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ON MATHEMATICAL ANALYSIS OF A NON-REGULAR ELECTRONIC CIRCUIT

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Abstract: In this paper, we present a mathematical modelling and analysis (existence, uniqueness...) of a non-regular electronic circuit using the superpotentiel of Moreau and the theories of variationnal inequalities and differential inclusions. Through this analysis example, we provide a new methodology for engineers to study a large class of applications. Some remarks are given to show perspectives of this work allowing to complete such analysis.

Key words: *variational inequalities, differential inclusions, recession functions, recession cones, set-valued ampere-volt characteristics, non-regular circuits in electronics*

Mathematics Subject Classification: *49A52, 49A29, 49J40, 49K24, 47N10, 94C05*

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1 Introduction

To deal with unilateral problems in mechanics it is now well established to use the notion of the superpotential for convex (but not differentiable) problems [16, 17]. This concept is generalized by Panagiotopoulos [18, 19] to the nonconvex case (see e.g. [8, 10]) and references cited therein). These approaches connect mainly the results of the variational and hemivariational theories to applications in Robotics, elasticity, plasticity... Our aim is to extend these results to the field of electronics circuits. Indeed, there is a class of these circuits that involves the so-called non-regular devices like semiconductors.

Semiconductors like diodes and Zener diodes are described through their Ampere-Volt characteristics [15] that are set-valued functions. In [1] it is shown how the approach of Moreau and Panagiotopoulos can be used to develop a rigorous formulation for such circuits. Indeed, in this paper, we first consider a dynamical system to describe the circuit then we adapt a result due to [11], to show the existence and uniqueness of the system. Afterwards, we will study the stationary solutions of the system given by the following variational inequality:

Find $u \in \mathbb{R}^n$ such that

$$\langle Mu + q, v - u\rangle + \Phi(v) - \Phi(u) \geq 0,\ \forall v \in \mathbb{R}^n, \tag{1.1}$$

where $M \in \mathbb{R}^{n\times n}$ is a real matrix, $q \in \mathbb{R}^n$ a vector and $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ a proper convex and lower semicontinuous function. This variational inequality will be denoted by $VI(M, q, \Phi)$.

To prove the existence (and possibly uniqueness) of the solution of $VI(M, q, \Phi)$, we follow the lines of [4]. Then we make some remarks on the numerical implementation, the stability and the sensitivity of the system.

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2 Circuit Modelling

2.1 Definitions and Notations

Let us first fix some notations and recall some tools in convex analysis which will be used in this paper.

For $x, y \in \mathbb{R}^n$, the notation $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is used to denote the euclidean scalar product on $\mathbb{R}^n$ and $\|x\| = \sqrt{\langle x, x\rangle}$ to denote the corresponding norm.

We denote by $\Gamma(\mathbb{R}^n; \mathbb{R} \cup \{+\infty\})$ the set of proper convex and lower semicontinuous functions $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with closed domain. The domain $D(\Phi)$ of Φ is defined by:

$$D(\Phi) = \{x \in \mathbb{R}^n : \Phi(x) < +\infty\}.$$

Convex subdifferential: Let $\Phi \in \Gamma(\mathbb{R}^n; \mathbb{R} \cup \{+\infty\})$ be given. The convex subdifferential $\partial\Phi(x)$ (see e.g. [12], [21]) of Φ at x is defined by:

$$\partial\Phi(x) = \{w \in \mathbb{R}^n : \Phi(v) - \Phi(x) \geq \langle w, v - x\rangle, \forall v \in \mathbb{R}^n\}.$$

The set $\partial\Phi(x)$ describes the differential properties of Φ by means of the supporting hyperplanes to the epigraph of Φ at $(x, \Phi(x))$.

Closed convex set: Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. We denote by Ψ_K the indicator function of K, that is:

$$\Psi_K(x) := \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases}, \quad (x \in \mathbb{R}^n). \tag{2.1}$$

Then

$$\partial\Psi_K(x) = \begin{cases} \{w \in \mathbb{R}^n : \langle w, v - x\rangle \leq 0, \forall v \in K\} & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}.$$

The dual cone of K is the nonempty closed convex cone K^* defined by:

$$K^* := \{w \in \mathbb{R}^n : \langle w, v\rangle \geq 0, \ \forall v \in K\}. \tag{2.2}$$

Recession function: Let x_0 be any element in $D(\Phi)$. The recession function of Φ is defined by

$$\Phi_\infty(x) = \lim_{\lambda \to +\infty} \frac{1}{\lambda}\Phi(x_0 + \lambda x) \ \ (x \in \mathbb{R}^n).$$

The function $\Phi_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function which describes the asymptotic behavior of Φ.

