Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL
Pacific Journal of Optimization
Vol. 5, No. 1, pp. 39–51, January 2009

A RELAXATION METHOD FOR SMOOTH TOMOGRAPHIC RECONSTRUCTION OF BINARY AXIALLY SYMMETRIC OBJECTS

M. Bergounioux and A. Srour

Abstract: In this paper we study a minimization problem which appears in tomographic reconstruction. The problem is known to be ill posed. The object to reconstruct is assumed to be binary so that the intensity function belongs to $\{0,1\}$. Therefore the feasible set is not convex and its interior is empty for most usual topologies. We propose a relaxed formulation of the problem. We prove existence of solutions and give optimality conditions.

Key words: *variational method, relaxation, Lagrange multipliers, mathematical programming method*

Mathematics Subject Classification: *49J20, 49M29*

---

1 Introduction

In this paper we study a minimization problem arising in tomographic reconstruction as described in [1]. We have to reconstruct a radially symmetric object with a single snapshot. In addition the image is assumed to be binary. This corresponds to the material density of the object to reconstruct (there is material or not). The problem is modelled in a standard way as follows:

$$(\mathcal{P}) \qquad \begin{cases} \min F_o(u) := \dfrac{1}{2}\|Hu-g\|^2_{L^2(\Omega)} + \lambda J(u), \\ u \in \mathcal{D} \ . \end{cases}$$

where

1. Ω is a bounded open domain in $\mathbb{R}^2$ with a smooth boundary,
2. $\mathcal{D} = \{u \in BV(\Omega) : u(u-1) = 0 \text{ a.e in } \Omega\}$, where $BV(\Omega)$ denotes the functions of bounded variation space defined by

   $$BV(\Omega) = \left\{u \in L^1(\Omega) : J(u) < \infty\right\},$$

   with

   $$J(u) = \sup\left\{\int_\Omega u(x)\mathrm{div}(\phi(x))dx : \phi \in C^1_c(\Omega,\mathbb{R}^2), \|\phi\|_\infty \le 1\right\}.$$

   Here $C^1_c(\Omega,\mathbb{R}^2)$ denotes the space of the C^1 functions with compact support in Ω with value in $\mathbb{R}^2$.

3. $H := BH_o$ where B is a linear blur (convolution) operator and H_o is the (linear) tomographic projection operator that can be written as in [2] :

$$(H_o u)(y, z) = 2 \int_{|y|}^{+\infty} u(r, z) \frac{r}{\sqrt{r^2 - y^2}} dr, \tag{1.1}$$

   for almost all $y, z \in \mathbb{R}$.

4. the observed image is $g \in L^2(\Omega)$.

In the sequel, we shall denote $||.||$ the $L^2(\Omega)$-norm. In the same way, $(.,.)_2$ denotes the $L^2(\Omega)$- scalar product, $(.,.)_{H^1}$ denotes the $H^1(\Omega)$- scalar product and $\langle .,. \rangle_{V',V}$, the duality product between V' and V, where V is a Banach space and V' is the dual space of V.

Similar problems have been studied by Aubert and Kornprobst in [5], Acar and Vogel in [3], Vogel and Oman in [13] and Chambolle, Caselles and Alter et al. in [4] . In the case of convex constraints, we may quote papers by E. Casas, K.Kunisch and C.Pola [9] and Aubert and Vese [6]. Here, the main difficulty comes from the fact that the feasible domain $\mathcal{D}$ is not convex and it's interior is empty for most usual topologies. In addition, the total variation $J(u)$ of a function u in $BV(\Omega)$ is not Frchet differentiable.

Abraham, Bergounioux and Trlat have also considered problem $(\mathcal{P})$ in [2]. They proved the existence of (at least) a solution of $(\mathcal{P})$ in $BV(\Omega)$ and gave first order optimality conditions. They use a penalization technique to deal with the binary constraint and considered

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \min \frac{1}{2}\|Hu - g\|^2 + \lambda J(u) + \frac{1}{2\varepsilon}\|u^2 - u\|^2, \\ u \in BV(\Omega). \end{cases}$$

where $\varepsilon > 0$. However the optimality system they obtained was not suitable for numerical purposes and they rather solved the penalized system. In addition, the proposed numerical algorithm is not very performing. The penalization term is difficult to handle and parameter tuning is quite delicate.

In this paper we look for different strategies to study problem $(\mathcal{P})$ from a numerical point of view. In a first step we want to avoid the difficulties mentioned before, namely the domain non convexity and the cost functional lack of differentiability. We could consider a problem where the total variation $J(u)$ is replaced by the $L^2(\Omega)$-norm of the gradient of $u \in H^1(\Omega)$. Therefore the underlying space is no longer $BV(\Omega)$ but $H^1(\Omega)$. The resulting problem would be

$$(\mathcal{P}_o) \quad \begin{cases} \min F(u) := \frac{1}{2}\|Hu - g\|^2 + \frac{\lambda}{2}\|\nabla u\|^2, \\ u \in \mathcal{D} \end{cases}$$

Unfortunately, this problem has no solutions except (may be) the two functions identically equal to 0 or 1. Indeed a binary function cannot belong to $H^1(\Omega)$ since its gradient (in the distribution sense) is a measure (it is a Dirac measure along the contours). We could choose $W^{1,1}(\Omega)$ instead of $BV(\Omega)$ and deal with the gradient L^1 norm instead of the L^2- norm, but we would like to overcome the lack of differentiability and avoid the use of subdifferentiability.

On the other hand, one can get rid of the binary constraint if we consider a relaxed formulation, that is $0 \leq u \leq 1, \quad (u, 1-u)_2 \leq \alpha$, where $\alpha > 0$ instead of $0 \leq u \leq 1, \quad (u, 1-u)_2 = 0$. The relaxation of the binary constraint is motivated and justified numerically: indeed, it is not possible to ensure $(u, 1-u)_2 = 0$ during computations but rather $|(u,1-u)_2| \leq \alpha$ where α may be chosen as small as wanted, but (strictly) positive. One can see (as in [2]) that the corresponding relaxed problem

$$\min F_o(u) \ , u \in BV(\Omega), \ 0 \leq u \leq 1, \ \ (u,1-u)_2 \leq \alpha \ ,$$

has a solution that converges (in $BV(\Omega)$) towards a solution to $(\mathcal{P})$. However, the problem of finding first order optimality conditions remains delicate since the total variation is not differentiable.

