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REVERSE CONVEX PROGRAMS: STABILITY AND GLOBAL OPTIMALITY

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Abstract: In this paper, we consider a reverse convex program for which we first give sufficient conditions ensuring stability. Then, we give necessary and sufficient conditions for global optimality. In particular, a necessary and sufficient optimality condition reduces the problem to a convex maximization problem constrained by a compact convex set.

Key words: *convex analysis, reverse convex programs, optimality conditions*

Mathematics Subject Classification: *90C25, 90C26, 49K35.*

1 Introduction

We consider the following global optimization problem

$$(\mathcal{P}) \qquad \operatorname*{Min}_{\substack{x\in\mathcal{D}\\ g(x)\geq 0}} f(x)$$

with $\mathcal{D} = \left\{x \in \mathbb{R}^n / \, G(x) \leq 0\right\}$, and $f, g : \mathbb{R}^n \to \mathbb{R}$, $G = (G_1, ..., G_p) : \mathbb{R}^n \to \mathbb{R}^p$, being convex functions, called a reverse convex program. Such a problem has many applications, for example engineering design, mechanics, communication networks, and economic management, and it was intensively studied in the literature in different contexts (see for instance [1-4,9-20]). For more motivation, we refer in particular to [10,17,18]. The presence of the constraint $g(x) \geq 0$, in the problem $(\mathcal{P})$, can destroy the convexity of the feasible set, and hence makes the resolution difficult. Horst-Tuy [10], Tuy [17], and Tuy-Thuong [18], were in particular interested in a concept of stability of reverse convex programs of the form of $(\mathcal{P})$. By virtue of this concept, the authors have established theoretical results and presented some algorithms for such problems (illustrative examples are given in [10,18]). A property of regularity of solutions is one of the main assumptions used in [10,18] to get stability results. In this paper, we first give sufficient conditions that ensure the stability of the problem $(\mathcal{P})$. Therefore, this result gives the possibility to apply some theoretical and numerical results given in the literature concerning stable reverse convex programs. Finally, we give necessary and sufficient conditions for global optimality. In particular, a necessary and sufficient optimality condition is derived as a consequence of the obtained stability results, which reduces the problem $(\mathcal{P})$ to a convex maximization problem constrained by a compact convex set.

The organization of the paper is as follows. In Section 2, we recall some definitions and results that will be required in the sequel. In Section 3, we give sufficient conditions that ensure the stability of $(\mathcal{P})$. Finally, in Section 4, we give necessary and sufficient conditions for global optimality.

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2 Preliminaries

Throughout the paper, the functions f, g and G are convex, and the following convention for inequalities will be used. For $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$,

$$x \geq y \Longleftrightarrow x_i \geq y_i \quad \text{and} \quad x > y \Longleftrightarrow x_i > y_i,\ i = 1, ..., n.$$

Before going further, let us recall the following definitions and results from [10,17,18], about a concept of stability of optimization problems.

Let $\hat{f}$, $\hat{g} : \mathbb{R}^n \to \mathbb{R}$, $\alpha \in \mathbb{R}$, and $\hat{\mathcal{D}}$ is a nonempty subset of $\mathbb{R}^n$. Consider the following optimization problem

$$(\hat{\mathcal{P}}_\alpha) \qquad \underset{\substack{x \in \hat{\mathcal{D}} \\ \hat{g}(x) \geq \alpha}}{\text{Min}}\ \hat{f}(x)$$

and let $\inf \hat{\mathcal{P}}_\alpha$ denote the value of $(\hat{\mathcal{P}}_\alpha)$.

Definition 2.1. **1)** The problem $(\hat{\mathcal{P}}_\alpha)$ is stable, if $\lim_{\substack{\alpha' \to \alpha \\ \alpha' > \alpha}} \inf \hat{\mathcal{P}}_{\alpha'} = \inf \hat{\mathcal{P}}_\alpha$.

2) A feasible point x of $(\hat{\mathcal{P}}_\alpha)$ is said to be regular for $(\hat{\mathcal{P}}_\alpha)$, if there exists a sequence (x_k) converging to x, such that $x_k \in \hat{\mathcal{D}}$, and $\hat{g}(x_k) > \alpha$, for large k.

We recall the following fundamental results that will be required in the sequel to establish our main results.

