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STABILITY OF GLOBAL MINIMUM POINTS OF LOWER SEMICONTINUOUS FUNCTIONS

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Abstract: We provide an extension of a result by Ioffe and Schwartzman [15], on the homotopical stability of global minimum points of continuous functions defined on complete metric spaces. Our approach is based on Ekeland's variational principle, rather than on the deformation techniques of nonsmooth (metric) critical point theory used in [15].

Key words: *variational principle, slopes, global minima, stability*

Mathematics Subject Classification: *58E30, 49J52, 49K40*

1 Introduction

Motivated by a series of studies by Bobylev and other Russian mathematicians (see [14]), on the homotopical stability of (local or global) minimum points for various classes of smooth functions on Banach spaces, Ioffe and Schwartzman initiated in [15] a critical point theory for continuous functions defined on complete metric spaces. One of the main results of [15] in that framework is a so-called Potential well theorem, providing an *a priori* estimate for the size of the potential well associated with a local minimum point, and allowing dealing with the homotopical stability of (isolated) local minimum points. These results were revisited by the first author and Hantoute [9], in the light of the nonsmooth critical point theory that had been developed independently in [8, 12], featuring in particular the notion of weak slope from [12], and the so-called Change-of-metric principle from [6]. It was also shown in [9] that if some of the arguments involved indeed employ the methods of critical point theory, part of the arguments can be established in a simpler way, relying on Ekeland's variational principle [13], thus on abstract results featuring the notion of strong slope from [11].

In [15], the question of the homotopical stability of global minimum points is also addressed, through a similar approach as in the local case. The purpose of this note is to show that in the global case, this question can be treated using Ekeland's variational principle only. Roughly speaking, and as we put in our previous paper [10], where this note was announced, the reason is that "when dealing with global minima we definitely know the 'size' of the potential well." As an immediate consequence, the results we obtain are valid in the lower semicontinuous case, rather than the continuous one as in [15].

In Section 2, we recall the basic form of Ekeland's principle and derive some simple lemmas involving the strong slope, to be used for the proof of our main results. In Section 3, we recall a basic deformation theorem in metric critical point theory, and we derive a

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criterion for a global minimum, similar to a result in [15], pointing out, in particular, that such a result is no more true replacing the weak slope by the strong slope. Our main results, on the (homotopical) stability of global minimum points, are in Section 4.

2 Ekeland's Principle and the Strong Slope

Throughout this note, X is a metric space endowed with the metric d. For $C \subset X$ and $\rho > 0$, we denote by $B_\rho(C)$ (resp., $\bar{B}_\rho(C)$) the open (resp., closed) ρ-neighborhood of C:

$$B_\rho(C) := \{y \in X : \ d(y, C) < \rho\}, \qquad \bar{B}_\rho(C) := \{y \in X : \ d(y, C) \leq \rho\},$$

where $d(y, C) := \inf\{d(y, x) : \ x \in C\}$, with the usual convention $d(y, \emptyset) = +\infty$ (according to the general convention $\inf \emptyset = +\infty$). We also set

$$\partial B_\rho(C) := \{y \in X : \ d(y, C) = \rho\},$$

while for $x \in X$ we simply write $B_\rho(x)$ for $B_\rho(\{x\})$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$. As usual, we say that f is *proper* if the set $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$ is nonempty. Recall from [17] that a *d-point* of f is a point $z \in X$ such that

$$f(z) < f(x) + d(x, z) \qquad \text{for every } x \in X, \ x \neq z.$$

Clearly, d-points of f belong to $\mathrm{dom} f$, and global minimum points of a proper f (if any) are d-points of f. Ekeland's variational principle [13], in its basic form, asserts that if (X, d) is complete, and if f is proper, lower semicontinuous, and bounded from below, then f has a d-point. This is proved using a simple iterative construction using (closed) sets of the type

$$M_{f,d}(x) := \{y \in X : \ f(y) + d(y, x) \leq f(x)\}.$$

Using the triangular inequality, it is readily seen that, given $x \in X$ we have

$$d\text{-points of the restriction of } f \text{ to } M_{f,d}(x) \text{ are } d\text{-points of } f. \tag{2.1}$$

Recall also from [11] that the *strong slope* of f at $x \in \mathrm{dom} f$ is defined and denoted by

$$|\nabla f|(x) := \begin{cases} 0 & \text{if } x \text{ is a local minimum point of } f \\ \displaystyle\limsup_{y \to x} \frac{f(x) - f(y)}{d(x, y)} & \text{otherwise} \end{cases},$$

while for $x \in X \setminus \mathrm{dom} f$, we set $|\nabla f|(x) := +\infty$.