So, in a first step we consider a "smooth version " of the above relaxed problem namely

$$(\mathcal{P}_\alpha) \quad \begin{cases} \min F(u) := \frac{1}{2}\|Hu-g\|^2 + \frac{\lambda}{2}\|\nabla u\|^2, \\ u \in \mathcal{D}_\alpha \end{cases}$$

where $\alpha > 0$ and $\mathcal{D}_\alpha$ is given by

$$\mathcal{D}_\alpha := \left\{ u \in H^1(\Omega) \mid 0 \leq u \leq 1 \ , \ (u,1-u)_2 \leq \alpha \ \right\} \tag{1.2}$$

Next section is devoted to existence results of solutions to $(\mathcal{P}_\alpha)$. Then we recall some general mathematical programming results to get some qualification conditions. In section 3, we define a "temporary" penalized problem that allows to decouple the binary constraint via a virtual control function. We give existence results and establish estimations to pass to the limit with respect to the penalization parameter. Then we get optimality conditions for $(\mathcal{P}_\alpha)$.

2 The Relaxed Problem

Now we consider the following relaxed problem

$$(\mathcal{P}_\alpha) \quad \begin{cases} \min F(u) := \frac{1}{2}\|Hu-g\|^2 + \frac{\lambda}{2}\|\nabla u\|^2, \\ u \in \mathcal{D}_\alpha \end{cases}$$

where $\alpha > 0$ and $\mathcal{D}_\alpha$ is given by (1.2). We first recall a important property of operator H

Proposition 2.1. *The operator H is continuous from $L^2(\Omega)$ to $L^2(\Omega)$.*

Proof. [2, Lemma 1]. □

We can give an existence result:

Theorem 2.2. *Problem $(\mathcal{P}_\alpha)$ has at least a solution u_α in $H^1(\Omega)$.*

Proof. Let $(u_n)_n \in H^1(\Omega)$ be a minimizing sequence. Since $0 \leq u_n \leq 1$, the sequence u_n is bounded in $L^\infty(\Omega)$ and in $L^2(\Omega)$ as well. As $\|\nabla u_n\|$ is bounded, $(u_n)_n$ is bounded in $H^1(\Omega)$ and weakly converges (up to a subsequence) to some u_α in $H^1(\Omega)$. As the functional $u \longmapsto \|\nabla u\|$ is convex and lower semi-continuous, it is weakly lower semi-continuous and we have

$$\|\nabla u_\alpha\|^2 \leq \liminf_{n\to\infty} \|\nabla u_n\|^2.$$

Now, we use the continuity property of H from $L^2(\Omega)$ to $L^2(\Omega)$ and the convexity of the $L^2(\Omega)$-norm, to write

$$\|Hu_\alpha - g\|^2 \leq \liminf_{n\to\infty} \|Hu_n - g\|^2.$$

So,

$$\begin{aligned}\inf(\mathcal{P}_\alpha) &= \lim_{n\to\infty}\left(\frac{1}{2}\|Hu_n - g\|^2 + \frac{\lambda}{2}\|\nabla u_n\|^2\right)\\ &\geq \frac{1}{2}\|Hu_\alpha - g\|^2 + \frac{\lambda}{2}\|\nabla u_\alpha\|^2. \end{aligned} \tag{2.1}$$

To finish the proof of Theorem 2.2, we must prove that $u_\alpha \in \mathcal{D}_\alpha$. As the set $\{u \in H^1(\Omega) : 0 \leq u \leq 1\}$ is convex and L^2- closed for the strong-topology of $L^2(\Omega)$, it is weakly L^2-closed as well. So $0 \leq u_\alpha \leq 1$. As $u_n \rightharpoonup u_\alpha$ in $H^1(\Omega)$ then $u_n \to u_\alpha$ strongly in $L^2(\Omega)$ and

$$(u_n, 1-u_n)_2 \longrightarrow (u_\alpha, 1-u_\alpha)_2 \ .$$

As $(u_n, 1-u_n)_2 \leq \alpha$ for every $n \in \mathbb{N}$, we conclude that $(u_\alpha, 1-u_\alpha)_2 \leq \alpha$ □

The constraint "$(u_\alpha, 1-u_\alpha)_2 \leq \alpha$" is not convex and it is not possible to find the "admissible" directions to compute derivatives. We need a general qualification assumption to derive optimality conditions. We are going to use general mathematical programming problems results and optimal control in Banach spaces.

2.1 A General Qualification Condition in Banach Spaces

The method we use has been mainly developed by Zowe and Kurcyusz [14] and Tröltzsch [11, 12]. Let us consider real Banach spaces $\mathcal{X}, \mathcal{U}, \mathcal{Z}_1, \mathcal{Z}_2$ and a convex closed "admissible" set $\mathcal{U}_{ad} \subseteq \mathcal{U}$. In $\mathcal{Z}_2$ a convex closed cone $\mathbf{P}$ is given so that $\mathcal{Z}_2$ is partially ordered by

$$x \geq y \Leftrightarrow x - y \in \mathbf{P}.$$

We also deal with:

$$f : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \ , \text{ Fréchet- differentiable functional },$$

$$T : \mathcal{X} \times \mathcal{U} \to \mathcal{Z}_1 \text{ and } G : \mathcal{X} \times \mathcal{U} \to \mathcal{Z}_2 \text{ continuously Fréchet-differentiable operators }.$$

Now, let be the abstract mathematical programming problem defined by:

$$\min \ \{ \ f(x,u) \mid T(x,u) = 0, \ G(x,u) \leq 0 \ , u \in \mathcal{U}_{ad} \ \} \ . \tag{2.2}$$

To shorten the text, we denote the partial Fréchet-derivative of f, T, and G with respect to x and u by a corresponding index x or u. We suppose that the problem (2.2) has an optimal solution that we call (x_o, u_o), and we introduce the sets:

$$\mathcal{U}_{ad}(u_o) = \{ \ u \in \mathcal{U} \mid \exists \lambda \geq 0, \exists u^* \in \mathcal{U}_{ad}, \ u = \lambda(u^* - u_o) \ \},$$

$$\mathbf{P}(G(x_o, u_o)) = \{ \ z \in \mathcal{Z}_2 \mid \exists \lambda \geq 0, \exists p \in -\mathbf{P}, \ z = p - \lambda G(x_o, u_o) \ \},$$

$$\mathbf{P}^+ = \{ \ y \in \mathcal{Z}_2^* \mid \ \langle y, p\rangle \geq 0 \ , \forall p \in \mathbf{P} \ \}.$$

