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ENTROPY SOLUTIONS TO THE OBSTACLE PROBLEM FOR NONLINEAR ELLIPTIC PROBLEMS WITH VARIABLE EXPONENT AND L^1-DATA

Stanislas Ouaro* and Sado Traore

Abstract: We establish for L^1-data, existence, uniqueness and continuous dependence results to the obstacle problem for nonlinear elliptic equations with variable exponent, where the variable exponent is supposed only measurable. Other classical results about stability properties of the corresponding coincidence set and the Lewy-Stampacchia inequalities are presented.

Key words: *obstacle problem, variable exponent, entropy solutions, weak energy solutions, coincidence set, Lewy-Stampacchia inequalities*

Mathematics Subject Classification: *35J85, 35J70, 35B30, 35R35*

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1 Introduction

In this paper, we consider the obstacle problem with L^1-data associated with a nonlinear elliptic differential operator in divergence form of $p(.)$-Laplacian type

$$\mathcal{A}u := -div(a(x, \nabla u)),$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, $N > 1$.

The theory of variational inequalities to which obstacle problems belong has been widely studied in the classical context of data in $W^{-1,p'}(\Omega)$. Indeed, if we consider for example, the obstacle problem associated with a nonlinear elliptic differential operator $\mathcal{A}$ of monotone type, mapping $W_0^{1,p}(\Omega)$, $p > 1$, into its dual $W^{-1,p'}(\Omega)$, for any datum $f \in W^{-1,p'}(\Omega)$, the unilateral problem relative to $\mathcal{A}$, f and the obstacle ψ is the problem of funding a function u such that

$$u \in W_0^{1,p}(\Omega), u \geq \psi$$

$$\langle \mathcal{A}u, v - u\rangle \geq \langle f, v - u\rangle$$

$$\forall v \in W_0^{1,p}(\Omega), v \geq \psi.$$

In the case of quasilinear operators in divergence form of $p(.)$-Laplacian type that we consider in this paper, the classical obstacle problem can be formulated, using the duality pairing

*Corresponding author

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between $W_0^{1,p(.)}(\Omega)$ and $W^{-1,p'(.)}(\Omega)$, in terms of variational inequality

$$u \in \mathcal{K}_\psi : \int_\Omega a(x, \nabla u).\nabla(v-u)dx \geq \langle f, v-u \rangle, \ \forall v \in \mathcal{K}_\psi \tag{1.1}$$

whenever $f \in W^{-1,p'(.)}(\Omega)$ and the convex subset

$$\mathcal{K}_\psi = \left\{ v \in W_0^{1,p(.)}(\Omega) : v \geq \psi \ a.e. \text{ in } \Omega \right\} \tag{1.2}$$

is nonempty.

As in the case of constant exponent p, for $f \in L^1(\Omega)$ and $1 < p(.) < N$, both sides of inequality (1.1) may have no meaning, so we are led, following [2, 4], to extend the formulation of the unilateral problem by replacing $v-u$ by its truncation $T_t(v-u)$, for every level $t > 0$, where T_t is defined by

$$T_t(s) := max\{-t, min\{t, s\}\}, \ s \in \mathbb{R};$$

and then, we may use the notion of entropy solution introduced in [2] for L^1-data.

In [15], the authors have considered the same problem as in the paper under the following assumptions on the vector field $a(.,.)$ and on the exponent $p(.)$:

$$a(x,\xi).\xi \geq \alpha |\xi|^{p(x)} \tag{1.3}$$

a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where α is a positive constant;

$$|a(x,\xi)| \leq \gamma(j(x) + |\xi|^{p(x)-1}), \tag{1.4}$$

a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(.)}(\Omega)$ and $\gamma > 0$;

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0, \tag{1.5}$$

a.e. $x \in \Omega$, for every ξ, $\xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

The exponent $p(.) : \Omega \to \mathbb{R}$ is a measurable function such that

$$p(.) \in W^{1,\infty}(\Omega) \text{ and } 1 < p_- \leq p_+ < N, \tag{1.6}$$

where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$ and N the dimension of the domain.