In the remainder of this section, we assume that the metric space (X, d) is complete, and that the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

Proposition 2.1. *If $m := \inf_X f \in \mathbb{R}$, and if $(x_h) \subset X$ is a sequence such that $f(x_h) \to m$, then there exists a sequence $(z_h) \subset X$ such that $d(z_h, x_h) \to 0$, $f(z_h) \to m$, and $|\nabla f|(z_h) \to 0$.*

Proof. If $f(x_h) = m$, set $z_h := x_h$; otherwise, let $0 < \varepsilon_h := \sqrt{f(x_h) - m} \to 0$. According to (2.1), f has an $\varepsilon_h d$-point z_h in the set

$$M_{f,\varepsilon_h d}(x_h) = \{y \in X : \ f(y) + \varepsilon_h d(y, x_h) \leq f(x_h)\},$$

so that $f(z_h) \leq f(x_h)$ and $d(z_h, x_h) \leq \varepsilon_h$, while $|\nabla f|(z_h) \leq \varepsilon_h$ in view of the definition of an $\varepsilon_h d$-point, and of the definition of the strong slope. $\square$

The following three simple lemmas provide some key ingredients needed for the proofs of our main results, in the next sections. For $b \in \mathbb{R}$, we set

$$[f \le b] := \{x \in X : \ f(x) \le b\}\,.$$

Lemma 2.2. *Let C be a nonempty subset of X. Assume that $\inf_C f \in \mathbb{R}$, and that for every $b \in \mathbb{R}$*

$$\inf_{[f \le b] \setminus C} |\nabla f| > 0\,.$$

Then f is bounded from below if and only if

$$\inf_X f = \inf_C f\,.$$

Proof. If f is bounded from below, assuming that $m := \inf_X f < \inf_C f =: m'$ yields

$$\inf_{[f \le b]} |\nabla f| > 0 \qquad \text{for } m < b < m'\,,$$

contradicting Proposition 2.1. The converse is obvious, since $m' \in \mathbb{R}$. $\square$

Lemma 2.3. *Let C be a nonempty subset of* $\mathrm{dom} f$ *such that the function f is constant on C, and let $\rho_0 > 0$. Assume that*

$$f(C) = \inf_{B_{\rho_0}(C)} f\,, \tag{2.2}$$

and that for every $0 < \rho < \rho_0$

$$\inf_{B_{\rho_0}(C) \setminus B_\rho(C)} |\nabla f| > 0\,. \tag{2.3}$$

Then for every $0 < r < \rho_0$ we have

$$\inf_{\partial B_r(C)} f > f(C)\,.$$

Proof. Set $\tilde{f} := \max\{f, f(C)\}$, so that $\inf\limits_X \tilde{f} = f(C)$, and $\tilde{f}$ coincides with f on the open set $B_{\rho_0}(C)$, according to (2.2). Assuming that $\inf_{\partial B_r(C)} f = f(C)$ for some $0 < r < \rho_0$, Proposition 2.1 yields a sequence $(z_h) \subset X$ such that $d(z_h, \partial B_r(C)) \to 0$ and $|\nabla \tilde{f}|(z_h) = |\nabla f|(z_h) \to 0$, which contradicts (2.3). $\square$

We say that a subset C of X is *bounded* if it is contained in a ball.