Proposition 2.2 ([18]). *If* $\inf \hat{\mathcal{P}}_\alpha > -\infty$, $\hat{f}$ *is upper semicontinuous on* $\hat{\mathcal{D}}$, *and if there exists at least one solution of* $(\hat{\mathcal{P}}_\alpha)$, *that is regular for* $(\hat{\mathcal{P}}_\alpha)$, *then* $(\hat{\mathcal{P}}_\alpha)$ *is stable.*

Consider the case where $\hat{\mathcal{D}} = \{x \in \mathbb{R}^n / h_i(x) \leq 0, i = 1, ..., m\}$, with $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, and set $\hat{\mathcal{G}} = \{x \in \mathbb{R}^n / \hat{g}(x) < 0\}$.

Proposition 2.3 ([17]). *Let* $\bar{x}$ *be a feasible point of* $(\hat{\mathcal{P}}_0)$ $(\alpha = 0)$. *Assume that the problem* $(\hat{\mathcal{P}}_0)$ *is stable and the following assumptions are satisfied.*

1) *The functions* $\hat{f}$, $\hat{g}$, *and* h_i, $i = 1, ..., m$, *are convex on* $\mathbb{R}^n$,

2) *The set* $\hat{\mathcal{G}}$ *is bounded,*

3) *There exists* $w \in \hat{\mathcal{D}} \cap \hat{\mathcal{G}}$, *such that* $\hat{f}(w) < \inf \hat{\mathcal{P}}_0$.

Then, $\bar{x}$ *is a solution of* $(\hat{\mathcal{P}}_0)$, *if and only if*

$$\max_{\substack{x \in \hat{\mathcal{D}} \\ \hat{f}(x) \leq \hat{f}(\bar{x})}} \hat{g}(x) = 0.$$

3 Stability Results

In this Section, we give sufficient conditions that ensure the stability of the problem $(\mathcal{P})$. As mentioned in the introduction, such a result gives the possibility to apply some theoretical and numerical results to $(\mathcal{P})$ given in the literature for stable reverse convex programs (see [10,17,18]). Notice that the concept of regularity will be crucial to give stability results. In the sequel, for a given convex function $h : \mathbb{R}^n \to \mathbb{R}$, we shall denote by $h'(x; d)$, and $\partial h(x)$, the directional derivative of h at x in the direction $d \in \mathbb{R}^n \setminus \{0\}$, and the subdifferential of h at x, respectively, i.e.,

$$h'(x; d) = \lim_{t \searrow 0^+} \frac{h(x + td) - h(x)}{t}$$

and

$$\partial h(x) = \big\{x^* \in \mathbb{R}^n /\, h(y) \geq h(x) + \langle x^*, y - x\rangle,\ \forall\, y \in \mathbb{R}^n\big\}.$$

An element x^* of $\partial h(x)$ is called a subgradient of h at x. For $x \in \mathcal{D}$, let $I(x) = \big\{i \in \{1, ..., p\} /\, G_i(x) = 0\big\}$, denote the set of active constraints G_i at x, and for $k \in \{1, ..., p\}$, set

$$I_k = \{1, ..., k\},\ \mathcal{D}_k = \bigcup_{i \in I_k} \big\{x \in \mathcal{D} /\, G_i(x) = 0\big\},\ \mathcal{F}_k = \mathcal{D} \setminus \mathcal{D}_k.$$

Remark that for a given problem $(\mathcal{P})$, a numeration of the constraints G_i is chosen and fixed. In the above notations, the sets I_k and $\mathcal{D}_k$ (and also $\mathcal{F}_k = \mathcal{D} \setminus \mathcal{D}_k$) depend on the numeration that we have chosen, and that we will keep for the rest of the paper. It follows that for a given integer $k \in \{1, ..., p\}$, the sets I_k, $\mathcal{D}_k$ and $\mathcal{F}_k$ are well defined (see examples 4.9 and 4.10 in section 4 for illustration). We will make the following assumptions.

(3.1) The set $\mathcal{D}$ is bounded,

(3.2) $\inf_{x \in \mathcal{D}} f(x) < \inf_{\substack{x \in \mathcal{D} \\ g(x) \geq 0}} f(x)$,

(3.3) There exists $l \in I_p$, such that $\inf_{\substack{x \in \mathcal{D} \\ g(x) \geq 0}} f(x) < \inf_{\substack{x \in \mathcal{D}_l \\ g(x) \geq 0}} f(x)$,

(3.4) $\forall x \in \mathcal{F}_l$, $0 \notin \partial g(x)$,

(3.5) $\forall x \in \mathcal{F}_l$, we have $\begin{cases} 0 \notin \bigcup_{i \in I(x)} \partial G_i(x), \\ \bigcap_{i \in I(x)} \big\{d \in \mathbb{R}^n /\, G_i'(x; d) < 0\big\} \cap \partial g(x) \neq \emptyset. \end{cases}$

Assumption (3.1) is ordinary and needs no explanation. Let us give some explanations and commentaries on assumptions (3.2)-(3.5) in the following remark.