The Lipschitz condition in (1.6) allowed them in particular to exploit some embedding theorems and also to perform some estimates needed since they can differentiate the exponent $p(.)$. For the assumptions (1.3)-(1.6), they allowed in particular Sanchon and Urbano in [18] to exploit the arguments in [8, Theorem 4.2] for the study of existence of entropy solution of the problem

$$\begin{cases} -div(a(x,\nabla u)) = f \text{ in } \Omega \\ \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

where $f \in L^1(\Omega)$; by using the classical approximation method. Note that the work in [16] is a direct consequence of [18]. But in [8], the authors studied

$$\begin{cases} -div(|\nabla u|^{p(x)-2} \nabla u) = f \text{ in } \Omega \\ \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.8}$$

where $f \in L^\infty(\Omega)$; and as one can see, problem (1.8) is a particular case of (1.7), and the use of the existence and uniqueness result in [8, Theorem 4.2] for the approximation method in the study of (1.7) seem not very clear. To avoid this lack of clarity in [18], we have made some additionnal assumptions on the vector field a in other to study the existence result of (1.7) (see [15] for more details) that we will present later. The interest of this work is that in general, the study of problem with variable exponent are made for a continuous variable exponent in order to exploit some continuous or compactly embedding results (as in the constant exponent case) but in this paper we show that results in [16] still holds for a variable exponent only measurable. It is well known that if $p(.)$ is not continuous, the generalized Sobolev space $W^{1,p(.)}(\Omega)$ is not embedded in $L^{p^*(.)}(\Omega)$, where $p^*(.) = \frac{Np(.)}{N-p(.)}$ is the variable Sobolev exponent (see [11, Ex. 3.2]). The novelty of this paper is on the assumption of the exponent $p(.)$ which is assumed only measurable.

The Lewy-Stampacchia inequalities was firstly proved by Lewy and Stampacchia [12] in the case of the obstacle problem for the Laplace operator. It was then extended by many authors to the case of linear and nonlinear elliptic operators and became a powerful tool for proving existence and regularity results, giving rise to numerous papers, some reference of which can be found e.g. in the book of Troianiello [19] and Rodrigues *et al* [16]. We show that the Lewy-Stampacchia inequalities still holds in the context of assumption on the exponent $p(.)$.

The paper is divided into four Sections. In Section 2, we introduce the assumptions and state the main results. In Section 3, we prove the existence and uniqueness of an entropy solution and its continuous dependence with respect to the data. Finally, in Section 4, we show that Lewy-Stampacchia inequalities and the stability of the coincidence sets to the context of entropy solutions and an exponent $p(.)$ only measurable holds true.

2 Assumptions, Preliminaries and Main Results

In this paper, we study the obstacle problem with less regularity on the variable exponent $p(.)$, more precisely, we assume that

$$\begin{cases} p(.) \text{ is a measurable function such that} \\ 1 < p_- \leq p_+ < +\infty. \end{cases} \tag{2.1}$$

For the vector fields $a(.,.)$, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that:

The following equality holds

$$A(x,0) = 0, \tag{2.2}$$

for almost every $x \in \Omega$.

There exists a positive constant C_1 such that

$$|a(x,\xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}) \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(.)}(\Omega)$, with $1/p(x) + 1/p'(x) = 1$.

The following inequality holds

$$(a(x,\xi) - a(x,\eta)) \,.\, (\xi - \eta) > 0 \tag{2.4}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$.

The following inequalities holds true

$$|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi) \tag{2.5}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

Those assumptions allow us to exploit minimax theory for the question of existence of weak energy solution of the problems (1.7) (cf. [15]) in the context of assumption (2.1). As examples of models with respect above assumptions for problem (1.7), we can give the following:

(i) Set $A(x,\xi) = (1/p(x))\,|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\,\xi$ where $p(x) \geq 2$. Then we get the $p(x)$-Laplace operator

$$div(|\nabla u|^{p(x)-2}\,\nabla u).$$

(ii) Set $A(x,\xi) = (1/p(x))\left[\left(1+|\xi|^2\right)^{p(x)/2} - 1\right]$, $a(x,\xi) = \left(1+|\xi|^2\right)^{(p(x)-2)/2}\xi$, where $p(x) \geq 2$. Then we obtain the generalized mean curvature operator

$$div\left(\left(1+|\nabla u|^2\right)^{(p(x)-2)/2}\nabla u\right).$$

As the exponent $p(.)$ appearing in (2.3) and (2.5) does not need to be constant but may depend on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponent.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\,dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} := inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)|u|_{p(.)}\,|v|_{p'(.)}\,, \tag{2.6}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Now, let

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}\,.$$

Next, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ under the norm

$$\|u\| := |\nabla u|_{p(.)}\,.$$

The space $\left(W_0^{1,p(.)}(\Omega), \|u\|\right)$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$.