Lemma 2.4. *Assume that $m := \inf_X f \in \mathbb{R}$, and that for every $b \in \mathbb{R}$, there exists a bounded subset C of X such that*

$$\inf_{[f \le b] \setminus C} |\nabla f| > 0\,.$$

Then f is coercive*, that is, for $B \subset X$ we have*

$$\sup_B f < +\infty \implies B \textit{ is bounded.}$$

Proof. Let $B \subset X$ be nonempty and such that $b := \sup_B f < +\infty$. Let further C be a bounded subset of X and $\sigma > 0$ be such that

$$\inf_{[f \le b] \setminus C} |\nabla f| > \sigma\,. \tag{2.4}$$

Assuming that B is not bounded, we find $x \in B$ such that $d(x,C) > \frac{b-m}{\sigma}$. According to (2.1), f has a σd-point $\bar{x}$ in the set

$$M_{f,\sigma d}(x) = \{y \in X : \ f(y) + \sigma d(y,x) \leq f(x)\}.$$

Since $M_{f,\sigma d}(x) \subset [f \leq b] \setminus C$ (by the choice of x), while $|\nabla f|(\bar{x}) \leq \sigma$ (in view of the definition of a σd-point and of the definition of the strong slope), we obtain a contradiction with (2.4). ☐

Remark 2.5. (a) It is readily seen that assumption (2.3) of Lemma 2.3 is equivalent to the following Palais-Smale type condition:

If $(x_h) \subset B_{\rho_0}(C)$ is a sequence such that $|\nabla f|(x_h) \to 0$, then $d(x_h, C) \to 0$.

This type of condition is discussed in detail in [10], where it is shown how it yields so-called *nonlinear error bound* estimates.

(b) Similarly, if f is bounded below, we see that the main assumption of Lemma 2.4 is equivalent to the following condition:

Every sequence $(x_h) \subset X$ such that $(f(x_h))$ is bounded and $|\nabla f|(x_h) \to 0$, is bounded.

The proof of Lemma 2.4, using the basic form of Ekeland's principle, could be used to recover, in a more straightforward way, the (more general) coercivity result of [5]. Of course, whenever $(X, \|\cdot\|)$ is a normed vector space, the coercivity of a function f on X amounts to $f(x) \to +\infty$ as $\|x\| \to +\infty$.

$\boxed{3}$ The Weak Slope and a Criterion for a Global Minimum

In this section, $f : X \to \mathbb{R}$ is continuous. Recall from [12] that the *weak slope* of f at $x \in X$, denoted by $|df|(x)$, is the upper bound of the set of nonnegative reals σ such that there exist $\delta > 0$ and a continuous $\mathcal{H} : B_\delta(x) \times [0,\delta] \to X$ with

$$d(\mathcal{H}(y,t), y) \leq t \qquad \text{and} \qquad f(\mathcal{H}(y,t)) \leq f(y) - \sigma t$$

for every $(y,t) \in B_\delta(x) \times [0,\delta]$. It is easy to see that $|df| \leq |\nabla f|$ (which accounts for the terminology employed for these notions). If X is a C^1 Finsler manifold and f is a C^1 function, then $|df|(x) = |\nabla f|(x) = \|f'(x)\|$ for every $x \in X$ (see [12]).

The following is a slight variant of the Noncritical Interval Theorem [8, Theorem (2.15)] (see also [7, Theorem 2]). For $a, b \in \mathbb{R}$ with $a < b$, we set

$$[a \leq f \leq b] := \{x \in X : \ a \leq f(x) \leq b\}.$$

Theorem 3.1. *Let (X,d) be a complete metric space, let $f : X \to \mathbb{R}$ be continuous, and let $a, b \in \mathbb{R}$ with $a < b$. Assume that*

$$\inf_{[a \leq f \leq b]} |df| > 0.$$

Then $[f \leq a]$ is a strong deformation retract *of $[f \leq b]$, that is, there exists a continuous $\eta : [f \leq b] \times [0,1] \to [f \leq b]$ such that:*

(a) $\eta(x,0) = x$ *for every* $x \in [f \leq b]$ *;*

(b) $\eta(x,t) = x$ *for every* $(x,t) \in [f \leq a] \times [0,1]$ *;*

(c) $\eta([f \leq b], 1) \subset [f \leq a]$.

Theorem 3.2. *Let (X, d) be a complete, arcwise connected metric space, let $f : X \to \mathbb{R}$ be continuous, and let C be a nonempty compact subset of X such that the function f is constant on C. Assume that*

$$C \text{ is a set of local minimum points of } f\,, \tag{3.1}$$

and that for every $\rho > 0$ and for all $a, b \in \mathbb{R}$ with $a < b$, we have

$$\inf_{[a\le f\le b]\setminus B_\rho(C)} |df| > 0\,. \tag{3.2}$$

Then C is a set of global minimum points of f, and f is coercive.