The main result about optimality conditions is the following :

Theorem 2.3. Let u_o be an optimal control with corresponding optimal state x_o and suppose that the following regularity condition is fulfilled:

$$\begin{array}{c} \forall (z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2 \text{ the system} \\ \begin{array}{rcl} T'(x_o, u_o)(x, u) & = & z_1, \\ G'(x_o, u_o)(x, u) - p & = & z_2, \end{array} \\ \text{is solvable with } (x, u, p) \in \mathcal{X} \times \mathcal{U}_{ad}(u_o) \times \mathbf{P}(G(x_o, u_o)) \ . \end{array} \tag{2.3}$$

Then a Lagrange multiplier $(y_1, y_2) \in \mathcal{Z}_1' \times \mathcal{Z}_2'$ exists such that

$$f_x'(x_o, u_o) + T_x'(x_o, u_o)^* y_1 + G_x'(x_o, u_o)^* y_2 = 0 \ , \tag{2.4}$$

$$\langle f_u'(x_o, u_o) + T_u'(x_o, u_o)^* y_1 + G_u'(x_o, u_o)^* y_2, u - u_o \rangle_{\mathcal{U}', \mathcal{U}} \geq 0 \ , \ \forall u \in \mathcal{U}_{ad} \ , \tag{2.5}$$

$$y_2 \in \mathbf{P}^+ \ , \ \ \langle y_2, G(x_o, u_o) \rangle_{\mathcal{Z}_2', \mathcal{Z}_2} = 0 \ . \tag{2.6}$$

Proof. See [14, 11, 12]. □

To apply this method to $(\mathcal{P}_\alpha)$, we introduce a virtual control variable. We can write the problem $(\mathcal{P}_\alpha)$ as,

$$(\mathcal{P}_\alpha) \qquad \begin{cases} \min F(u) \\ (u, v) \in \mathcal{C}_\alpha, \end{cases}$$

where

$$\mathcal{C}_\alpha := \{(u, v) \in H^1(\Omega) \times H^1(\Omega) \mid u \geq 0, \ v \geq 0, \ u + v = 1 \ \text{ a.e. on } \Omega, \ \ (\mathrm{u}, \mathrm{v})_2 \leq \alpha\} \ . \tag{2.7}$$

We apply the previous theorem to $(\mathcal{P}_\alpha)$ with $\mathcal{X} = H^1(\Omega), \mathcal{U} = H^1(\Omega)$, $\mathcal{Z}_2 = H^1(\Omega) \times \mathbb{R}$

$$\mathbf{P} = \{(u, \tau) \in H^1(\Omega) \times \mathbb{R} \mid u \geq 0 \quad \text{a.e. in } \Omega, \ \tau \geq 0 \ \} \ .$$

In addition $T(u, v) = u + v - 1$ and f, G are given by

$$f(u, v) = \frac{1}{2}\|Hu - g\|^2 + \frac{\lambda}{2}\|\nabla u\|^2,$$

and

$$\begin{array}{rcl} G : H^1(\Omega) \times L^2(\Omega) & \rightarrow & H^1(\Omega) \times \mathbb{R} \\ (u, v) & \mapsto & (-u, (u, v)_2 - \alpha) \, . \end{array}$$

In this case f and G are $\mathcal{C}^1$ and

$$G'(u_\alpha, v_\alpha)(u, v) = (-u, (u_\alpha, v)_2 + (v_\alpha, u)_2) \, .$$

Moreover

$$\mathcal{V}_{ad} = \left\{ v \in H^1(\Omega) : v \geq 0 \text{ a.e} \right\}$$

and $\mathcal{V}_{ad}(v_\alpha) = \{\lambda(v - v_\alpha) : \lambda \geq 0, v \in \mathcal{V}_{ad}\}$. Finally

$$\mathbf{P}(G(u_\alpha, v_\alpha)) = \left\{ (-p + \lambda u_\alpha, -\gamma - \lambda \left[(u_\alpha, v_\alpha) - \alpha) \right) \right] \in H^1(\Omega) \times \mathbb{R} \right\},$$

where p, λ and γ are real nonnegative constants. Let us write condition (2.3): for any (z_1, z_2, θ) in $H^1(\Omega) \times H^1(\Omega) \times \mathbb{R}$, we must solve the system:

$$\begin{array}{c} u + \mu(v - v_\alpha) = z_1, \\ -u + p - \lambda u_\alpha = z_2, \\ (u_\alpha, \mu\,(v - v_\alpha)) + (u, v_\alpha)_2 + \delta + \lambda\,[(u_\alpha, v_\alpha) - \alpha] = \theta, \end{array}$$

where $(\mu, \delta, \lambda, p) \in \mathbb{R}_+^4$, $v \in \mathcal{V}_{ad}, u \in H^1(\Omega)$. This condition appears to be difficult to satisfy. Therefore we adopt a different strategy that is to penalize the constraint $u + v = 1$.

$\boxed{2.2}$ Penalization of the Relaxed Problem

From now, we fix $\alpha > 0$ as small as wanted. We first penalize $(\mathcal{P}_\alpha)$ to get an approximate problem $(\mathcal{P}_\alpha^\varepsilon)$ such that the family of solutions to $(\mathcal{P}_\alpha^\varepsilon)$ converges to a solution to $(\mathcal{P}_\alpha)$ with respect to ε. Once we prove the existence of a solution of $(\mathcal{P}_\alpha^\varepsilon)$, we use the above techniques and derive penalized optimality conditions. Then we pass to the limit with respect to ε. Let us consider the following minimization problem, where $(u_\alpha, v_\alpha := 1 - u_\alpha)$ is a (fixed) solution to $(\mathcal{P}_\alpha)$:

$$(\mathcal{P}_\alpha^\varepsilon) \quad \begin{cases} \min F_\alpha^\varepsilon(u,v) := & \dfrac{1}{2}\|Hu - g\|^2 + \dfrac{\lambda}{2}\|\nabla u\|^2 + \dfrac{1}{2\varepsilon}\|u + v - 1\|^2 \\ & +\dfrac{1}{2}\|u - u_\alpha\|_{H^1}^2 + \dfrac{1}{2}\|v - v_\alpha\|^2, \\ (u,v) \in \mathcal{K}_\alpha & \end{cases}$$

where

$$\mathcal{K}_\alpha := \{(u,v) \in H^1(\Omega) \times L^2(\Omega) \mid u \geq 0,\ v \geq 0,\ (u,v)_2 \leq \alpha\} \tag{2.8}$$

The term $\dfrac{1}{\varepsilon}\|u + v - 1\|^2$ is the penalization term of the constraint $u + v - 1 = 0$. The other terms $\|u - u_\alpha\|_{H^1}^2$ and $\|v - v_\alpha\|^2$ are adapted penalization terms which ensure the strong convergence of the penalized solution towards the desired solution (u_α, v_α) (see Barbu [7]).