We have the following result (cf. [9]):

Lemma 2.1. *If* $u_n, u \in L^{p(\cdot)}(\Omega)$ *and* $p_+ < +\infty$ *then the following relations holds true:*

(i) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p+}$;

(ii) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p-}$;

(iii) $|u|_{L^{p(\cdot)}(\Omega)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho(u) < 1$ (*respectively* $= 1; > 1$);

(iv) $|u_n|_{L^{p(\cdot)}(\Omega)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow \rho(u_n) \to 0$ (*respectively* $\to +\infty$);

(v) $\rho\left(u/\,|u|_{L^{p(\cdot)}(\Omega)}\right) = 1$.

The resulting notion of entropy solution for the obstacle problem is made precise in the following definition.

Definition 2.2. An entropy solution of the obstacle problem for $\{f, \psi\}$ is a measurable function u such that $u \geq \psi$ a.e. in Ω, and, for every $t > 0$, $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega a(x, \nabla u).\nabla T_t(\varphi - u)dx \geq \int_\Omega f T_t(\varphi - u)dx, \qquad (2.7)_{f,\psi}$$

for all $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$.

In this paper, concerning the right hand side of $(2.7)_{f,\psi}$ and the obstacle ψ, we make the following assumptions:

$$f \in L^1(\Omega),\ \ \psi \in W^{1,p(\cdot)}(\Omega),\ \text{ and } \psi^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega). \qquad (2.8)$$

In particular, the last assumption guarantees that $\mathcal{K}_\psi \cap L^\infty(\Omega) \neq \emptyset$.

Our first result concerns the existence and uniqueness of an entropy solution in the sens of definition 2.2, to the obstacle problem. We recall from [15, 18] that it is still possible, as in the case of a constant p (cf. [2]), to define the weak gradient of a measurable function u such that $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$, for all $t > 0$. In fact, there exists a unique measurable vector field $v : \Omega \to \mathbb{R}^N$ such that

$$v\chi_{\{|u|<t\}} = \nabla T_t(u) \text{ for a.e. } x \in \Omega, \text{ and for all } t > 0,$$

where χ_B denotes the characteristic function of a measurable set B. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u

Theorem 2.3. *Assume (2.1)-(2.5) and (2.8). Then there exists a unique entropy solution u to the obstacle problem* $(2.7)_{f,\psi}$.

Now, consider a sequence $\{f_n, \psi_n\}_n$ and the corresponding obstacle problems $(2.7)_{f_n,\psi_n}$. The next result states that, under adequate assumptions, the limit of an entropy solution u_n of $(2.7)_{f_n,\psi_n}$ is the solution of the limit obstacle problem $(2.7)_{f,\psi}$.

Theorem 2.4. *Let $\{f_n, \psi_n\}_n$ be a sequence in $L^1(\Omega) \times W^{1,p(.)}(\Omega)$. Assume (2.1)-(2.5), (2.8) and that $\psi_n^+ \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, for all n. Let u_n be the entropy solution of the obstacle problem $(2.7)_{f_n,\psi_n}$. If*

$$f_n \to f \text{ in } L^1(\Omega) \text{ and } \psi_n \to \psi \text{ in } W^{1,p(.)}(\Omega), \tag{2.9}$$

then

$$u_n \to u \text{ in measure},$$

where u is the unique entropy solution of the obstacle problem $(2.7)_{f,\psi}$.

We also show that the so-called Lewy-Stampacchia inequalities and their consequences as the stability of coincidence set, the L^1-contraction property for the obstacle problem holds true.

Theorem 2.5. *Assume (2.1)-(2.5), (2.8) and $\mathcal{A}\psi \in L^1(\Omega)$. Let u be the entropy solution of the obstacle problem $(2.7)_{f,\psi}$. Then $\mathcal{A}u \in L^1(\Omega)$ and the following Lewy-Stampacchia inequalities holds*

$$f \le \mathcal{A}u \le f + (\mathcal{A}\psi - f)^+, \text{ a.e. in } \Omega. \tag{2.10}$$

Using the Lewy-Stampacchia inequalities and showing that $\mathcal{A}u = f$, a.e. in $\{u > \psi\}$, we show that the entropy solution of $(2.7)_{f,\psi}$ satisfies an equation involving the coincidence set $\{u = \psi\}$.

