g - Tf \big\|^2_G \\ \text{s.t.} & f \in L^2(V_1). \end{array}\right.$$

Since $T^+TT^+ = T^+$, the latter problem is equivalent to the classical least squares problem:

$$(\mathscr{P}_0)\quad \left|\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\big\| g - Tf \big\|^2_G \\ \text{s.t.} & f \in L^2(V_1), \end{array}\right.$$

whose solution is nothing but T^+g whenever $g \in \mathscr{D}(T^+)$.

The following theorem is an extension of Theorem 2.1. Since many technicalities of the proof given in [1] remain valid in our extended setting, we shall often refer to the latter reference.

Theorem 4.1. *Consider Problem $(\mathscr{P}_\beta)$ above, in which ϕ_β is as in Equation (2.1).*

I. *Let $\alpha > 0$ and $\beta \in (0,1]$ be fixed. Assume that Problem $(\mathscr{Q}_\beta)$ has a solution Φ_β. Then $(\mathscr{P}_\beta)$ has a unique solution f_β. Moreover, f_β depends continuously on $g \in G$.*

II. *Assume that $\hat\phi(\xi) \neq 1$ for all $\xi \in \mathrm{R}^d \setminus \{0\}$, and that there exist positive numbers K and s such that $|1 - \hat\phi(\xi)| \sim_{\xi\to 0} K \big\|\xi\big\|^s$. Assume in addition that, for every $\beta \in (0,1]$, Problem $(\mathscr{Q}_\beta)$ has a solution Φ_β. Let $g \in \mathscr{D}(T^+)$ be such that T^+g belongs to E. Then, the unique solution f_β of Problem $(\mathscr{P}_\beta)$ converges strongly to T^+g in $L^2(\mathrm{R}^d)$.*

We shall need the following lemma, whose proof can be found in [1].

Lemma 4.2. *Let ϕ be as in Theorem 4.1, and let*

$$m_\beta := \min_{\|\xi\|=1} |1-\hat{\phi}(\beta\xi)|^2 \quad \text{and} \quad M_\beta := \max_{\|\xi\|=1} |1-\hat{\phi}(\beta\xi)|^2$$

Then, the following hold:

(i) *For all $\beta > 0$, one has $0 < m_\beta \leq M_\beta \leq \left(1 + \|\phi\|_{L^1(\mathrm{R}^d)}\right)^2$;*

(ii) $\sup_{\beta>0}(M_\beta/m_\beta) < \infty$ *and $M_\beta \to 0$ as $\beta \downarrow 0$;*

(iii) *there exist two positive constants ν_0 et A_0 such that, for all $\beta \in (0,1]$ and all $\xi \in \mathrm{R}^d \setminus \{0\}$,*

$$\nu_0 \left(\|\xi\|^{2s} 1_{B_{1/\beta}}(\xi) + \frac{1}{M_\beta} 1_{B^c_{1/\beta}}(\xi) \right) \leq \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2} \leq A_0 \|\xi\|^{2s}.$$

The last estimate is the corner stone of the proof of Theorem 4.1. In particular, it will provide us with a bound on the norm of $I - C_\beta$.

Proposition 4.3. *In the context of Theorem 4.1 and Lemma 4.2, the following holds:*

$$\forall \beta \in (0,1], \quad \|I - C_\beta\|^2_{L(E,L^2(\mathrm{R}^d))} \leq M_\beta A_0,$$

where I stands for the canonical continuous injection of E into $L^2(\mathrm{R}^d)$.