Remark 3.1. **1)** Assumption (3.2) means that the constraint $g(x) \geq 0$, is essential (see [10]). This definition is given in the sense that if assumption (3.2) is not satisfied, then the problem $(\mathcal{P})$ is equivalent to minimize f over $\mathcal{D}$ (which is a convex minimization problem). So, it is natural to make such an assumption for reverse convex programs. As in [17], we deduce that assumption (3.2), in particular, implies that

- there exists $x_0 \in \mathcal{D}$, verifying $g(x_0) < 0$, and $f(x_0) < \inf_{\substack{x \in \mathcal{D} \\ g(x) \geq 0}} f(x)$,
- for any $x \in \mathcal{D}$, such that $g(x) > 0$, there exists $\pi(x) \in [x_0, x]$, verifying

$$g(\pi(x)) = 0,\ G(\pi(x)) \leq 0, \text{ and } f(\pi(x)) < f(x),$$

where for $a, b \in \mathbb{R}^n$, $[a, b] = \big\{x \in \mathbb{R}^n /\, x = a + t(b - a), 0 \leq t \leq 1, t \in \mathbb{R}\big\}$. It follows that if $\bar{x}$ is a solution of $(\mathcal{P})$, then $g(\bar{x}) = 0$.

2) Assumption (3.3) implies that if $\mathcal{M}$ denotes the set of solutions to $(\mathcal{P})$, then $\mathcal{M} \subset \mathcal{F}_l \cap \big\{x \in \mathbb{R}^n /\, g(x) = 0\big\}$. Note that a procedure is given in [18], to compute the point $\pi(x)$, where the line segment $[x_0, x]$, meets the boundary of the set $\big\{x \in \mathbb{R}^n /\, g(x) < 0\big\}$ (obviously the boundary is $\big\{x \in \mathbb{R}^n /\, g(x) = 0\big\}$). On the other hand, from the definition of $\mathcal{F}_l$, we have $\bigcup_{x \in \mathcal{F}_l} I(x) \subset \{l + 1, ..., p\}$.

3) Assumption (3.4) requires that any point of $\mathcal{F}_l$ must not be a minimizer of the function g on $\mathbb{R}^n$.

4) Assumption (3.5) is composed of two properties. The first property : $\forall x \in \mathcal{F}_l$, $0 \notin \bigcup_{i \in I(x)} \partial G_i(x)$, means that for any x in $\mathcal{F}_l$, any active constraint G_i at x, must not attain its minimum on $\mathbb{R}^n$ at x. The second property says that for any point x in $\mathcal{F}_l$, there exists a subgradient of the constraint g at x, which is a common descent direction of active constraints G_i at x. This property will be required for establishing stability and optimality results. Furthermore, it is implicitly supposed that for any $x \in \mathcal{F}_l$, we have

$$\bigcap_{i \in I(x)} \left\{d \in \mathbb{R}^n / \, G_i^{'}(x; d) < 0\right\} \neq \emptyset.$$

In order to give stability results under appropriate assumptions, we will first prove that any solution of $(\mathcal{P})$ is regular. Then, under additional assumptions, we will conclude by Proposition 2.2. Hence, it is natural to begin by the following result on the existence of solutions of $(\mathcal{P})$ by using the theorem of Weierstrass. Mention that all the functions f, g and G_i, $i \in \{1, ..., p\}$, are continuous throughout $\mathbb{R}^n$, as convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ (see for instance [5]).

Proposition 3.2. *Let assumption (3.1) hold. Then, the problem $(\mathcal{P})$ has at least one solution.*

Proposition 3.3. *Let assumptions (3.1)-(3.5) hold. Then, the problem $(\mathcal{P})$ is stable.*

Proof. Let $\hat{x}$ be any solution of $(\mathcal{P})$, and let us show that it is regular. First, mention that Proposition 3.2 implies that $\inf \mathcal{P} > -\infty$. On the other hand, according to 2) of Remark 3.1, we have $\hat{x} \in \mathcal{F}_l$, and $g(\hat{x}) = 0$. Let us consider the following cases.
(i) If $I(\hat{x}) = \emptyset$, i.e., $G(\hat{x}) < 0$, then let $x^* \in \partial g(\hat{x})$. From assumption (3.4), it follows that $x^* \neq 0$. Define the sequence $x_k = \hat{x} + \alpha_k x^*$, $k \in \mathbb{N}$, where $\alpha_k \searrow 0^+$. Whence, $x_k \to \hat{x}$, as $k \to +\infty$, and $G(x_k) < 0$, for large k. Since $x^* \in \partial g(\hat{x})$, then