Theorem 2.4. *Problem $(\mathcal{P}_\alpha^\varepsilon)$ admits at least a solution $(u_\alpha^\varepsilon, v_\alpha^\varepsilon) \in H^1(\Omega) \times L^2(\Omega)$.*

Proof. Let $\varepsilon > 0$ be fixed and (u_n, v_n) be a minimizing sequence. The sequence (u_n) is H^1- bounded and (up to a subsequence) weakly converges to some u_α^ε in $H^1(\Omega)$. Similarly, the sequence (v_n) is L^2-bounded and (up to a subsequence) weakly converges to v_α^ε. The functional $F_\alpha^\varepsilon(u,v) : H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is convex and continuous, so it is weakly lower semi-continuous and

$$F_\alpha^\varepsilon(u_\alpha^\varepsilon, v_\alpha^\varepsilon) \leq \liminf_{n \to +\infty} F_\alpha^\varepsilon(u_n, v_n) = \inf(\mathcal{P}_\alpha^\varepsilon)\ .$$

As the $\mathcal{K}_\alpha$ is a convex, closed subset of $H^1(\Omega) \times L^2(\Omega)$ it is also weakly closed. Therefore $(u_\alpha^\varepsilon, v_\alpha^\varepsilon) \in \mathcal{K}_\alpha$. This ends the proof. $\square$

Theorem 2.5. *When ε converges to 0, the family $(u_\alpha^\varepsilon, v_\alpha^\varepsilon)$ strongly converges to (u_α, v_α) in $H^1(\Omega) \times L^2(\Omega)$.*

Proof. As (u_α, v_α) is a solution to $(\mathcal{P}_\alpha)$ it is always feasible for $(\mathcal{P}_\alpha^\varepsilon)$ and we get

$$F_\alpha^\varepsilon(u_\alpha^\varepsilon, v_\alpha^\varepsilon) \leq F(u_\alpha)\ . \tag{2.9}$$

Therefore u_α^ε is H^1 bounded and v_α^ε is L^2 bounded independently of ε. So (up to a subsequence) u_α^ε weakly converges to u_0 in $H^1(\Omega)$ and v_α^ε weakly converges to v_0 in $L^2(\Omega)$. As $\mathcal{K}_\alpha$ is weakly closed we get

$$u_0 \geq 0, v_0 \geq 0, (u_0, v_0)_2 \leq \alpha. \tag{2.10}$$

Moreover, (2.9) gives

$$\|u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1\|^2 \leq 2\varepsilon F(u_\alpha)\ ,$$

so that

$$\|u_0 + v_0 - 1\|^2 \leq \liminf_{\varepsilon \to 0} \|u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1\|^2 = 0.$$

We conclude with the lower semi-continuity of the functional that

$$\frac{1}{2}\|Hu_0 - g\|^2 + \frac{\lambda}{2}\|\nabla u_0\|^2 + \frac{1}{2}\|v_0 - v_\alpha\|^2 + \frac{1}{2}\|u_0 - u_\alpha\|^2_{H^1}$$
$$\leq F^\varepsilon_\alpha(u^\varepsilon_\alpha, v^\varepsilon_\alpha) \leq F(u_\alpha) = \frac{1}{2}\|Hu_\alpha - g\|^2 + \frac{\lambda}{2}\|\nabla u_\alpha\|^2. \tag{2.11}$$

As u_0 and v_0 are also feasible for $(\mathcal{P}_\alpha)$ we have

$$\frac{1}{2}\|Hu_\alpha - g\|^2 + \frac{\lambda}{2}\|\nabla u_\alpha\|^2 \leq \frac{1}{2}\|Hu_0 - g\|^2 + \frac{\lambda}{2}\|\nabla u_0\|^2. \tag{2.12}$$

We use (2.11) and (2.12), we conclude easily that $u_0 = u_\alpha$, then

$$\frac{1}{2}\|Hu_0 - g\|^2 + \frac{\lambda}{2}\|\nabla u_0\|^2 = \inf(\mathcal{P}_\alpha).$$

This implies also strong convergence of u^ε_α to u_α in $H^1(\Omega)$ and v^ε_α to v_α in $L^2(\Omega)$ since

$$\|u^\varepsilon_\alpha - u_\alpha\|^2_{H^1} + \|v^\varepsilon_\alpha - v_\alpha\|^2 \leq \|Hu_\alpha - g\|^2 + \lambda\|\nabla u_\alpha\|^2 - \|Hu^\varepsilon_\alpha - g\|^2_{L^2(\Omega)} - \lambda\|\nabla u^\varepsilon_\alpha\|^2,$$

and

$$\lim_{\varepsilon\to 0}\left(\|Hu^\varepsilon_\alpha - g\|^2 + \|\nabla u^\varepsilon_\alpha\|^2\right) = |Hu_\alpha - g\|^2 + \|\nabla u_\alpha\|^2.$$

We just proved that the only cluster point is (u_α, v_α): the whole family $(u^\varepsilon_\alpha, v^\varepsilon_\alpha)$ (strongly) converges to (u_α, v_α) as $\varepsilon \to 0$. □

Now, we apply Theorem 2.3 to problem $(\mathcal{P}^\varepsilon_\alpha)$ with $\mathcal{X} = H^1(\Omega)$, $\mathcal{Z}_2 = H^1(\Omega) \times \mathbb{R}$ and $\mathbf{P}$ as in the previous subsection. Now, we set $\mathcal{U} = L^2(\Omega)$, $T = 0$, $f = F^\varepsilon_\alpha$ and

$$\begin{array}{rcl} G : H^1(\Omega) \times L^2(\Omega) & \to & H^1(\Omega) \times \mathbb{R} \\ (u, v) & \mapsto & \left(-u, (u, v)_2 - \alpha\right). \end{array}$$

Functions f and G are $\mathcal{C}^1$ and

$$G'(u^\varepsilon_\alpha, v^\varepsilon_\alpha)(u, v) = \left(-u, (u^\varepsilon_\alpha, v)_2 + (v^\varepsilon_\alpha, u)_2\right).$$