Proof. Since C is compact and f is continuous, (3.1) implies (2.2) for some $\rho_0 > 0$ such that f is bounded above on $B_{\rho_0}(C)$, and since $|\nabla f| \ge |df|$, (3.2) then implies (2.3). We deduce from Lemma 2.3 that for every $r > 0$ small enough we have

$$\inf_{\partial B_r(C)} f > f(C)\,. \tag{3.3}$$

Let $x_0 \in C$ and assume, for a contradiction, that there exists $x_1 \in X$ such that $f(x_1) < f(x_0)$. Let Γ denote the set of continuous $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$, and set:

$$c := \inf_{\gamma\in\Gamma} \max_{[0,1]} (f \circ \gamma)\,,$$

so that $c > f(x_0)$, according to (3.3). Let $0 < \varepsilon < c - f(x_0)$, and let $\rho > 0$ be such that $f(x) < c - \varepsilon$ for every $x \in B_\rho(C)$. Then

$$\inf_{[c-\varepsilon\le f\le c+\varepsilon]} |df| > 0\,,$$

according to (3.2). Applying Theorem 3.1 with $a := c - \varepsilon > f(x_0)$ and $b := c + \varepsilon$, we find a continuous $\eta : [f{\le}c{+}\varepsilon] \times [0,1] \to [f{\le}c{+}\varepsilon]$ such that

$$\eta(x_0, 1) = x_0\,, \quad \eta(x_1, 1) = x_1\,, \quad \text{and } \eta([f{\le}c{+}\varepsilon], 1) \subset [f{\le}c{-}\varepsilon]\,.$$

Let $\gamma \in \Gamma$ with $\gamma([0,1]) \subset [f{\le}c{+}\varepsilon]$, according to the definition of c. Defining $\tilde\gamma : [0,1] \to X$ by $\tilde\gamma(t) := \eta(\gamma(t), 1)$ we thus have $\tilde\gamma \in \Gamma$, while $\tilde\gamma([0,1]) \subset [f{\le}c{-}\varepsilon]$, contradicting the definition of c.

Since f is bounded from below, and since $|\nabla f| \ge |df|$, so that for every $b \in \mathbb{R}$ we have

$$\inf_{[f\le b]\setminus B_1(C)} |\nabla f| > 0$$

according to (3.2), we obtain from Lemma 2.4 that f is coercive. □

Remark 3.3. (a) In critical point theory, a sequence $(x_h) \subset X$ is called a *Palais-Smale sequence* for the (continuous) $f : X \to \mathbb{R}$ if

$$(f(x_h)) \text{ is bounded and } |df|(x_h) \to 0\,,$$

and f is said to satisfy the *Palais-Smale condition* if every Palais-Smale sequence for f has a convergent subsequence. Due to the (obvious) lower semicontinuity of $|df|$, a cluster point x of a Palais-Smale sequence for f is a *critical point* of f, that is: $|df|(x) = 0$. Assumptions (3.1) and (3.2) are clearly equivalent to

The set of critical points of f is a set C of local minimum points, and f satisfies the Palais-Smale condition.

(Note in particular that under (3.1) and (3.2), every Palais-Smale sequence for f must converge to a point of C.) The argument of the proof of Theorem 3.2 is that of the celebrated Mountain pass theorem of Ambrosetti and Rabinowitz [1], see also [8, Theorem (3.7)] in our nonsmooth setting.

(b) Thanks to the methods initiated in [12, 8], and refined in [3], Theorem 3.2 can be extended to functions belonging to appropriate *classes* of lower semicontinuous functions. This involves, in particular, an extension of the notion of weak slope, as we explain after this remark. However, we stress that deformation results of the type of Theorem 3.1 do not hold for arbitrary lower semicontinuous f. Similarly, considering $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) := -x$ for $x \leq 0$, $f_1(x) := 1 - x$ for $x > 0$ (so that 0 is a local minimum point of f_1 and $|df_1|(x) = 1$ for $x \neq 0$), shows that Theorem 3.2 does not hold for arbitrary lower semicontinuous functions. Considering the continuous $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_2(x) := |x|$ for $|x| \leq 1$, $f_2(x) := 2 - |x|$ for $|x| \geq 1$ (so that 0 is a local minimum point of f_2 and $|\nabla f_2|(x) = 1$ for $x \neq 0$), shows that Theorem 3.2 does not hold replacing the weak slope by the strong one. Finally, consider the C^1 function $f_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_3(x_1, x_2) := \frac{3x_1^2 - 2x_1^3 - 1}{1 + x_2^2} + (3x_1^2 - 2x_1^3)e^{-x_2}\,.$$