$$g(x_k) \geq g(\hat{x}) + \alpha_k \|x^*\|^2 > g(\hat{x}) = 0, \quad \text{for all } k,$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$. That is $\hat{x}$ is regular.
(ii) If $I(\hat{x}) \neq \emptyset$, then let $d^* \in \bigcap_{i \in I(\hat{x})} \left\{d \in \mathbb{R}^n / \, G_i^{'}(\hat{x}; d) < 0\right\} \cap \partial g(\hat{x})$. Define a new sequence $u_k = \hat{x} + \alpha_k d^*$, $k \in \mathbb{N}$ (note that $d^* \neq 0$), where $\alpha_k \searrow 0^+$. Then, $u_k \to \hat{x}$, as $k \to +\infty$. As stated above, we have $g(u_k) > 0$, for all k. We distinguish the following subcases.
• For $i \in I(\hat{x})$, we have $G_i(\hat{x}) = 0$. Since

$$G_i^{'}(\hat{x}; d^*) = \lim_{k \to +\infty} \frac{G_i(u_k) - G_i(\hat{x})}{\alpha_k} < 0,$$

it follows that

$$\frac{G_i(u_k) - G_i(\hat{x})}{\alpha_k} < 0, \text{ for large } k.$$

Hence

$$G_i(u_k) < 0, \quad \text{for large } k.$$

• For $i \notin I(\hat{x})$, we have $G_i(\hat{x}) < 0$. Since $u_k \to \hat{x}$, as $k \to +\infty$, it follows that

$$G_i(u_k) < 0, \text{ for large } k.$$

Whence, from the two subcases we deduce that $G(u_k) < 0$, for large k, and hence $\hat{x}$ is regular. Finally, by using Proposition 2.2, we conclude that $(\mathcal{P})$ is stable. □

Let us consider the case where for any $x \in \mathcal{F}_l$, the functions g and G_i, $i \in I(x)$, are differentiable at x. Then, we shall make the following assumptions.

(3.6) $\forall x \in \mathcal{F}_l$, the functions g and G_i, $i \in I(x)$, are differentiable at x,

$$\textbf{(3.7)}\ \forall x \in \mathcal{F}_l, \text{we have} \begin{cases} 0 \notin \{\nabla G_i(x), i \in I(x)\}, \\ \\ \langle \nabla g(x), \nabla G_i(x) \rangle < 0, \forall\, i \in\ I(x), \end{cases}$$

where $\nabla g(x)$ and $\nabla G_i(x)$ denote the gradients of g and G_i at x, respectively.

Remark 3.4. As remarked above, assumptions (3.4)-(3.7) are related to assumption (3.3) by the integer l. This integer and the set $\mathcal{F}_l$, which are fixed by assumption (3.3) ($\mathcal{F}_l = \mathcal{D} \setminus \mathcal{D}_l$) will be used throughout the rest of the paper (see examples 4.9 and 4.10 in section 4 for illustration).

Then, we can replace assumption (3.5) by assumptions (3.6) and (3.7).

Proposition 3.5. *Let assumptions (3.1)-(3.4), (3.6) and (3.7) hold. Then, the problem* $(\mathcal{P})$ *is stable.*

Proof. Let $\hat{x}$ be a solution of $(\mathcal{P})$, and let us prove that $\hat{x}$ is regular. As previously noted in the proof of Proposition 3.3, we have $\inf \mathcal{P} > -\infty$. Let $x_k = \hat{x} + \alpha_k \nabla g(\hat{x})$, $k \in \mathbb{N}$ ($\nabla g(\hat{x}) \neq 0$, by assumption (3.4) since $\hat{x} \in \mathcal{F}_l$), with $\alpha_k \searrow 0^+$, which converges to $\hat{x}$, as $k \to +\infty$. Then, the proof is identical to the proof of Proposition 3.3, except the second case, which is replaced by the following.
If $I(\hat{x}) \neq \emptyset$, then let $i \in I(\hat{x})$. Since the function G_i is differentiable at $\hat{x}$, then, there exists a function $\beta(\hat{x}, .) : \mathbb{R}^n \to \mathbb{R}$, satisfying $\beta(\hat{x}, x - \hat{x}) \to 0$, as $x \to \hat{x}$, such that

$$G_i(x) = G_i(\hat{x}) + \langle \nabla G_i(\hat{x}), x - \hat{x} \rangle + \|x - \hat{x}\| \beta(\hat{x}, x - \hat{x}),\ \forall x \in \mathbb{R}^n.$$