Recession cone: Let x_0 be some arbitrary element of K. The recession cone of K is defined by

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda}(K - x_0).$$

The set K_∞ is a nonempty closed convex cone that is described in terms of the directions which recede from K.

$\boxed{2.2}$ Circuit Modelling

We consider an RLC circuit containing two non-regular devices: the diodes D_1 and D_2 (see Figure 1). Applying Kirchhoff's laws to this circuit yield:

$$\left\{\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= -\tfrac{1}{L_3C}x_1-\tfrac{(R_1+R_3)}{L_3}x_2+\tfrac{R_1}{L_3}x_3-\tfrac{1}{L_3}V_1-\tfrac{1}{L_3}V_2\\ \frac{dx_3}{dt} &= \tfrac{R_1}{L_2}x_2-\tfrac{(R_1+R_2)}{L_2}x_3+\tfrac{1}{L_2}V_1+\tfrac{1}{L_2}u\end{aligned}\right. \tag{2.3}$$

where $R_1 > 0, R_2 > 0, R_3 > 0$ are resistors, $L_2 > 0$, $L_3 > 0$ are inductors, $C > 0$ is a capacitor, x_1 is the time integral of the current across the capacitor, x_2 is the current across the capacitor, x_3 is the current across the inductor L_2 and resistor R_2, V_1 is the voltage of the diode D_1 and V_2 is the voltage of the diode D_2.

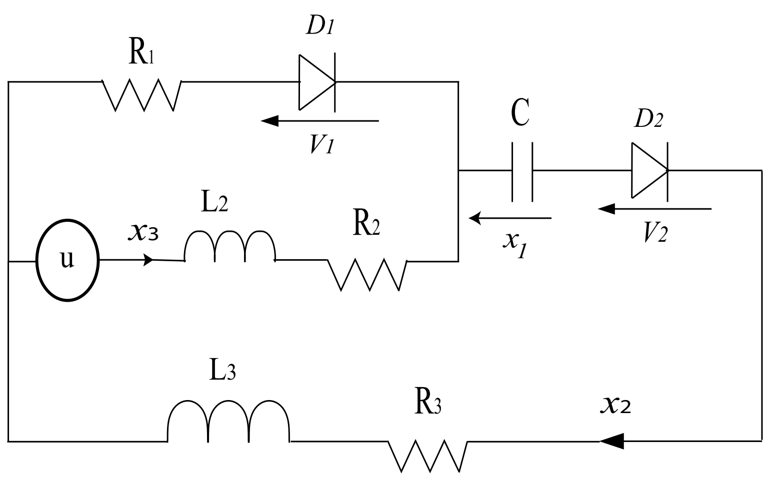


Figure 1: RLC circuit with diodes

Let us recall that diodes are described through multivalued functions. We consider here the ideal and the practical diodes which charateristics are depicted in Figures 2 and 3 respectively. Let us first consider the ideal diode Ampere-Volt charcteristic given in Figure 2.

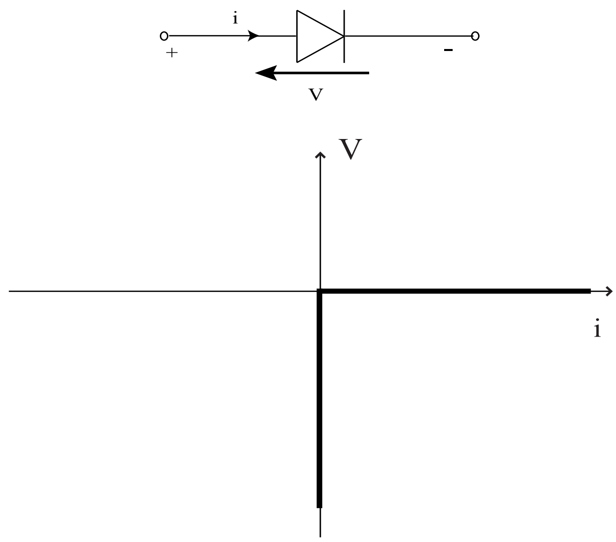


Figure 2: Ideal diode characteristic

The ideal diode plays a simple switch role. If $V < 0$ then $i = 0$ and the diode is blocking. If $i > 0$ then $V = 0$ and the diode is conducting. It can be easily seen that the ideal diode is described by the complementarity relation