Here $\mathcal{V}_{ad} = \left\{v \in L^2(\Omega) : v \geq 0 \text{ a.e}\right\}$,

$$\mathcal{V}_{ad}(v^\varepsilon_\alpha) = \left\{\lambda(v - v^\varepsilon_\alpha) : \lambda \geq 0, v \in \mathcal{V}_{ad}\right\},$$

and

$$\mathbf{P}(G(u^\varepsilon_\alpha, v^\varepsilon_\alpha)) = \left\{(-p + \lambda u^\varepsilon_\alpha, -\gamma - \lambda\left[(u^\varepsilon_\alpha, v^\varepsilon_\alpha)_2 - \alpha\right)\right] \in H^1(\Omega) \times \mathbb{R}\right\},$$

where p, λ and γ are real nonnegative constants.
Let us write condition (2.3): for any (z, w) in $H^1(\Omega) \times L^2(\Omega)$, we have to solve the system:

$$\begin{array}{c} -u + p - \lambda u^\varepsilon_\alpha = z, \\ (u^\varepsilon_\alpha, \mu\,(v - v^\varepsilon_\alpha))_2 + (u, v^\varepsilon_\alpha)_2 + \delta + \lambda[(u^\varepsilon_\alpha, v^\varepsilon_\alpha)_2 - \alpha] = w, \end{array}$$

where $(\mu, \delta, \lambda, p) \in \mathbb{R}^4_+$, $v \in \mathcal{V}_{ad}, u \in H^1(\Omega)$. Taking u from the first equation into the second gives:

$$(u^\varepsilon_\alpha, \mu(v - v^\varepsilon_\alpha))_2 + (p, v^\varepsilon_\alpha)_2 + \delta - \lambda\alpha = w + (z, v^\varepsilon_\alpha)_2 := \rho,$$

with $\mu, \delta, \lambda \geq 0, p \geq 0$, $v \in \mathcal{V}_{ad}$. We see that we may take: $\mu = 1, v = v^\varepsilon_\alpha, p = 0$, and

$$\begin{cases} \lambda = 0, \quad \delta = \rho, \quad \text{if } \rho \geq 0, \\ \\ \lambda = -\frac{\rho}{\alpha}, \quad \delta = 0, \quad \text{if } \rho \leq 0. \end{cases}$$

So the condition (2.3) is always satisfied. Since f, G are Frchet-differentiable, we can apply Theorem 2.3: there exists $y_\alpha^\varepsilon \in (H^1)^{'}(\Omega)$ and $r_\alpha^\varepsilon \in \mathbb{R}$ such that

$$\begin{aligned} \forall u \in H^1(\Omega) \quad & (H^\star H u_\alpha^\varepsilon - g, u - u_\alpha^\varepsilon)_2 + \lambda\,(\nabla u_\alpha^\varepsilon, \nabla(u - u_\alpha^\varepsilon))_2 + (u_\alpha^\varepsilon - u_\alpha, u - u_\alpha^\varepsilon)_{H^1} + \\ & (v_\alpha^\varepsilon - v_\alpha, v - v_\alpha^\varepsilon)_2 + \tfrac{1}{\varepsilon}\,(u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1, u - u_\alpha^\varepsilon)_2 + \\ & r_\alpha^\varepsilon\,(u - u_\alpha^\varepsilon, v_\alpha^\varepsilon) + \langle y_\alpha^\varepsilon, -u + u_\alpha^\varepsilon\rangle_{(H^1)',H^1} = 0, \end{aligned}$$

$$\forall v \in \mathcal{V}_{ad} \quad \frac{1}{\varepsilon}\,(u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1, v - v_\alpha^\varepsilon)_2 + r_\alpha^\varepsilon\,(u_\alpha^\varepsilon, v - v_\alpha^\varepsilon)_2 \geq 0,$$

$$r_\alpha^\varepsilon \geq 0, \qquad r_\alpha^\varepsilon\,((u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 - \alpha) = 0,$$

and for all $u \in H^1(\Omega)$ such that $u \geq 0$, we have

$$\langle y_\alpha^\varepsilon, u\rangle_{(H^1)',H^1} \geq 0, \qquad \langle y_\alpha^\varepsilon, u_\alpha^\varepsilon\rangle_{(H^1)',H^1} = 0.$$

We set

$$p_\alpha^\varepsilon = H^\star H u_\alpha^\varepsilon - g, \qquad q_\alpha^\varepsilon = \frac{1}{\varepsilon}(u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1). \tag{2.13}$$

Theorem 2.6. *The solution $(u_\alpha^\varepsilon, v_\alpha^\varepsilon)$ of problem $(\mathcal{P}_\alpha^\varepsilon)$ satisfies the following optimality system*

$$\begin{aligned} \forall u \in H^1(\Omega) \qquad & (p_\alpha^\varepsilon + q_\alpha^\varepsilon, u - u_\alpha^\varepsilon)_2 + \lambda\,(\nabla u_\alpha^\varepsilon, \nabla(u - u_\alpha^\varepsilon))_2 + r_\alpha^\varepsilon\,(u - u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 \\ & + (u_\alpha^\varepsilon - u_\alpha, u - u_\alpha^\varepsilon)_{H^1} + (v_\alpha^\varepsilon - v_\alpha, v - v_\alpha^\varepsilon)_2 \geq 0, \end{aligned} \tag{2.14a}$$

$$\forall v \in \mathcal{V}_{ad} \qquad (q_\alpha^\varepsilon, v - v_\alpha^\varepsilon)_2 + r_\alpha^\varepsilon\,(u_\alpha^\varepsilon, v - v_\alpha^\varepsilon)_2 \geq 0, \tag{2.14b}$$

$$r_\alpha^\varepsilon \in \mathbb{R}^+, \qquad r_\alpha^\varepsilon((u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 - \alpha) = 0, \tag{2.14c}$$

$$\forall u \in H^1(\Omega),\ u \geq 0 \qquad \langle y_\alpha^\varepsilon, u\rangle_{(H^1)',H^1} \geq 0, \qquad \langle y_\alpha^\varepsilon, u_\alpha^\varepsilon\rangle_{(H^1)',H^1} = 0. \tag{2.14d}$$

$\boxed{3}$ Optimality System for $(\mathcal{P}_\alpha)$

Now we want to pass to the limit in the above inequalities. So we need estimates for $q_\alpha^\varepsilon, r_\alpha^\varepsilon$ and p_α^ε with respect to ε and we use [8] techniques. We already know that, if ε tends to zero, u_α^ε converges to u_α strongly in $H^1(\Omega)$. The continuity properties of H and $H^\star$ from $L^2(\Omega)$ to $L^2(\Omega)$ [2] imply that

$$p_\alpha^\varepsilon = H^\star(H u_\alpha^\varepsilon - g) \rightarrow H^\star(H u_\alpha - g) = p_\alpha \quad \text{strongly in } \ L^2(\Omega),$$

and p_α^ε is bounded independently of ε in $L^2(\Omega)$.