Proof. By definition,

$$\begin{aligned}
&\|I - C_\beta\|^2_{L(E,L^2(\mathrm{R}^d))} \\
&= \sup_{\substack{h\in E\\ \|h\|_s=1}} \|h - \phi_\beta * h\|^2_{L^2(\mathrm{R}^d)} \\
&= \sup_{\substack{h\in E\\ \|h\|_s=1}} \|(1-\hat{\phi}_\beta)\hat{h}\|^2_{L^2(\mathrm{R}^d)} \\
&= \sup_{\substack{h\in E\\ \|h\|_s=1}} \int_{\mathrm{R}^d} |1-\hat{\phi}(\beta\xi/\|\xi\|)|^2 \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2} |\hat{h}(\xi)|^2 \, \mathrm{d}\xi \\
&\leq \sup_{\substack{h\in E\\ \|h\|_s=1}} M_\beta A_0 \int_{\mathrm{R}^d} \|\xi\|^{2s} |\hat{h}(\xi)|^2 \, \mathrm{d}\xi \\
&\leq M_\beta A_0 \sup_{\substack{h\in E\\ \|h\|_s=1}} \|h\|^2_s \\
&= M_\beta A_0,
\end{aligned}$$

where the first inequality stems from Lemma 4.2. □

The following application of the above lemma is in accordance with the intuition about the behavior of Φ_β as $\beta \downarrow 0$.

Proposition 4.4. *In the context of Theorem 4.1 and Lemma 4.2,*

$$\|\Phi_\beta T - T\|_{L(E,G)} \to 0 \quad \text{as} \quad \beta \downarrow 0.$$

Proof. We shall use the continuity of T on $L^2(\mathrm{R}^d)$. Since Φ_β is a solution to $(\mathscr{Q}_\beta)$, one has:

$$\begin{aligned}\|\Phi_\beta T - T\|_{L(E,G)} &\leq \|\Phi_\beta T - TC_\beta\|_{L(E,G)} + \|T - TC_\beta\|_{L(E,G)}\\ &\leq 2\|T - TC_\beta\|_{L(E,G)}\\ &\leq 2\|T\|_{L(L^2(\mathrm{R}^d),G)}\|I - C_\beta\|_{L(E,L^2(\mathrm{R}^d))}.\end{aligned}$$

The conclusion then follows from Lemma 4.2. □

Since Φ_β is designed so as to vanish on $(\operatorname{ran} T_{\lfloor E})^\perp$, we see that, Φ_β converges in the weak sense to the orthogonal projection onto the closure of $\operatorname{ran} T_{\lfloor E}$.

We are ready to prove our main theorem.

Proof of the Theorem. Following [1], we divide the proof into three steps. In Step 1, the L^2-norm of f_β is bounded above by a quantity which does not depend on β. The weak convergence of f_β to T^+g is then established in Step 2. Finally, in Step 3, some of the estimates of Step 1 allow us to call on a classical compactness theorem to show that the convergence is in fact strong.

Step 1: L^2-estimate. Since f_β is the solution to $(\mathscr{P}_\beta)$, the following inequality is satisfied by all $f \in L^2(V_1)$:

$$\frac{1}{2}\left\|\Phi_\beta g - Tf_\beta\right\|_G^2 + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_\beta\right\|^2_{L^2(\mathrm{R}^d)} \leq \frac{1}{2}\left\|\Phi_\beta g - Tf\right\|_G^2 + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}\right\|^2_{L^2(\mathrm{R}^d)}.$$

The choice $f = T^+g$ yields

$$\frac{1}{2}\left\|\Phi_\beta g - Tf_\beta\right\|_G^2 + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_\beta\right\|^2_{L^2(\mathrm{R}^d)} \leq \frac{1}{2}\left\|\Phi_\beta g - TT^+g\right\|_G^2 + \frac{\alpha}{2}\left\|(I-C_\beta)T^+g\right\|^2_{L^2(\mathrm{R}^d)}.$$

Recall that Φ_β vanishes on $T(E)^\perp = T(L^2(V_1))^\perp$, so that $\Phi_\beta g = \Phi_\beta TT^+g$. The above inequality then implies that

$$\frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_\beta\right\|^2_{L^2(\mathrm{R}^d)} \leq \frac{1}{2}\left\|(\Phi_\beta T - T)T^+g\right\|_G^2 + \frac{\alpha}{2}\left\|(I-C_\beta)T^+g\right\|^2_{L^2(\mathrm{R}^d)}.$$