$$V \leq 0, \quad i \geq 0, \quad Vi = 0$$

that is also

$$\min\{-V, i\} = 0.$$

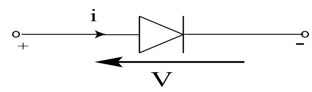
The electrical superpotential of the ideal diode is then

$$\varphi_D(x) = \Psi_{R_+}(x), \quad (x \in \mathbb{R}).$$

The subdifferential (see e.g. [12, 21]) of the electrical superpoteniel is given by:

$$\partial\varphi_D(x) := \begin{cases} \mathbb{R}_- \text{ if } x = 0 \\ 0 \text{ if } x > 0 \\ \emptyset \text{ if } x < 0 \end{cases}, \quad (x \in \mathbb{R}).$$

Figure 3 illustrates the ampere-volt characteristic of a practical diode model.



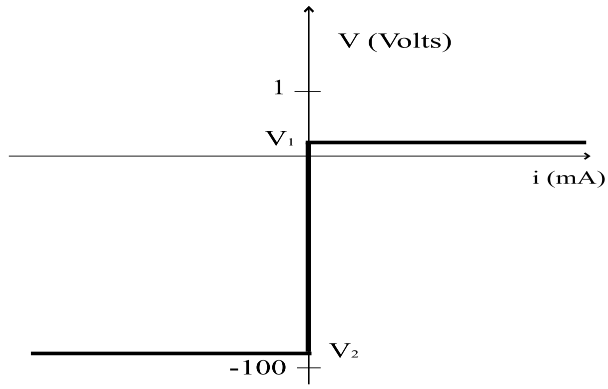


Figure 3: Practical diode characteristic

There is a voltage point, called the knee voltage V_1, at which the diode begins to conduct and a maximum reverse voltage, called the peak reverse voltage V_2, that will not force the diode to conduct. When this voltage is exceeded, the depletion may breakdown and allow the diode to conduct in the reverse direction.

The electrical superpotential of the practical diode is

$$\varphi_{PD}(x) = \begin{cases} V_1 x & \text{if } x \geq 0 \\ V_2 x & \text{if } x < 0 \end{cases}, \quad (x \in \mathbb{R}).$$

and the recession function of the electrical superpotential is given by:

$$(\varphi_{PD})_\infty(x) = \varphi_{PD}(x), \ (x \in \mathbb{R}).$$

The subdifferential of the superpotential is given by:

$$\partial\varphi_{PD}(x) = \begin{cases} V_2 & \text{if } x < 0 \\ [V_2, V_1] & \text{if } x = 0 \\ V_1 & \text{if } x > 0 \end{cases}, \quad (x \in \mathbb{R}).$$

Combining the system (2.3) and the multivalued characteristics of the diodes, we obtain the following model:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{L_3 C} & -\frac{(R_1+R_3)}{L_3} & \frac{R_1}{L_3} \\ 0 & \frac{R_1}{L_2} & -\frac{(R_1+R_2)}{L_2} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$- \overbrace{\begin{pmatrix} 0 & 0 \\ \frac{1}{L_3} & \frac{1}{L_3} \\ -\frac{1}{L_2} & 0 \end{pmatrix}}^{B} \begin{pmatrix} y_{L,1} \\ y_{L,2} \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{1}{L_2} \end{pmatrix}}^{D} u, \tag{2.4}$$

and

$$\begin{cases} y_{L,1} \in \partial j_{D_1}(-x_3 + x_2) \\ y_{L,2} \in \partial j_{D_2}(x_2) \end{cases} \tag{2.5}$$

where $y_{L,1}$ is the voltage of the diode D_1, $y_{L,2}$ is the voltage of the diode, j_{D_1} is the electrical superpotential of the diode D_1 and j_{D_2} is the electrical superpotential of the diode D_2. Setting

$$y = \overbrace{\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and defining the function $j_D : \mathbb{R}^2 \to \mathbb{R}; X \mapsto j(X)$ by the formula:

$$j_D(X) = j_{D_1}(X_1) + j_{D_2}(X_2),$$

The mapping $x \mapsto j_D(x)$ is proper, convex and lower semicontinuous function for all $t \geq 0$.