$\boxed{3.1}$ Estimate on r_α^ε

Let us choose $(u,v) \in H^1(\Omega) \times L^2(\Omega)$ such that $u \geq 0$, $v \geq 0$ a.e and add relations (2.14a) and (2.14b), to obtain

$$(q_\alpha^\varepsilon, -(u+v-1))_2 - r_\alpha^\varepsilon[(u_\alpha^\varepsilon, v)_2 + (v_\alpha^\varepsilon, u)_2] + 2r_\alpha^\varepsilon\,(u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 \leq$$

$$(\nabla u_\alpha^\varepsilon, \nabla(u_\alpha^\varepsilon - u))_2 + (p_\alpha^\varepsilon, u_\alpha^\varepsilon - u)_2 - \varepsilon\|q_\alpha^\varepsilon\|^2 + (u_\alpha^\varepsilon - u_\alpha, u - u_\alpha^\varepsilon)_{H^1} + (v_\alpha^\varepsilon - v_\alpha, v - v_\alpha^\varepsilon)_2 \leq$$

$$(\nabla u_\alpha^\varepsilon, \nabla(u_\alpha^\varepsilon - u))_2 + (p_\alpha^\varepsilon, u_\alpha^\varepsilon - u)_2 + (u_\alpha^\varepsilon - u_\alpha, u - u_\alpha^\varepsilon)_{H^1} + (v_\alpha^\varepsilon - v_\alpha, v - v_\alpha^\varepsilon)_2\,.$$

As p_α^ε, v_α^ε are L^2 bounded and u_α^ε is H^1 bounded (with respect to ε), the left side is uniformly bounded with respect to ε by a constant $C_{(u,v)}$ which only depends on u, v. Moreover relation (2.14c) implies

$$r_\alpha^\varepsilon\,[(u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 - \alpha] = 0,$$

so that we finally obtain:

$$(q_\alpha^\varepsilon, -(u+v-1))_2 - r_\alpha^\varepsilon\,[(u_\alpha^\varepsilon, v)_2 + (v_\alpha^\varepsilon, u)_2] + 2r_\alpha^\varepsilon\alpha \leq C_{(u,v)}. \tag{3.1}$$

Now, we distinguish two cases:
First case: assume $(u_\alpha, v_\alpha)_2 < \alpha$. As $(u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 \to (u_\alpha, v_\alpha)_2$ in $\mathbb{R}$, there exists ε_0 such that

$$\forall \varepsilon \leq \varepsilon_0,\ (u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 < \alpha,$$

and relation (2.14c) implies that $r_\alpha^\varepsilon = 0$. So the limit value is $r_\alpha = 0$.
Second case: $(u_\alpha, v_\alpha)_2 = \alpha$. We cannot conclude immediately and need the following Lemma.

Lemma 3.1. *Let α be such that $(u_\alpha, v_\alpha) = \alpha$. Then, there exist $\bar{u} \in H^1(\Omega), \bar{u} \geq 0$ and $\bar{v} \in \mathcal{V}_{ad}$ such that*

$$\bar{u} + \bar{v} - 1 = 0 \text{ and } (\bar{u}, v_\alpha)_2 + (\bar{v}, u_\alpha)_2 < 2\alpha.$$

Proof. It is enough to prove that there exists $\bar{u} \in H^1(\Omega)$ such that $0 \leq \bar{u} \leq 1$ and

$$\int_\Omega \bar{u}(x)(1 - u_\alpha(x))\,dx + \int_\Omega (1 - \bar{u}(x))u_\alpha(x)\,dx < 2\alpha. \tag{3.2}$$

From now, we denote $\int_\Omega u(x)v(x)dx = \int_\Omega uv$ for all $(u,v) \in L^2(\Omega) \times L^2(\Omega)$.
Since $\int_\Omega u_\alpha(1 - u_\alpha) = \alpha$, we have

$$\begin{aligned}
\int_\Omega \bar{u}(1-u_\alpha) + \int_\Omega (1-\bar{u})u_\alpha < 2\alpha &\Leftrightarrow \int_\Omega \bar{u} - 2\int_\Omega \bar{u}u_\alpha - \int_\Omega u_\alpha + 2\int_\Omega u_\alpha^2 < 0 \\
&\Leftrightarrow \int_\Omega \bar{u}(1-2u_\alpha) - \int_\Omega u_\alpha(1-2u_\alpha) < 0 \\
&\Leftrightarrow \int_\Omega (\bar{u} - u_\alpha)(1-2u_\alpha) < 0 \\
&\Leftrightarrow \int_\Omega (u_\alpha - \bar{u})(2u_\alpha - 1) < 0.
\end{aligned}$$

For every $x \in \Omega$, we set

$$\bar{u}(x) = \begin{cases} 1 & \text{if} \quad \mathrm{u}_\alpha(\mathrm{x}) \geq \dfrac{3}{4}, \\ 0 & \text{if} \quad \mathrm{u}_\alpha(\mathrm{x}) \leq \dfrac{1}{4}, \\ 2u_\alpha(x) - \dfrac{1}{2} & \text{if} \quad \dfrac{1}{4} \leq \mathrm{u}_\alpha(\mathrm{x}) \leq \dfrac{3}{4}. \end{cases}$$

It is clear that $0 \leq \bar{u} \leq 1$ and $\bar{u} \in H^1(\Omega)$ [10, Theorem 7.8]

$$\int_\Omega (u_\alpha - \bar{u})(2u_\alpha - 1) =$$

$$\underbrace{\int_{\{u_\alpha \leq \frac{1}{4}\}} (u_\alpha - \bar{u})(2u_\alpha - 1)}_{I_1} + \underbrace{\int_{\{u_\alpha \geq \frac{3}{4}\}} (u_\alpha - \bar{u})(2u_\alpha - 1)}_{I_2} + \underbrace{\int_{\{\frac{1}{4} \leq u_\alpha \leq \frac{3}{4}\}} (u_\alpha - \bar{u})(2u_\alpha - 1)}_{I_3}.$$