Whence,

$$G_i(x_k) = G_i(\hat{x}) + \alpha_k \langle \nabla G_i(\hat{x}), \nabla g(\hat{x}) \rangle + \alpha_k \|\nabla g(\hat{x})\| \beta(\hat{x}, \alpha_k \nabla g(\hat{x})).$$

Since $\langle \nabla G_i(\hat{x}), \nabla g(\hat{x}) \rangle < 0$, it follows that

$$G_i(x_k) < G_i(\hat{x}) = 0, \quad \text{for large } k.$$

On the other hand, for $i \notin I(\hat{x})$, we have $G_i(x_k) < 0$, for large k. That is $G(x_k) < 0$, for large k. Moreover, since $\nabla g(\hat{x}) \neq 0$, then

$$g(x_k) \geq g(\hat{x}) + \alpha_k \|\nabla g(\hat{x})\|^2 > g(\hat{x}) = 0, \quad \text{for all } k.$$

Therefore, $\hat{x}$ is regular, and the result follows from Proposition 2.2. □

4 Conditions for Global Optimality

In this Section, we shall be interested in necessary and sufficient conditions for global optimality for $(\mathcal{P})$. Some conditions will be expressed in terms of ϵ-subdifferentials of g and ϵ-normal directions to a level set that will be specified. For this, let us recall the following definitions.

(i) For $\epsilon \geq 0$, and $x \in \mathbb{R}^n$, the ϵ-subdifferential of g at x, $\partial_\epsilon g(x)$, is defined by

$$\partial_\epsilon g(x) = \big\{x^* \in \mathbb{R}^n / \, g(y) \geq g(x) + \langle x^*, y - x\rangle - \epsilon, \, \forall \, y \in \mathbb{R}^n\big\}.$$

When $\epsilon = 0$, we get the subdifferential of g at x.

(ii) Let C be a nonempty closed convex subset of $\mathbb{R}^n$. Let $\epsilon \geq 0$, and $\bar{x} \in C$. The ϵ-normal directions to C at $\bar{x}$, $\mathcal{N}_\epsilon(C, \bar{x})$, is defined by

$$\mathcal{N}_\epsilon(C, \bar{x}) = \big\{x^* \in \mathbb{R}^n / \, \langle x^*, x - \bar{x}\rangle \leq \epsilon, \forall \, x \in C\big\}.$$

For $\epsilon = 0$, we obtain the normal cone to C at $\bar{x}$, which is denoted by $\mathcal{N}(C, \bar{x})$. Finally, for $\alpha \in \mathbb{R}$, we consider the level set

$$L_\alpha = \big\{x \in \mathcal{D} / \, f(x) \leq \alpha\big\}$$

of the function f that we will need to express our results.

4.1 Necessary Conditions for Global Optimality

For a feasible point x of $(\mathcal{P})$, we will give a necessary condition for global optimality expressed in terms of $\partial_\epsilon g(x)$ and $\mathcal{N}_\epsilon(L_{f(x)}, x)$. More precisely, we have the following

Proposition 4.1. *Let assumptions (3.1) and (3.2) hold. Suppose that $\hat{x}$ is a solution of $(\mathcal{P})$. Then, for every $\epsilon \geq 0$, we have*

$$\partial_\epsilon g(\hat{x}) \subset \mathcal{N}_\epsilon(L_{f(\hat{x})}, \hat{x}).$$

Proof. Let $\epsilon \geq 0$. Suppose that there exists $u^* \in \partial_\epsilon g(\hat{x})$, verifying $u^* \notin \mathcal{N}_\epsilon(L_{f(\hat{x})}, \hat{x})$. Then, there exists $\bar{x} \in L_{f(\hat{x})}$, such that $\langle u^*, \bar{x} - \hat{x}\rangle > \epsilon$. On the other hand, we have

$$g(\bar{x}) \geq g(\hat{x}) + \langle u^*, \bar{x} - \hat{x}\rangle - \epsilon > g(\hat{x}) = 0.$$

According to 1) of Remark 3.1, there exists $\pi(\bar{x}) \in \mathbb{R}^n$, such that

$$f(\pi(\bar{x})) < f(\bar{x}) \leq f(\hat{x}), \; G(\pi(\bar{x})) \leq 0, \quad \text{and} \quad g(\pi(\bar{x})) = 0,$$

which contradicts the optimality of $\hat{x}$ to the problem $(\mathcal{P})$. □

4.2 Sufficient Conditions for Global Optimality

In order to give sufficient conditions for global optimality, we begin by the following remark that will be used in the sequel.