On the one hand,

$$\begin{aligned}&\frac{1}{2}\left\|(\Phi_\beta T - T)T^+g\right\|_G^2\\ &\leq \frac{1}{2}\left\|T^+g\right\|_s^2\left\|\Phi_\beta T - T\right\|^2_{L(E,G)}\\ &\leq \frac{1}{2}\left\|T^+g\right\|_s^2\left(\left\|\Phi_\beta T - TC_\beta\right\|_{L(E,G)} + \left\|T - TC_\beta\right\|_{L(E,G)}\right)^2\\ &\leq \frac{1}{2}\left\|T^+g\right\|_s^2\left(\left\|T - TC_\beta\right\|_{L(E,G)} + \left\|T - TC_\beta\right\|_{L(E,G)}\right)^2\\ &\leq 2\left\|T^+g\right\|_s^2\left\|T\right\|^2_{L(L^2(\mathrm{R}^d),G)}\left\|I - C_\beta\right\|^2_{L(E,L^2(\mathrm{R}^d))}\\ &\leq 2\left\|T^+g\right\|_s^2\left\|T\right\|^2_{L(L^2(\mathrm{R}^d),G)}A_0M_\beta,\end{aligned}$$

in which the last inequality results from Proposition 4.3. On the other hand,

$$\begin{aligned}\frac{\alpha}{2}\left\|(I-C_\beta)T^+g\right\|^2_{L^2(\mathrm{R}^d)} &\le \frac{\alpha}{2}\left\|T^+g\right\|^2_s\left\|(I-C_\beta)\right\|^2_{L(E,L^2(\mathrm{R}^d))}\\ &\le \frac{\alpha}{2}\left\|T^+g\right\|^2_s A_0M_\beta,\end{aligned}$$

by Proposition 4.3 again. Consequently,

$$\begin{aligned}&\frac{\alpha}{2}\left\|(1-\hat\phi_\beta)\hat f_\beta\right\|^2_{L^2(\mathrm{R}^d)}\\ &\le \left(2\left\|T^+g\right\|^2_s\left\|T\right\|^2_{L(L^2(\mathrm{R}^d),G)}+\frac{\alpha}{2}\left\|T^+g\right\|^2_s\right)A_0M_\beta.\end{aligned}$$

Furthermore, it can be shown (see [1]) that $\|(1-\hat\phi_\beta)\hat f_\beta\|^2_{L^2(\mathrm{R}^d)}$ is bounded below by

$$m_\beta\left(\nu_0\int_{\|\xi\|\le 1/\beta}\|\xi\|^{2s}\left|\hat f_\beta(\xi)\right|^2\,\mathrm{d}\xi+\frac{\nu_0}{M_\beta}\int_{\|\xi\|>1/\beta}\left|\hat f_\beta(\xi)\right|^2\,\mathrm{d}\xi\right)$$

Finally, putting things together, we obtain the following estimate: for all $\beta\in(0,1]$,

$$\nu_0\int_{\|\xi\|\le 1/\beta}\|\xi\|^{2s}\left|\hat f_\beta(\xi)\right|^2\,\mathrm{d}\xi+\frac{\nu_0}{M_\beta}\int_{\|\xi\|>1/\beta}\left|\hat f_\beta(\xi)\right|^2\,\mathrm{d}\xi\le A,$$

where

$$A:=A_0\left(\sup_{\beta>0}\frac{M_\beta}{m_\beta}\right)\left(\frac{4}{\alpha}\left\|T^+g\right\|^2_s\left\|T\right\|^2_{L(L^2(\mathrm{R}^d),G)}+\left\|T^+g\right\|^2_s\right).$$

Notice that Lemma 4.2 (ii) implies that A is a finite (positive) real number. We proceed again as in [1] to deduce that

$$\forall\beta\in(0,1],\quad \left\|f_\beta\right\|^2_{L^2(\mathrm{R}^d)}\le\left\|F^{-1}_{B_1^c}\right\|^2\left(1+\|\phi\|_{L^1(\mathrm{R}^d)}\right)^2\frac{A}{\nu_0},$$

where

$$\begin{array}{llcl}F_{B_1^c}: & L^2(\mathrm{R}^d) & \longrightarrow & L^2(B_1^c)\\ & f & \longmapsto & 1_{B_1^c}\hat f.\end{array}$$

has a bounded inverse.