It is readily checked that $(0, 0)$ is a strict local minimum and the unique critical point of f_3, but not a global minimum point; thus, f_3 does not satisfy the Palais-Smale condition (as is also easily observed by computing its gradient).

(c) The latter example is given in the monograph by Emelyanov *et al.* [14], and Theorem 3.2 is a variant of several results therein, stated for various classes of "smooth" functions on Banach spaces. Theorem 3.2 is indeed a (refined) version of Ioffe and Schwartzman's [15, Proposition 9], where C is a singleton, and where the method of proof is more in the line of that of [14]; it is derived from [15, Theorem 1], which can partly be seen as a "quantitative", local variant of Theorem 3.2. The word "quantitative" refers to so-called *nonlinear error bound estimates* for the function f with respect to the (critical) set C, see [9, 10] for a detailed analysis of such results, as already mentioned in the introduction, and evoked in Remark 2.5 (a).

We now give the extension of the notion of weak slope for an arbitrary proper function $f : X \to \overline{\mathbb{R}}$, as given by Campa and Degiovanni [3], and that we shall use in the next section, dealing again with lower semicontinuous functions.

We consider the epigraph of f

$$\mathrm{epi}f := \{(x, \mu) \in X \times \mathbb{R} : \ f(x) \leq \mu\}$$

as endowed with the metric

$$d((x, \mu), (y, \xi)) := d(x, y) + |\mu - \xi|\,.$$

For $x \in X$ with $f(x) \in \mathbb{R}$, the *weak slope* of f at x, denoted by $|df|(x)$, is the upper bound of the nonnegative reals σ such that there exist $\delta > 0$ and a continuous $\mathcal{H} : (B_\delta(x, f(x)) \cap \mathrm{epi}f) \times [0, \delta] \to X$ with

$$d(\mathcal{H}((y, \xi), t), y) \leq t \qquad \text{and} \qquad f(\mathcal{H}((y, \xi), t)) \leq \xi - \sigma t\,.$$

It is easy to see that $|\nabla f| \geq |df|$. As shown in [3, Proposition 2.2, Proposition 2.3] (taking into account the choice of a different—but equivalent—metric on epi$f$ therein), the above definition of $|df|$ agrees with the "basic" one in the case when f is (finite-valued and) continuous, while in the general case we have

$$|df|(x) = \begin{cases} \dfrac{|d\mathcal{G}_f|(x, f(x))}{1 - |d\mathcal{G}_f|(x, f(x))} & \text{if } |d\mathcal{G}_f|(x, f(x)) < 1 \\ +\infty & \text{if } |d\mathcal{G}_f|(x, f(x)) = 1 \end{cases}, \tag{3.4}$$

where $\mathcal{G}_f : \text{epi}f \to \mathbb{R}$ is defined by $\mathcal{G}_f(x, \mu) := \mu$ (note that $\mathcal{G}_f$ is 1-Lipschitz continuous, so that $|d\mathcal{G}_f| \leq 1$.) Indeed, formula (3.4) was used in [8, 12] as the *definition* of the weak slope in the lower semicontinuous case. Echoing Remark 3.3 (b), a class $\mathcal{C}$ of lower semicontinuous functions is "appropriate" if for $f \in \mathcal{C}$ we have

$$\inf\{|d\mathcal{G}_f|(x, \mu) : \ \mu > f(x)\} > 0 \,.$$

Under such condition, existence results of critical points for f can indeed be obtained from corresponding results for the continuous $\mathcal{G}_f$. As an example, if $(X, \|\cdot\|)$ is a normed space, and if $f = g + h$ with g locally Lipschitz continuous and h proper, lower semicontinuous, and convex, then $|d\mathcal{G}_f|(x, \mu) = 1$ if $\mu > f(x)$, while

$$|df|(x) \geq \inf\{\|\alpha + \beta\|_* : \ \alpha \in \partial g(x) \,, \ \beta \in \partial h(x)\} \,,$$

where ∂ denotes the Clarke-Rockafellar subdifferential (see [4, 16]), and $\|\cdot\|_*$ denotes the dual norm. In particular, if $|df|(x) = 0$ then $0 \in \partial f(x) \, (\subset \partial g(x) + \partial h(x))$— but not *vice versa*, that is, the above inequality is strict, in general, see, e.g., [3, Example 4.14]. For more on the connections between nonsmooth critical point theory and "classical" nonsmooth analysis, see [3], containing in particular a new notion of subdifferential operator.