The mapping $u : [0, +\infty[\to \mathbb{R}; t \mapsto u(t)$ describes the input of the model . We suppose that:

$$u \in C^0([0, +\infty); \mathbb{R}), \ \frac{du}{dt} \in L^\infty_{loc}(0, +\infty; \mathbb{R}).$$

For $x_0 \in \mathbb{R}^3$, we consider the problem $P(x_0)$: Find a function $x : [0, +\infty[\to \mathbb{R}^3;\ t \mapsto x(t)$ and a function $y_L : [0, +\infty[\to \mathbb{R}^2;\ t \mapsto y_L(t)$ such that:

$$x \in C^0([0, +\infty[; \mathbb{R}^3), \tag{2.6}$$

$$B y_L \in L^\infty_{loc}(0, +\infty; \mathbb{R}^3), \tag{2.7}$$

$$\frac{dx}{dt} \in L^\infty_{loc}(0, +\infty; \mathbb{R}^3), \tag{2.8}$$

$$x(0) = x_0, \tag{2.9}$$

$$\frac{dx}{dt}(t) = Ax(t) - By_L(t) + Du(t), \text{ a.e. } t \geq 0 \tag{2.10}$$

$$y(t) = Cx(t), \ \forall\, t \geq 0, \tag{2.11}$$

and

$$y_L(t) \in \partial j_D(y(t)), \text{ a.e. } t \geq 0. \tag{2.12}$$

Let us note that the relations in (2.5) are equivalent to

$$y_L \in \partial j_D(Cx).$$

3 Dynamics Analysis

In this section we will give an existence and uniqueness result for our circuit in the ideal and practical case.

Let us consider the following symmetric and invertible matrix R

$$R = \begin{pmatrix} \frac{1}{\sqrt{C}} & 0 & 0 \\ 0 & \sqrt{L_3} & 0 \\ 0 & 0 & \sqrt{L_2} \end{pmatrix}$$

satisfying

$$R^{-2}C^T = B,$$

where $R^{-2} = (R^{-1})^2$.

By using the Kalman-Yakoubovich-Popov lemma [13, 20, 22] and following the lines of [1], we can reduce the problem $P(x_0)$ to the differential inclusion $Q(z_0)$ defined by:

Find a function $z : [0, +\infty[\to \mathbb{R}^3;\ t \mapsto z(t)$ such that:

$$z \in C^0([0, +\infty[; \mathbb{R}^3), \tag{3.1}$$

$$\frac{dz}{dt} \in L^\infty_{loc}(0, +\infty; \mathbb{R}^3), \tag{3.2}$$

$$z(0) = Rx_0, \tag{3.3}$$

$$\frac{dz}{dt}(t) \in RAR^{-1}z(t) + RDu(t) - R^{-1}C^T\partial j_D(CR^{-1}z(t)), \text{ a.e. } t \geq 0. \tag{3.4}$$

where $z_0 = z(0)$.

Let us recall the following theorem, due to [11] (consequence of Kato's theorem [6]).

Theorem 3.1. *Let H be a real Hilbert and $T : D(T) \subset H \to 2^H$ be a maximal monotone operator. Let $t_0 \in \mathbb{R}, \sigma \in \mathbb{R}, z_0 \in D(T)$ be given and suppose that $f : [t_0, +\infty) \to H$ satisfies*

$$f \in C^0([0, +\infty); H), \frac{df}{dt} \in L^\infty_{loc}(0, \infty; H).$$

Then there exists unique $z \in C^0([0,\infty); H)$ satisfying

$$\frac{dz}{dt} \in L^{\infty}_{loc}(0,+\infty; H);$$

$$z \ \ is \ \ right-differentiable \ \ on \ \ [t_0,\infty);$$

$$z(t) \in D(T), \ \ t \geq t_0;$$

$$z(t_0) = z_0;$$

$$\sigma z(t) + f(t) \in \frac{dz}{dt}(t) + Tz(t), \quad a.e.t \geq t_0.$$

As a consequence of Theorem 3.1 we obtain the following proposition:

Proposition 3.2. *The problem $Q(z_0)$ has a unique solution.*

Proof. Here $\sigma = 0$ As j_D is proper, convex and lower semicontinuous, its subdifferential ∂j_D is maximal monotone [6].

The domain $D(j)$ is nonempty for both ideal and practical diodes.

We consider the input function $f = -RDu \in C^0(0,+\infty;\mathbb{R}^3)$ and $RD\frac{du}{dt} \in L^{\infty}_{loc}(0,+\infty;\mathbb{R})$.