Remark 4.2. Let assumptions (3.2) and (3.3) hold. According to 2) of Remark 3.1, the problem $(\mathcal{P})$ is equivalent to minimize f over the set $\mathcal{F}_l \cap \big\{x \in \mathbb{R}^n / \, g(x) = 0\big\}$.

Then, we have the following sufficient condition for global optimality.

Proposition 4.3. *Let assumptions (3.2)-(3.5) hold. Let $\hat{x}$ be a feasible point of $(\mathcal{P})$. Suppose that $g(\hat{x}) = 0$, and that the following inclusion holds for any $\epsilon > 0$,*

$$\partial_\epsilon g(\hat{x}) \subset \mathcal{N}_\epsilon(L_{f(\hat{x})}, \hat{x}).$$

Then, $\hat{x}$ is a solution of $(\mathcal{P})$.

Proof. Suppose that $\hat{x}$ is not a solution of $(\mathcal{P})$. Hence, according to Remark 4.2, there exists $\bar{x} \in \mathcal{F}_l$, such that $g(\bar{x}) = 0$, and $f(\bar{x}) < f(\hat{x})$. Then, we consider the following cases.
(i) If $G(\bar{x}) < 0$, then let $x^* \in \partial g(\bar{x})$, and define $x_k = \bar{x} + \alpha_k x^*$, $k \in \mathbb{N}$, with $\alpha_k \searrow 0^+$. Whence, $x_k \to \bar{x}$, as $k \to +\infty$,

$$g(x_k) > 0, \forall k, \quad \text{and} \quad G(x_k) < 0, \quad \text{for large } k.$$

Since we have $f(\bar{x}) < f(\hat{x})$, it follows that $f(x_k) < f(\hat{x})$, for large k. Hence, $x_k \in L_{f(\hat{x})}$, for large k. On the other hand, since $g(\hat{x}) = 0$, then, we have $g(x_k) - g(\hat{x}) > 0$, for large k. Furthermore, from [5,Theorem 1.3.6 (vol. 2)] (see also [6]), we have

$$g(x_k) - g(\hat{x}) = \sup_{\epsilon > 0, u^* \in \partial_\epsilon g(\hat{x})} \{\langle u^*, x_k - \hat{x}\rangle - \epsilon\} > 0.$$

Therefore, for every k, there exists $\epsilon_k > 0$, $u_k^* \in \partial_{\epsilon_k} g(\hat{x})$, such that $\langle u_k^*, x_k - \hat{x}\rangle > \epsilon_k$. Then, it follows that $u_k^* \notin \mathcal{N}_{\epsilon_k}(L_{f(\hat{x})}, \hat{x})$, which is a contradiction.
(ii) Otherwise, $I(\bar{x}) \neq \emptyset$. Let $d^* \in \bigcap_{i \in I(\bar{x})} \{d \in \mathbb{R}^n / \, G_i'(\bar{x}; d) < 0\} \cap \partial g(\bar{x})$. Define a new sequence $u_k = \bar{x} + \alpha_k d^*$, $k \in \mathbb{N}$, $\alpha_k \searrow 0^+$. Then, $u_k \to \bar{x}$, as $k \to +\infty$,

$$g(u_k) > 0, \forall k, \quad \text{and} \quad G_i(u_k) < 0, \quad \text{for large } k, i \in I(\bar{x}).$$

Since for $i \notin I(\bar{x})$, we have $G_i(\bar{x}) < 0$, then $G_i(u_k) < 0$, for large k. Hence,

$$G_i(u_k) < 0, \text{ for large } k, \forall i.$$

That is $G(u_k) < 0$, for large k. On the other hand, we have $f(\bar{x}) < f(\hat{x})$. It follows that $f(u_k) < f(\hat{x})$, for large k. Then, the sequence $(u_k)_k$ satisfies

$$u_k \to \bar{x}, \text{ as } k \to +\infty, f(u_k) < f(\hat{x}), G(u_k) < 0, g(u_k) > 0, \text{ for large } k.$$

Consequently, the result follows by applying the first case to the sequence (u_k) instead of (x_k). □

By virtue of Propositions 4.1 and 4.3, we have the following characterization of global solutions of $(\mathcal{P})$.