In Section 4, we also need the following notion from [2]: Given $f : X \to \mathbb{R} \cup \{+\infty\}$ and $C \subset X$, we let

$$r_C(f) := \sup_{\rho > 0} \big(\inf_{B_\rho(C)} f \big) = \lim_{\rho \to 0} \big(\inf_{B_\rho(C)} f \big)$$

denote the *uniform infimum* of f on C. Of course, $r_C(f) \leq \inf_C f$. Various cases when equality holds are listed in [2, Proposition 3.2]; that is the case, for example, when f is uniformly continuous on a uniform neighborhood of C (see Remark 4.4 below).

4 Stability of Global Minimum Points

In this section, we consider a family $(f_\lambda)_{\lambda \in [0,1[}$ of lower semicontinuous functions $f_\lambda : X \to \mathbb{R} \cup \{+\infty\}$ satisfying the following properties:

(f.1) For every $\lambda \in [0, 1[$, every $x \in \text{dom} f_\lambda$, and every sequence $(\lambda_h) \subset [0, 1[$ with $\lambda_h \to \lambda$, there exists $(x_h) \subset X$ such that $f_{\lambda_h}(x_h) \to f_\lambda(x)$;

(f.2) For every $\lambda \in [0, 1[$, every bounded set B in X, and every sequence $(\lambda_h) \subset [0, 1[$ with $\lambda_h \to \lambda$, we have

$$\liminf_{h \to \infty} \big(\inf_B f_{\lambda_h} \big) \geq \inf_B f_\lambda \,.$$

Theorem 4.1. *Let (X, d) be a complete metric space, let $(f_\lambda)_{\lambda \in [0,1[}$ be a family of proper, lower semicontinuous functions on X satisfying (f.1) and (f.2), and let $(C_\lambda)_{\lambda \in [0,1[}$ be a family of subsets of X satisfying the following properties:*

(c.1) $(C_\lambda)_{\lambda\in[0,1[}$ *is locally bounded;*

(c.2) $\big(\inf_{C_\lambda} f_\lambda\big)_{\lambda\in[0,1[}$ *is locally bounded (in* $\mathbb{R}$*);*

(c.3) *For every* $\lambda \in [0,1[$ *and for every* $b\in\mathbb{R}$*, we have*

$$\inf_{[f_\lambda\le b]\setminus C_\lambda} |\nabla f_\lambda| > 0\,.$$

Then $\inf_X f_\lambda = \inf_{C_\lambda} f_\lambda$ *for every* $\lambda\in[0,1[$*, provided it is true for* $\lambda := 0$*.*

Proof. Set

$$\Lambda := \{\lambda\in[0,1[:\ \inf_X f_\lambda = \inf_{C_\lambda} f_\lambda\}\,,$$

and assume that $0\in\Lambda$. Thanks to (c.2) and (c.3), Lemma 2.2 yields

$$\Lambda = \{\lambda\in[0,1[:\ f_\lambda \text{ is bounded below}\}. \tag{4.1}$$

We show that Λ is closed in $[0,1[$. Let $(\lambda_h)\subset\Lambda$ with $\lambda_h\to\lambda\in[0,1[$, and let $m\in\mathbb{R}$ be such that

$$\big(\inf_X f_{\lambda_h} = \big)\inf_{C_{\lambda_h}} f_{\lambda_h} \ge m \qquad \text{for large } h\,,$$

according to (c.2). Let $x\in \mathrm{dom} f_\lambda$. Considering $(x_h)\subset X$ such that $f_{\lambda_h}(x_h)\to f_\lambda(x)$, according to ($f$.1), we obtain that $f_\lambda(x)\ge m$. Thus, f_λ is bounded below, so that $\lambda\in\Lambda$ according to (4.1).