On the other hand, it is clear that the operator T, given by $Tx = L.x + R^{-1}C^T\partial j_D(CR^{-1}x)$, $\forall x \in D(\partial j_D)$ where $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear operator defined by $Lx = RAR^{-1}x$, is maximal monotone. □

4 Qualitative Properties of the Stationary Solutions

Let us first consider the stationary solutions of (2.4) satisfying the problem: Find $(x, y_L) \in \mathbb{R}^3 \times \mathbb{R}^2$ such that

$$Ax - By_L + Du = 0, \tag{4.1}$$

$$y = Cx, \tag{4.2}$$

and

$$y_L \in \partial j_D(y). \tag{4.3}$$

Let us denote this problem by $NRM(A, B, C, D, u, j_D)$.

$(H1)$: $j_D \in \Gamma(\mathbb{R}^2; \mathbb{R} \cup \{+\infty\})$.

$(H2)$: *There exists $\bar{x}_0 \in \mathbb{R}^3$ such that j_D is finite and continuous at $\bar{y}_0 = C\bar{x}_0$.*

$(H3)$: *There exists an invertible matrix $P \in \mathbb{R}^{3\times3}$ such that*

$$PB = C^T.$$

We set

$$\Phi(x) = j_D(Cx), \ (\forall x \in \mathbb{R}^3). \tag{4.4}$$

Let us consider the following result [4]

Proposition 4.1. *Suppose that assumptions* $(H1)-(H3)$ *are satisfied and let* Φ *be defined as in (4.4).*

1) *If* (x, y_L) *is solution of Problem* $\mathbf{NRM(A,B,C,D,u,j_D)}$ *then* x *is solution of problem* $\mathbf{VI(-PA,-PDu,\Phi)}$ *defined by*

$$\langle -PAx - PDu, v-x\rangle + \Phi(v) - \Phi(x) \geq 0, \forall v \in \mathbb{R}^n. \tag{4.5}$$

2) *If* x *is solution of problem* $VI(-PA,-PDu,\Phi)$ *then there exists* $y_L \in \mathbb{R}^m$ *such that* (x, y_L) *is solution of Problem* $NRM(A,B,C,D,u,j_D)$.

Let us check that the assumptions of Proposition 4.1 are satisfied. Indeed, it is clear that j_D satisfies $(H1)$. In the case of ideal and practical diodes, it suffices to consider $\begin{pmatrix} 1 & 2 \end{pmatrix}^T = C\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}^T \in \operatorname{int}\{\mathbb{R}^2_+\}$ a point at which j_D is finite and continuous therefore Assumption $(H2)$ holds. Finally, the matrix

$$P = \begin{pmatrix} 1/C & 0 & 0 \\ 0 & L_3 & 0 \\ 0 & 0 & L_2 \end{pmatrix}$$

allows to check Assumption $(H3)$.

Let us now consider the problem $VI(-PA,-PDu,\Phi)$ where

$$-PA = \begin{pmatrix} 0 & -1/C & 0 \\ 1/C & R_1+R_3 & -R_1 \\ 0 & -R_1 & R_1+R_2 \end{pmatrix}$$

Here we study separatly the ideal case then the practical one. We first start by recalling the following result adapted from [3].

Corollary 4.2. *Let* $\Phi \in \Gamma(\mathbb{R}^n; \mathbb{R}\cup\{+\infty\})$ *and* $M \in \mathbb{R}^{n\times n}$ *a positive semidefinite matrix. If there exists* $x_0 \in D(\Phi)$ *such that:*

$$\langle q - M^T x_0, v\rangle + \Phi_\infty(v) > 0,\ \forall v \in\ D(\Phi)_\infty \cap \ker\{M+M^T\} \cap \{x \in \mathbb{R}^n : Mx \in D(\Phi_\infty)^*\},\ v \neq 0, \tag{4.6}$$

then problem $\mathbf{VI(M,q,\Phi)}$ *has at least one solution.*

□

This result allows to prove that the problem $VI(-PA,-PDu,\Phi)$ has at least one solution in the case of ideal diodes where

$$\Phi(x) = \Psi_{(\mathbb{R}_+)^2}(x), \quad \forall x \in \mathbb{R}^2.$$

Indeed, the matrix $-PA$ is positive semidefinite. Moreover, we have, $D(\Phi)_\infty = D(\Phi) = (\mathbb{R}_+)^2$,