We start with I_1:

$$I_1 = \int_{\{u_\alpha \leq \frac{1}{4}\}} (u_\alpha - \bar{u})(2u_\alpha - 1) = \int_{\{u_\alpha \leq \frac{1}{4}\}} u_\alpha(2u_\alpha - 1) < 0,$$

since $u_\alpha \geq 0$ and $2u_\alpha - 1 < 0$ in $\{u_\alpha \leq \frac{1}{4}\}$. Similarly

$$I_2 = \int_{\{u_\alpha \geq \frac{3}{4}\}} (u_\alpha - \bar{u})(2u_\alpha - 1) = \int_{\{u_\alpha \geq \frac{3}{4}\}} (u_\alpha - 1)(2u_\alpha - 1) < 0$$

since $u_\alpha \leq 1$ and $2u_\alpha - 1 \geq 0$. At last

$$I_3 = \int_{\{\frac{1}{4} \leq u_\alpha \leq \frac{3}{4}\}} (u_\alpha - \bar{u})(2u_\alpha - 1) = \int_{\{\frac{1}{4} \leq u_\alpha \leq \frac{3}{4}\}} \frac{-1}{2}(2u_\alpha - 1)^2 \leq 0.$$

□

This Lemma provides test functions $(\bar{u}, \bar{v})$ which do not depend on ε and

$$(\bar{u}, v_\alpha^\varepsilon)_2 + (\bar{v}, u_\alpha^\varepsilon)_2 \to (\bar{u}, v_\alpha)_2 + (\bar{v}, u_\alpha)_2 < 2\alpha.$$

Therefore, there exist $\rho \in]0, 2\alpha[$ and ε_0 such that

$$\forall \varepsilon \leq \varepsilon_0, (\bar{u}, v_\alpha^\varepsilon)_2 + (\bar{v}, u_\alpha^\varepsilon)_2 \leq 2\alpha - \rho. \tag{3.3}$$

Now, we use (3.1) and (3.3): $\forall \varepsilon \leq \varepsilon_0$, we have

$$r_\alpha^\varepsilon \left[2\alpha - (2\alpha - \rho)_2\right] \leq r_\alpha^\varepsilon \left[2\alpha - (\bar{u}, v_\alpha^\varepsilon)_2 + (\bar{v}, u_\alpha^\varepsilon)_2\right] \leq C_{(\bar{u},\bar{v})}.$$

So

$$\forall \varepsilon \leq \varepsilon_0 \qquad r_\alpha^\varepsilon \leq \frac{C_{(\bar{u},\bar{v})}}{\rho},$$

where $C_{(\bar{u},\bar{v})}$ is a positive constant that depends only on $\bar{u}$ and $\bar{v}$. We have proved the following

Proposition 3.2. *Let α be fixed. There exists $C_\alpha > 0$ and $\varepsilon_{0,\alpha} > 0$ such that*

$$\forall \varepsilon \geq \varepsilon_{0,\alpha} \qquad 0 \leq r_\alpha^\varepsilon \leq C_\alpha \ .$$

3.2 q_α^ε Estimates

Once we have r_α^ε estimates, we use relation (3.1) to get

$$\forall u \in H^1(\Omega) \text{ such that } u \geq 0,\ \forall v \in L^2(\Omega), v \geq 0 \quad -(q_\alpha^\varepsilon, u+v-1)_2 \leq C_{(u,v)}. \tag{3.4}$$

Let us choose $\chi \in L^\infty(\Omega)$ such that $\|\chi\|_\infty \leq 1$, set $u_\chi = 0$ and $v_\chi = 1 + \dfrac{\chi}{2}$ in relation (3.4) to obtain

$$\forall \chi \in L^\infty(\Omega), \|\chi\|_{L^\infty(\Omega)} \leq 1, \quad (q_\alpha^\varepsilon, \chi)_2 \leq C_\chi \leq C.$$

The positive constant C is independent of χ: indeed, from the previous computation it is easy to check that C_χ does not depend on χ as soon as $\|\chi\|_\infty \leq 1$. So q_α^ε is bounded in $(L^\infty)'(\Omega)$ i.e. in $\mathcal{M}(\Omega)$ by a constant independent of ε. Here $\mathcal{M}(\Omega)$ denotes the space of radon measures on Ω.

From the previous estimates on r_α^ε and q_α^ε, there exist r_α in $\mathbb{R}$ and $q_\alpha \in \mathcal{M}(\Omega)$ such that $r_\alpha^\varepsilon \to r_\alpha$ in $\mathbb{R}$ and $q_\alpha^\varepsilon \overset{*}{\rightharpoonup} q_\alpha$ weakly-star in $\mathcal{M}(\Omega)$.

3.3 $(\mathcal{P}_\alpha)$ Optimality System

We may pass to the limit in the penalized optimality system :

Theorem 3.3. *Assume u_α is a solution to $(\mathcal{P}_\alpha)$ and $v_\alpha = 1 - u_\alpha$. There exists a Lagrange multiplier $(q_\alpha, r_\alpha) \in \mathcal{M}(\Omega) \times \mathbb{R}^+$ such that for all $u \in H^1(\Omega) \cap L^\infty(\Omega), u \geq 0$ and for all $v \in \mathcal{V}_{ad} \cap L^\infty(\Omega)$ we have*

$$\begin{aligned}(H^*(Hu_\alpha - g), u - u_\alpha)_2 + \langle q_\alpha, u+v-1\rangle_{\mathcal{M},L^\infty} + \lambda\,(\nabla u_\alpha, \nabla(u-u_\alpha))_2 & \\ +r_\alpha\left[(u-u_\alpha, v_\alpha)_2 + (u_\alpha, v - v_\alpha)_2\right] & \geq 0,\end{aligned} \tag{3.5a}$$

$$r_\alpha\left[(u_\alpha, v_\alpha)_2 - \alpha\right] = 0. \tag{3.5b}$$

Proof. Let be $u \in H^1(\Omega) \cap L^\infty(\Omega), u \geq 0$ and $v \in \mathcal{V}_{ad} \cap L^\infty(\Omega)$. We add (2.14a) and (2.14b):

$$(p_\alpha^\varepsilon, u - u_\alpha^\varepsilon)_2 + (q_\varepsilon, u+v-1)_2 - (q_\varepsilon, u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1)_2 + r_\alpha^\varepsilon\left[(u_\alpha^\varepsilon, v - v_\alpha^\varepsilon)_2\,(u - u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2\right]$$
$$+ (u_\alpha^\varepsilon - u_\alpha, u - u_\alpha^\varepsilon)_{H^1} + (v_\alpha^\varepsilon - v_\alpha, v - v_\alpha^\varepsilon)_2 + \lambda\,(\nabla u_\alpha^\varepsilon, \nabla(u - u_\alpha^\varepsilon)) \geq 0.$$