Step 2: weak convergence. Let $(\beta_n)_{n\in\mathrm{N}^*}$ be any positive sequence converging to zero. If we can prove that the sequence $(f_n)_{n\in\mathrm{N}^*}$ defined by

$$f_n:=f_{\beta_n}$$

has a subsequence which converges weakly to T^+g, then the weak convergence of f_β to T^+g will be established, because the sequence $(\beta_n)_{n\in\mathrm{N}^*}$ is arbitrary.

From Step 1, we know that such a sequence (f_n) is bounded, and thus from the Weak Compactness Theorem that it has an accumulation point in $L^2(V_1)$, which we call f'. It is now sufficient to prove that $f'=T^+g$. Recall that T^+g is the unique solution to

$$(\mathscr{P}_0)\quad\left|\begin{array}{rl}\text{minimize} & \frac{1}{2}\left\|g-Tf\right\|^2_G\\ \text{s.t.} & f\in L^2(V_1).\end{array}\right.$$

Let $\Phi_{n_k} := \Phi_{\beta_{n_k}}$. Since f_{n_k} is the solution to $(\mathscr{P}_{\beta_{n_k}})$, for every $f \in L^2(V_1)$,

$$\frac{1}{2}\left\| \Phi_{n_k} g - T f_{n_k} \right\|_G^2 + \frac{\alpha}{2}\left\| (1-\hat{\phi}_{n_k})\hat{f}_{n_k} \right\|_{L^2(\mathrm{R}^d)}^2 \leq \frac{1}{2}\left\| \Phi_{n_k} g - T f \right\|_G^2 + \frac{\alpha}{2}\left\| (1-\hat{\phi}_{n_k})\hat{f} \right\|_{L^2(\mathrm{R}^d)}^2.$$

We have:

$$\left\| \Phi_{n_k} g - T f_{n_k} \right\|_G^2 \leq \left\| \Phi_{n_k} g - T f \right\|_G^2 + \alpha \left\| (I - C_{\beta_{n_k}}) f \right\|_{L^2(\mathrm{R}^d)}^2. \tag{4.1}$$

Since $g \in \mathscr{D}(T^+)$, it can be written as $g = g_1 + g_2$ with $g_1 \in \operatorname{ran} T$ and $g_2 \in (\operatorname{ran} T)^\perp$, and we have $\Phi_{n_k} g = \Phi_{n_k} g_1$. Moreover recall that $T^+ g = T^+ g_1$, so that we can assume that $g \in \operatorname{ran} T$. Let k tend to ∞ in this inequality. Proposition 4.4 implies that $\Phi_{n_k} g \longrightarrow g$. Moreover, it is clear that $T f_{n_k}$ converges weakly to Tf'. It follows that $\Phi_{n_k} g - T f_{n_k}$ converges weakly to $g - Tf'$, so that

$$\left\| g - Tf' \right\|_G^2 \leq \liminf_{k\to\infty} \left\| \Phi_{n_k} g - T f_{n_k} \right\|_G^2.$$

On the other hand, $\Phi_{n_k} g - Tf$ converges strongly to $g - Tf$ and by Proposition 4.3 again $\|(1-\hat{\phi}_{n_k})\hat{f}\|_{L^2} = \|(I - C_{\beta_{n_k}})f\|_{L^2}$ tends to 0. We deduce that the right hand side of (4.1) converges to $\|g - Tf\|_G^2$, so that

$$\left\| g - Tf' \right\|_G^2 \leq \left\| g - Tf \right\|_G^2.$$

Since this holds for every $f \in L^2(V_1)$, we see that f' must be the unique solution to $(\mathscr{P}_0)$.