We have also:

$$\ker\{-PA - (PA)^T\} = \{(\alpha, 0, 0)^T : \alpha \in \mathbb{R}\}$$

$$D(\Phi)_\infty \cap \ker\{-PA - (PA)^T\} \cap \{x \in \mathbb{R}^3 : Mx \in D(\Phi_\infty)^*\} = \{(\alpha, 0, 0)^T : \alpha \in \mathbb{R}_+\}$$

On the other hand, $-PDu = (0, 0, -u)^T$ and $\Phi_\infty = \Psi_{(\mathbb{R}_+)^2}$. It suffices to consider $x_0 = (0, 1, 0)^T$ to check (4.6) and consequently to conclude from Corollary 4.2 that our problem $VI(-PA, -PDu, \Phi)$ has at least one solution.

We now turn to the practical diodes case where the superpotentiel Φ is defined by

$$\Phi(x) = |x_1| + |x_2|, \ \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

We can easily see that $-PA$ is a P_0-matrix and $\Phi \in SD\Gamma(\mathbb{R}^2; \mathbb{R} \cup \{+\infty\})$, where $SD\Gamma(\mathbb{R}^n; \mathbb{R} \cup \{+\infty\})$ denotes the the set of functions $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfying

$$\Phi(x) = \Phi_1(x_1) + \Phi_2(x_2) + \ldots + \Phi_n(x_n), \ \forall x \in \mathbb{R}^n, \tag{4.7}$$

where, for all $1 \leq i \leq n$, we have

$$\Phi_i \in \Gamma(\mathbb{R}; \mathbb{R} \cup \{+\infty\}), \tag{4.8}$$

$$\Phi_i(\lambda x) = \lambda \Phi_i(x), \ \forall \lambda \geq 0, \ \forall x \in D(\Phi_i), \tag{4.9}$$

and such that the functions Φ_i $(1 \leq i \leq n)$ in (4.7) are strictly convex.

Here

$$D(\Phi_\infty) = D(\Phi)_\infty = \mathbb{R}^2$$

and

$$D(\Phi_\infty)^* = \{0\}.$$

Therefore we have

$$\{x \in \mathbb{R}^3 : -PAx \in D(\Phi_\infty)^*\} = \{0\}$$

and thus

$$D(\Phi)_\infty \cap \{x \in \mathbb{R}^3 : \langle -PAx, x\rangle = 0\} \cap \{x \in \mathbb{R}^3 : -PAx \in D(\Phi_\infty)^*\} = \{0\},$$

so that we can apply the following corollary to show that $VI(-PA, -PDu, \Phi)$ has a unique solution.

Corollary 4.3. *Suppose that $\Phi \in SD\Gamma(\mathbb{R}^n; \mathbb{R} \cup \{+\infty\})$ and let $M \in \mathbb{R}^{n \times n}$ be a P_0-matrix. If*

$$D(\Phi)_\infty \cap \{x \in \mathbb{R}^n : \langle Mx, x\rangle = 0\} \cap \{x \in \mathbb{R}^n : Mx \in D(\Phi_\infty)^*\} = \{0\}$$

then for each $q \in \mathbb{R}^n$, problem $\mathbf{VI}(\mathbf{M}, \mathbf{q}, \mathbf{\Phi})$ has a unique solution.

Remark 4.4. The circuit 1 is also used with a practical Zener diode (see Figure 4) instead of the diode D_1. Zener diodes are made to permit current to flow in the reverse direction if the voltage is larger than the rated breakdown or "Zener voltage" V_3.