We show that Λ is open in $[0,1[$. Let $\lambda\in\Lambda$, let $(\lambda_h)\subset[0,1[$ with $\lambda_h\to\lambda$, and let C be a bounded subset of X and $M\in\mathbb{R}$ be such that

$$C_{\lambda_h}\subset C \quad\text{and}\quad \inf_{C_{\lambda_h}} f_{\lambda_h} < M \qquad \text{for large } h\,, \tag{4.2}$$

according to (c.1) and (c.2). Let $x\in X$. We need to show that

$$f_{\lambda_h}(x) \ge \inf_{C_{\lambda_h}} f_{\lambda_h} \qquad \text{for large } h\,,$$

so that we may assume, without loss of generality, that $f_{\lambda_h}(x)\le M$ for every h. Applying Lemma 2.4 to f_λ, thanks to (4.1) and to (c.3), we find $\rho > d(x,C)$ such that

$$\inf_{\partial B_\rho(C)} f_\lambda > M$$

(note that the (bounded) set $\partial B_\rho(C)$ may be empty). Using (f.2) with $B := \partial B_\rho(C), B_\rho(C)$, we obtain that for large h:

$$\inf_{\partial B_\rho(C)} f_{\lambda_h} > M \quad\text{and}\quad \inf_{B_\rho(C)} f_{\lambda_h} > -\infty$$

(since f_λ is lower bounded). We thus have, for such h:

$$X_h := \bar{B}_\rho(C)\cap[f_{\lambda_h}\le M] = B_\rho(C)\cap[f_{\lambda_h}\le M]\,,$$

with (X_h,d) complete, f_{λ_h} bounded from below on X_h, and

$$|\nabla \tilde{f}_{\lambda_h}|(y) = |\nabla f_{\lambda_h}|(y) \qquad \text{for every } y\in X_h\,, \tag{4.3}$$

where $\tilde{f}_{\lambda_h}$ is the restriction of f_{λ_h} to X_h. Setting also $\tilde{C}_{\lambda_h} := C_{\lambda_h} \cap X_h$, we further have

$$\inf_{\tilde{C}_{\lambda_h}} \tilde{f}_{\lambda_h} = \inf_{C_{\lambda_h}} f_{\lambda_h} \in \mathbb{R} \qquad \text{for large } h\,,$$

according to (4.2), and we infer from (4.3) and $(c.3)$ that for every $b \in \mathbb{R}$

$$\inf_{[\tilde{f}_{\lambda_h} \le b] \setminus \tilde{C}_{\lambda_h}} |\nabla \tilde{f}_{\lambda_h}| \ge \inf_{[f_{\lambda_h} \le b] \setminus C_{\lambda_h}} |\nabla f_{\lambda_h}| > 0 \qquad \text{for large } h\,.$$

Applying Lemma 2.2 (to X_h, $\tilde{f}_{\lambda_h}$, and $\tilde{C}_{\lambda_h}$), and since $x \in X_h$, we conclude that

$$f_{\lambda_h}(x) \ge \inf_{X_h} f_{\lambda_h} = \inf_{C_{\lambda_h}} f_{\lambda_h}$$

for large h, as desired. The overall conclusion is that $\Lambda = [0, 1[\,$. □

Remark 4.2. In [9, Theorem 4.1], dealing with the homotopical stability of an isolated local minimum point z, the openness of the corresponding set Λ was established using Ekeland's principle, but the closedness required the deformation techniques of critical point theory (the Potential well theorem, as in [15]). From a "technical" point of view, this is due to the fact that the existence of a $\rho > 0$ such that z would be a minimum point of f_λ on $B_\rho(z)$ for every λ (a common "size" of the potential well) is not *a priori* known (assumed)—on the contrary, it is a conclusion of the Potential well theorem.