Corollary 4.4. *Let assumptions (3.1)-(3.5) hold. Let $\hat{x}$ be a feasible point of $(\mathcal{P})$. Then, $\hat{x}$ is a solution of $(\mathcal{P})$, if and only if*

$$\begin{cases} g(\hat{x}) = 0, \\ \partial_\epsilon g(\hat{x}) \subset \mathcal{N}_\epsilon(L_{f(\hat{x})}, \hat{x}), \forall \epsilon \geq 0. \end{cases}$$

Proof. Use Propositions 4.1 and 4.3. □

Remark 4.5. Optimality conditions using ϵ-subdifferentials and ϵ-normal directions were also given for convex maximization problems in [6-8]. On the other hand, in this paper, the main hypotheses that we have used to derive such optimality conditions are different from those existing in the literature concerning the study of optimality for reverse convex programs.

As a direct consequence of stability results established in Section 3, we derive the following necessary and sufficient optimality condition which reduces the problem $(\mathcal{P})$ to a convex maximization problem constrained by a compact convex set.

Corollary 4.6. *Suppose that assumptions of Proposition 3.3 or 3.5 are satisfied. Let $\hat{x}$ be a feasible point of $(\mathcal{P})$. Then, $\hat{x}$ is a solution of $(\mathcal{P})$ if and only if*

$$\max_{\substack{x\in\mathcal{D}\\ f(x)\le f(\hat{x})}} g(x) = 0.$$

Proof. Use Proposition 2.3. □

Before giving some examples where the main assumptions are satisfied, let us recall the following definition.

Definition 4.7. Let A be a nonempty convex cone of $\mathbb{R}^n$. The polar of A denoted by A°, is defined by $A^\circ = \big\{x^* \in \mathbb{R}^n / \, \forall x \in A, \langle x^*, x\rangle \le 0\big\}$.

Remark 4.8. Suppose that assumptions (3.3) and (3.5) are satisfied. Let $x \in \mathcal{F}_l$, and $i \in I(x)$. We have

$$\begin{aligned}\big\{d \in \mathbb{R}^n / \, G_i^{'}(x;d) \le 0\big\} &= \big\{d \in \mathbb{R}^n / \, \langle x_i^*, d\rangle \le 0, \forall x_i^* \in \partial G_i(x)\big\}\\ &= \big\{d \in \mathbb{R}^n / \, \langle \lambda x_i^*, d\rangle \le 0, \forall \lambda \ge 0, \forall x_i^* \in \partial G_i(x)\big\}\\ &= [\mathbb{R}^+ \partial G_i(x)]^\circ,\end{aligned}$$

with $\mathbb{R}^+ = \{\lambda \in \mathbb{R} / \, \lambda \ge 0\}$, and $\mathbb{R}^+\partial G_i(x) = \big\{\lambda x_i^* / \, \lambda \in \mathbb{R}^+, x_i^* \in \partial G_i(x)\big\}$. Then, the first condition of assumption (3.5) implies that

$$\big\{d \in \mathbb{R}^n / \, G_i^{'}(x;d) < 0\big\} = \operatorname{int}[\mathbb{R}^+ \partial G_i(x)]^\circ,$$

where $\operatorname{int}[\mathbb{R}^+ \partial G_i(x)]^\circ$ denotes the interior of $\big[\mathbb{R}^+ \partial G_i(x)\big]^\circ$.

Let the following example where assumptions (3.1)-(3.5) are satisfied.

Example 4.9. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$, and $G : \mathbb{R}^2 \to \mathbb{R}^4$, be the functions defined by

$$f(x) = \|x^T\| + 3(x_1 + x_2), \;\; g(x) = \|x^T\| + 2(x_1 + x_2) - 1,$$

$$G = (G_1, G_2, G_3, G_4),$$

$$G(x) = (x_1 + x_2 - 3, |x_1 + x_2 - 1| - 2(x_1 + x_2) - 1, -x_1, -x_2),$$

with $x = (x_1, x_2)$. Then, the constraint set $\mathcal{D}$ is compact. Let us verify assumptions (3.2)-(3.5). Let $\mathcal{D}_1 = \big\{x \in \mathcal{D} / \, G_1(x) = 0\big\}$. Then, it is easy to see that

$$\inf_{\substack{x\in\mathcal{D}_1\\ g(x)\ge 0}} f(x) = 9 + 3\sqrt{2}/2, \quad \inf_{x\in\mathcal{D}} f(x) = 0, \text{ and}$$

$$0 < \inf_{\substack{x\in\mathcal{D}\\ g(x)\ge 0}} f(x) < 9 + 3\sqrt{2}/2.$$

Hence, assumptions (3.2) and (3.3) are satisfied. Let $x \in \mathcal{F}_1$. Then, we have $I(x) \subset \{2, 3, 4\}$, and

$$\partial G_2(x) = \begin{cases} \{-\alpha(1,1)^T, \, 1 \le \alpha \le 3\} & \text{if } x_1 + x_2 = 1,\\ \{(-1,-1)^T\} & \text{if } x_1 + x_2 > 1,\\ \{(-3,-3)^T\} & \text{if } x_1 + x_2 < 1.\end{cases}$$