Let us use the notation of Figure 4. It is here assumed that

$$I_1 < 0 < I_2, \ \ V_1 < V_3 < 0 < V_4 < V_2.$$

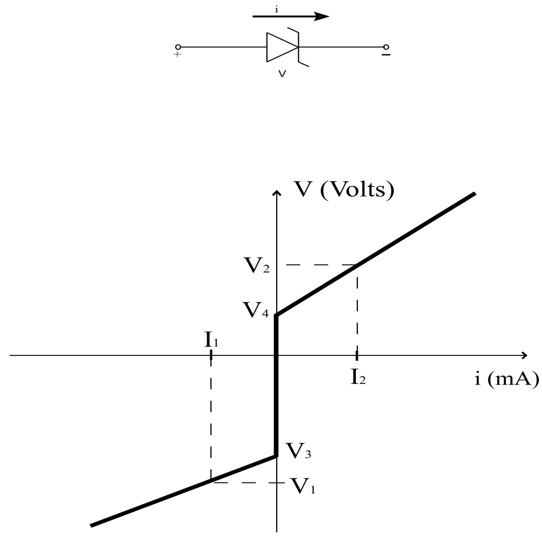


Figure 4: Practical Zener diode model

The electrical superpotential of the Zener diode is

$$\varphi_Z(x) = \begin{cases} \frac{(V_1 - V_3)}{2I_1} x^2 + V_3 x & \text{if } x < 0 \\ \\ \frac{(V_2 - V_4)}{2I_2} x^2 + V_4 x & \text{if } x \geq 0 \end{cases}, \quad (x \in \mathbb{R}).$$

The recession function of the electrical superpotential is given by:

$$(\varphi_Z)_\infty(x) = \Psi_{\{0\}}(x), \ (x \in \mathbb{R}).$$

Moreover

$$\partial\varphi_Z(x) = \begin{cases} \frac{(V_1 - V_3)}{I_1} x + V_3 & \text{if } x < 0 \\ \\ [V_3, V_4] & \text{if } x = 0 \\ \\ \frac{(V_2 - V_4)}{I_2} x + V_4 & \text{if } x > 0 \end{cases}, \quad (x \in \mathbb{R})$$

Here we can use Proposition 3.1 to assert that the problem $P(x_0)$ has a unique solution in the case where a Zener diode replaces the diode D_1.

Concerning the equilibrium study, we may use directly, Corollary 4.3, to prove the existence and uniqueness of the solution. Indeed, it is easy to check that the characteristics multi-valued functions of the practical diode and practical Zener diode have the same domain and belong to $SD\Gamma(\mathbb{R}^2; \mathbb{R} \cup \{+\infty\})$. Moreover, the matrix M is the same for both cases i.e. $M = -PA$.

Remark 4.5. 1. To simulate the stationnary solution, we can use appropriate methods developed for the variational inequalities and differential inclusions theories. For the ideal diodes case, where $\Phi(x) = \Psi_{(\mathbb{R}_+)^2}(x), \ \ \forall x \in \mathbb{R}^2$, we can reduce $\mathbf{VI}(\mathbf{M}, \mathbf{q}, \mathbf{\Phi})$ to a Linear Complementarity Problem (LCP). Indeed, let us consider the system (4.1)-(4.3), where we denote by $y$the the vector $y = (y_1 \ \ y_2)^T$, that we reduce to the following sytem:

$$y = Hy_L + b, \tag{4.10}$$

where

$$H = \begin{pmatrix} R_1 + R_2 & -2R_1 - R_2 \\ & \\ -2R_1 - R_3 - 1/C & 2R_1 + R_2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -u \\ u \end{pmatrix}.$$

The system (4.10) and the ideal diode characteristic i.e.

$$-y_L \geq 0, \ \ y \geq 0, \ \ y_L^T y = 0,$$

is an LCP that can be implemented using Lemke's algorithm [14].

2. The paper [2] deals with the sensitivity of the problem $VI(M, q, \Phi, K)$ defined by:

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \langle Mu + q, v - u\rangle + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in K \end{cases}$$

The sensitivity is defined in the sense of the following definition:

One says that the variational inequality $VI(M, q, \Phi, K)$ is stable provided that there exists $\varepsilon > 0$ such that for any symmetric and positive semidefinite matrix M_ε, any vector $q_\varepsilon \in q + \varepsilon\mathbb{B}_n$ (here $\mathbb{B}_n$ denotes the open unit ball in $\mathbb{R}^n$), any proper lower semicontinuous bounded from below convex function Φ_ε, and any nonempty closed convex set K_ε satisfying the following conditions

$$0 \in D(\Phi_\varepsilon \cap K_\varepsilon) \text{ and } \ker(M) \cap \ker(\Phi_\infty) \cap K_\infty = \ker(M_\varepsilon) \cap \ker((\Phi_\varepsilon)_\infty) \cap (K_\varepsilon)_\infty, \tag{4.11}$$

the perturbed problem $VI(M_\varepsilon, q_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ has at least one solution.

The stability result presented in [2] is to be extended to the nonsymmetric case.

3. In [5, 7, 9], we can find results concerning the stability of the trivial solution of Problem $VI(M, q, \Phi)$, in the ideal diode case.

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