For our last result below, we further need a (lower semicontinuous) function $f_1 : X \to \mathbb{R} \cup \{+\infty\}$, together with the property:

$(f.3)$ For every sequence $(\lambda_h) \subset [0, 1[$ with $\lambda_h \to 1$, the sequence (f_{λ_h}) Γ-converges to f_1, that is, for every $x \in X$ we have

$$\begin{array}{rclr} \liminf\limits_{h\to\infty} f_{\lambda_h}(x_h) & \ge & f_1(x) & \text{for every } x_h \to x\,; \\ \lim\limits_{h\to\infty} f_{\lambda_h}(x_h) & = & f_1(x) & \text{for some } x_h \to x\,. \end{array}$$

For $\lambda \in [0, 1[\,$, we denote by

$$K_\lambda := \{x \in X : \ |df_\lambda|(x) = 0\}$$

the set of *critical points* of f_λ (with respect to the weak slope). The following corollary of Theorem 4.1 is the main result of this note.

Theorem 4.3. *Let (X, d) be a complete metric space, and let $(f_\lambda)_{\lambda\in[0,1[}$ be a family of (proper) lower semicontinuous functions on X satisfying $(f.1)$ and $(f.2)$. Assume that:*

$(k.1)$ *$(K_\lambda)_{\lambda\in[0,1[}$ is locally bounded;*

$(k.2)$ *For every $\lambda \in [0, 1[\,$, there is a point $z_\lambda \in K_\lambda$ such that $f_\lambda(z_\lambda) = r_{K_\lambda}(f_\lambda)$, and $(f_\lambda(z_\lambda))_{\lambda\in[0,1[}$ is locally bounded;*

$(k.3)$ *For every $\lambda \in [0, 1[\,$, every $\rho > 0$, and every $b \in \mathbb{R}$, we have*

$$\inf_{[f_\lambda \le b] \setminus B_\rho(K_\lambda)} |\nabla f_\lambda| > 0\,.$$

Then z_λ is a global minimum point of f_λ for all $\lambda \in [0,1[$, provided z_0 is a global minimum point of f_0. If, moreover, $(f.3)$ holds and $z_\lambda \to z_1$ as $\lambda \to 1$, then z_1 is also a global minimum point of f_1.

Proof. For $\lambda \in [0,1[$, let $0 < \rho_\lambda \leq 1$ be such that

$$(r_{K_\lambda}(f_\lambda) \geq) \inf_{B_{\rho_\lambda}(K_\lambda)} f_\lambda \geq r_{K_\lambda}(f_\lambda) - 1\,,$$

and set $C_\lambda := B_{\rho_\lambda}(K_\lambda)$. From the above inequalities and from $(k.2)$, we see that property $(c.2)$ of Theorem 4.1 is satisfied. On the other hand, since $(\rho_\lambda)_{\lambda\in[0,1[} \subset]0,1]$, assumptions $(k.1)$ and $(k.3)$ readily imply properties $(c.1)$ and $(c.3)$ of that result. Thus, if

$$f_0(z_0) = \inf_X f_0 = \inf_{C_0} f_0\,,$$

applying Theorem 4.1 we obtain

$$\inf_X f_\lambda = \inf_{C_\lambda} f_\lambda \qquad \text{for every } \lambda \in [0,1[\,.$$

Letting $\rho_\lambda \to 0$ for each $\lambda \in [0,1[$, and according to $(k.2)$, this yields

$$\inf_X f_\lambda = f_\lambda(z_\lambda) \qquad \text{for every } \lambda \in [0,1[\,,$$

from which the last conclusion of the theorem is a well known fact (or is easily checked). □

Remark 4.4. Theorem 4.1 and Theorem 4.3 are extensions of Theorem 5 and Theorem 6 in [15], respectively. In Ioffe and Schwartzman' results, it is assumed that the (complete) metric space X is connected, and that the function $f : [0,1] \times X \to \mathbb{R}$ defined by $f(\lambda, x) := f_\lambda(x)$ is continuous, and uniformly continuous (hence bounded) on $[0,\mu] \times B$ for every $\mu \in [0,1[$ and every bounded $B \subset X$. Note also that in this case we have

$$r_{K_\lambda}(f_\lambda) = \inf_{K_\lambda} f_\lambda \qquad \text{for every } \lambda \in [0,1[\,.$$

These assumptions, stronger than ours, are somewhat "natural" due to the approach of [15], based on deformation techniques of critical point theory (recall Section 3). However, note that since $|\nabla f| \geq |df|$ (with strict inequality in general, even for continuous f), our assumption $(k.3)$ is weaker than the corresponding one in [15].

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