According to Remark 4.8, we have,

$$\{d \in \mathbb{R}^2 / \, G_2^{'}(x; d) < 0\} = \operatorname{int}[\mathbb{R}^+ \partial G_2(x)]^\circ =]0, +\infty[\times]0, +\infty[, \text{ and}$$

$$\{d \in \mathbb{R}^2 / \, G_2^{'}(x; d) \leq 0\} = \mathbb{R}^2_+.$$

We have

$$\nabla G_3(x) = (-1, 0)^T, \, \nabla G_4(x) = (0, -1)^T,$$

$$\partial g(x) = \begin{cases} B(0,1) + \{(2,2)^T\} & \text{if } x = 0, \\ \{x^T / \|x^T\| + (2,2)^T\} & \text{if } x \neq 0, \end{cases}$$

where $B(0,1)$ denotes the euclidean closed unit ball of $\mathbb{R}^2$. Then, $0 \notin \partial g(x)$,

$$\partial g(x) \subset \{d \in \mathbb{R}^2 / \, G_2^{'}(x; d) < 0\},$$

$$\forall x^* \in \partial g(x), \, \langle x^*, \nabla G_i(x) \rangle < 0, \, i = 3, 4, \text{ and}$$

$$0 \notin \bigcup_{i \in I(x)} \partial G_i(x) \subset \partial G_2(x) \cup \{\nabla G_3(x), \nabla G_4(x)\}.$$

Hence, assumptions (3.4) and (3.5) are satisfied.

Now, let us give an example in the differentiable case.

Example 4.10. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$, and $G : \mathbb{R}^2 \to \mathbb{R}^4$, be the functions defined by

$$f(x) = \|x^T\|^2 + 3(x_1 + x_2), \;\; g(x) = \|x^T\|^2/20 + 2(x_1 + x_2) - 1,$$

$$G = (G_1, G_2, G_3, G_4),$$

$$G(x) = (x_1 + x_2 - 3, x_1 - 2x_2 - 2, -x_1, -x_2),$$

with $x = (x_1, x_2)$. Then, the constraint set $\mathcal{D}$ is compact. Let us verify assumptions (3.2)-(3.4), (3.6) and (3.7). Let $\mathcal{D}_1 = \{x \in \mathcal{D} / \, G_1(x) = 0\}$. Then,

$$\inf_{\substack{x \in \mathcal{D}_1 \\ g(x) \geq 0}} f(x) = 27/2, \quad \inf_{x \in \mathcal{D}} f(x) = 0, \text{ and}$$

$$0 < \inf_{\substack{x \in \mathcal{D} \\ g(x) \geq 0}} f(x) < 27/2.$$

That is assumptions (3.2) and (3.3) are satisfied. Let $x = (x_1, x_2) \in \mathcal{F}_1$. We have

$$\nabla g(x) = x^T/10 + (2,2)^T \neq 0.$$

Then, assumption (3.4) is satisfied. On the other hand, we have $I(x) \subset \{2, 3, 4\}$,

$$\nabla G_2(x) = (1, -2)^T, \, \nabla G_3(x) = (-1, 0)^T, \, \nabla G_4(x) = (0, -1)^T,$$

$$0 \notin \bigcup_{i \in I(x)} \partial G_i(x) \subset \{\nabla G_2(x), \nabla G_3(x), \nabla G_4(x)\}.$$

Then, we verify that

$$\begin{cases} \langle \nabla g(x), \nabla G_2(x) \rangle = x_1/10 - x_2/5 - 2 < 0, \\ \\ \langle \nabla g(x), \nabla G_3(x) \rangle = x_1/10 - 2 < 0, \\ \\ \langle \nabla g(x), \nabla G_4(x) \rangle = x_2/10 - 2 < 0. \end{cases}$$

Hence, assumptions (3.6) and (3.7) are satisfied.

5 Conclusions

In this paper, we have considered a reverse convex program ($\mathcal{P}$), for which we have first given sufficient conditions ensuring stability. This gives the possibility to apply some theoretical and numerical results to ($\mathcal{P}$) given in the literature for stable reverse convex programs. The obtained stability results are based on the notion of regular solutions. Finally, we have given necessary and sufficient conditions for global optimality for ($\mathcal{P}$). In particular, by virtue of a necessary and sufficient optimality condition, the problem ($\mathcal{P}$) is reduced to a convex maximization problem constrained by a compact convex set.

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