Step 3: strong convergence. Since $(f_n)_{n\in\mathrm{N}^*} \subset L^2(V_1)$ is bounded and so is V_1, it is clear that

$$\lim_{R\to\infty} \sup_{n\in\mathrm{N}^*} \int_{\|x\|>R} \left| f_n(x) \right|^2 \, \mathrm{d}x = 0.$$

Furthermore, it was proved in [1] that, under the estimates of Step 1,

$$\sup_{n\in\mathrm{N}^*} \left\| \mathscr{T}_h f_n - f_n \right\|_{L^2(\mathrm{R}^d)} \to 0 \quad \text{as} \quad \|h\| \to 0, \tag{4.2}$$

where for every $h \in \mathrm{R}^d$ and every $f \in L^2(\mathrm{R}^d)$, $\mathscr{T}_h f$ denote the translated function $x \mapsto f(x-h)$.

Finally, it results from [4, Theorem 3.8 page 175], that the sequence $(f_n)_{n\in\mathrm{N}^*}$ defined in Step 2 is relatively compact, and thus that the convergence of $(f_n)_{n\in\mathrm{N}^*}$ to T^+g obtained in Step 2 is in fact strong. □

Remark 4.5. In order to remain coherent with our regularization principle, the data g should be taken in the smaller subspace

$$T(L^2(V) \cap H^s(\mathrm{R}^d)) + T(L^2(V_1))^\perp.$$

In this case, f_β converges to an element of $L^2(V)$. The only reason for replacing V by V_1 was to avoid possible boundary effects in the solution to $(\mathscr{P}_\beta)$. The whole construction, including Theorem 4.1, is valid with any compact set $V' \supset V$.

5 Conclusion

The regularization by mollification was originally designed for problems of Fourier synthesis. In this paper, we have extended this regularization principle to a wider class of ill-posed operator equations. Our main result is the extension of a convergence result obtained by Alibaud *et al.* [1], which demonstrates the asymptotic coherence of this regularization scheme.

Our extension relies on the definition of an operator Φ_β (depending on the new regularization parameter β) which can be regarded as a surrogate of the data regularization $g \mapsto \hat{\phi}_\beta g$ appearing naturally in Fourier synthesis. As pointed out in Section 3, Φ_β is defined through an ill-posed problem, and numerical difficulties can be expected in the computation of Φ_β. However, we believe that these difficulties are not insurmountable, for the following reasons.

Observe first that, in order to compute (the finite dimensional approximation of) $\Phi_\beta g = TC_\beta T^+ g$, it may be easier to first compute $T^+ g$, via an ill-posed least squares problem, and then apply TC_β. This may be the only possible approach whenever the dimension of the problem does not allow for the computation and storage of $TC_\beta T^+$. Again, ill-posedness of the computation of $T^+ g$ must be addressed.

In fact, the ill-posedness we are facing encompasses two aspects: first, the difficulty to reach an accurate numerical estimation of $T^+ g$; second, the sensitivity of $T^+ g$ to perturbations δg of the data g (which involves the norm of T^+). Concerning the numerical accuracy, let us merely mention that high accuracy may be reached with bad (but not dramatic) condition: for example, a proximal strategy (which is a particular case of *iterative refinement*) may be considered. Concerning the sensitivity to perturbations, one should keep in mind that the norm of T^+ is not really what matters, since the *regularized data* is rather $TC_\beta T^+ g$. Recall that the operators of interest here are such that $TC_\beta T^+$ is continuous (see Proposition 3.1). One can reasonably expect that, in the finite dimensional version of our extended regularization principle, the continuity of $TC_\beta T^+$ will result in a reasonable norm of the corresponding matrix.

The analysis presented in this paper provides the basis for the implementation of a reconstruction algorithm for a wide class of inverse problems. The actual design of an algorithm is currently under study, and it is deferred to future publication.

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