As $-(q_\alpha^\varepsilon, u_\alpha^\varepsilon + v_\alpha^\varepsilon - 1) = -\varepsilon||q_\varepsilon||^2$ is always negative, we have

$$(p_\alpha^\varepsilon, u - u_\alpha^\varepsilon)_2 + (q_\alpha^\varepsilon, u+v-1) + \lambda\,(\nabla u_\alpha^\varepsilon, \nabla(u - u_\alpha^\varepsilon))$$
$$+ (u_\alpha^\varepsilon - u_\alpha, u - u_\alpha^\varepsilon)_{H^1} + (v_\alpha^\varepsilon - v_\alpha, v - v_\alpha^\varepsilon) + r_\alpha^\varepsilon\left[(u - u_\alpha^\varepsilon, v_\alpha^\varepsilon)_2 + (u_\alpha^\varepsilon, v - v_\alpha^\varepsilon)\right] \geq 0.$$

Passing to the limit in the above relations gives the result. □

The above system may be decoupled taking respectively $u = u_\alpha$ and $v = v_\alpha$:

Corollary 3.4. *Assume u_α is a solution to $(\mathcal{P}_\alpha)$ and $v_\alpha = 1 - u_\alpha$. There exists a Lagrange multiplier $(q_\alpha, r_\alpha) \in \mathcal{M}(\Omega) \times \mathbb{R}^+$ such that*

$$\begin{aligned}&\forall u \in H^1(\Omega) \cap L^\infty(\Omega) \text{ such that } u \geq 0 \\ &(H^*(Hu_\alpha - g) + r_\alpha v_\alpha, u - u_\alpha)_2 + \lambda\,(\nabla u_\alpha, \nabla(u-u_\alpha))_2 + \langle q_\alpha, u - u_\alpha\rangle_{\mathcal{M},L^\infty} \geq 0,\end{aligned} \tag{3.6a}$$

$$\forall v \in \mathcal{V}_{ad} \cap L^\infty(\Omega) \qquad \langle q_\alpha, v - v_\alpha\rangle_{\mathcal{M},L^\infty} + r_\alpha\,(u_\alpha, v - v_\alpha)_2 \geq 0, \tag{3.6b}$$

$$r_\alpha\left[(u_\alpha, v_\alpha)_2 - \alpha\right] = 0. \tag{3.6c}$$

$\boxed{4}$ Conclusion

We have obtained optimality conditions for the relaxed problem. However it is impossible to get convergence results as $\alpha \to 0$. Indeed the binary constraint does not allow H^1 functions and the limit function would not fit the functional framework. A generalization is to consider in a first step $W^{1,1}$ functions and use the L^1 norm of the gradient instead of the L^2 one. A regularization process is necessary but still possible (via Moreau-Yosida approximation) and we believe that the same techniques remain applicable. For the general case (BV norm) we need different tools, for instance sharp non smooth analysis. However, the relaxed H^1 formulation is suitable for numerical purpose. As we get the optimality system satisfied by solutions of $(\mathcal{P}_\alpha)$, next issue is to perform computations to solve it.

References

[1] I. Abraham, R. Abraham and M. Bergounioux, An active curve approach for tomographic reconstruction of binary radially symmetric objects, in *Numerical Mathematics and Advanced Applications*, K. Kunisch, G. Of and O. Steinbac (eds.), 2008, pp. 663–670.

[2] R. Abraham, M. Bergounioux and E. Trlat. A penalization approach for tomographic reconstruction of binary axially symmetric objects, *Appl. Math. Optim.* 2008 DOI 10.1007/s00245-008-9039-8

[3] R. Acar and C. R. Voge, Analysis of bounded variation penalty methods for ill-posed problems, *Inverse Problems* 10 (1994) 1217–1229.

[4] F. Alter, V. Caselles and A. Chambolle, Evolution of characteristic functions of convex sets in the plane by the minimizing total variation flow, *Interfaces Free Bound.* 7 (2005) 29–53.

[5] G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing*, Applied Mathematical Sciences, Vol. 147, Springer, New York, second edition, 2006.

[6] G. Aubert and L. Vese, A variational method in image recovery, *SIAM J. Numer. Anal.* 34 (1997) 1948–1979.

[7] V. Barbu. *Optimal Control of Variational Inequalities*, Research Notes in Mathematics, Vol. 147, Pitman, Boston, MA, 1984.

[8] M. Bergounioux, Optimal control of an obstacle problem, *Appl. Math. Optim.* 36 (1997) 147–172.

[9] E. Casas, K. Kunisch and C. Pola, Regularization by functions of bounded variation and applications to image enhancement, *Appl. Math. Optim.* 40 (1999) 229–257.

[10] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[11] F. Tröltzsch, A modification of the Zowe and Kurcyusz regularity condition with application to the optimal control of Noether operator equations with constraints on the control and the state, *Math. Operationforsch. Statist., Ser. Optimization* 14 (1983) 245–253.

[12] F. Tröltzsch, *Optimality Conditions for Parabolic Control Problems and Applications*, Teubner Texte, Leipzig 1984.

[13] C.R. Vogel and M.E. Oman, Iterative methods for total variation denoising, *SIAM J. Sci. Comput.* 17 (1996) 227–238.

[14] J. Zowe and S. Kurcyusz, Regularity and stability for the mathematical programming problem in banach spaces, *Appl. Math. Optim.* 42 (1979) 49–62.

Manuscript received 1 March 2008
revised 4 October 2008
accepted for publication 22 November 2008

M. Bergounioux
Laboratoire MAPMO, UMR 6628, Université d'Orléans, Fédération Denis Poisson
Route de Chartres, BP 6759, 45067 Orléans cedex 2, France
E-mail address: maitine.bergounioux@univ-orleans.fr

A. Srour
Laboratoire de Mathématiques et Physique Théorique Université Fraçois-Rabelais Tours
Fédération Denis Poisson -UMR CNRS 6083, Parc de Grandmont, 37200 Tours, France
E-mail address: ali.srour@univ-tours.fr