Theorem 2.6. *Assume (2.1)-(2.5), (2.8) and $\mathcal{A}\psi \in L^1(\Omega)$. The entropy solution u of the obstacle problem $(2.7)_{f,\psi}$ satisfies the equation*

$$\mathcal{A}u - (\mathcal{A}\psi - f)\chi_{\{u=\psi\}} = f, \text{ a.e. in } \Omega. \tag{2.11}$$

Note that (2.10) and (2.11) imply, in particular,

$$(\mathcal{A}\psi - f)\chi_{\{u=\psi\}} = (\mathcal{A}\psi - f)^+\chi_{\{u=\psi\}}, \text{ a.e. in } \Omega.$$

The next result concerns the convergence of the coincidence set of a sequence of entropy solutions to the limit coincidence set.

Theorem 2.7. *Under the assumptions of Theorem 2.4, assume that*

$$\mathcal{A}\psi_n \to \mathcal{A}\psi \text{ in } L^1(\Omega) \text{ and } \mathcal{A}\psi \ne f, \text{ a.e. in } \Omega.$$

Then

$$\chi_{\{u_n=\psi_n\}} \to \chi_{\{u=\psi\}} \text{ in } L^q(\Omega), \tag{2.12}$$

for all $1 \le q < +\infty$.

Finally, we obtain an L^1-contraction property for the obstacle problem and an estimate for the stability of two coincidence sets I_1 and I_2 in terms of their symetric difference

$$I_1 \div I_2 := (I_1 \backslash I_2) \cup (I_2 \backslash I_1).$$

Theorem 2.8. *Assume (2.1)-(2.5), let f_1, $f_2 \in L^1(\Omega)$, ψ satisfy (2.8) and $\mathcal{A}\psi \in L^1(\Omega)$. Let u_1 and u_2 be the entropy solutions of the obstacle problems $(2.7)_{f_1,\psi}$ and $(2.7)_{f_2,\psi}$, respectively. If $\xi_i := f_i - \mathcal{A}u_i$, $i = 1, 2$, then*

$$\|\xi_1 - \xi_2\|_1 \le \|f_1 - f_2\|_1. \tag{2.13}$$

If, in addition, the non-degeneracy condition

$$f_i - \mathcal{A}\psi \leq -\lambda < 0, \;\; a.e. \;\; in \; D, \;\; i = 1, 2, \tag{2.14}$$

holds in a measurable subset $D \subset \Omega$, then, for $I_i := \{u_i = \psi\}$,

$$meas((I_1 \div I_2) \cap D) \leq \frac{1}{\lambda} \|f_1 - f_2\|_1 . \tag{2.15}$$

3 Existence, Uniqueness and Continuous Dependence of Entropy Solutions

We first give a priori estimates results in Lebesgue and Sobolev spaces with variable exponent for an entropy solution of the obstacle problem.

Lemma 3.1. *Assume (2.1)-(2.5), (2.8) and let $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$. If u is an entropy solution of the variational inequality $(2.7)_{f,\psi}$ then*

$$\int_{\{|u| \leq t\}} |\nabla u|^{p(x)} \, dx \leq C((t + \|\varphi\|_\infty) \|f\|_1 + \int_\Omega (|\nabla \varphi|^{p(x)} + j(x)^{p'(x)}) dx), \tag{3.1}$$

for all $t > 0$, where C is a constant depending only on C_1 and $p(.)$.

Proof. Take $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ in the variational inequality $(2.7)_{f,\psi}$ to obtain

$$\int_{\{|u-\varphi| \leq t\}} a(x, \nabla u).\nabla(u - \varphi) dx \leq \int_\Omega f T_t(u - \varphi) dx \leq t \|f\|_1 ,$$

for all $t > 0$.

On the other hand, by assumptions (2.3), (2.5) and Young's inequality, we have, for all $t > 0$,

$$\begin{cases} \displaystyle\int_{\{|u-\varphi| \leq t\}} a(x, \nabla u).\nabla(u - \varphi) dx \geq \int_{\{|u-\varphi| \leq t\}} |\nabla u|^{p(x)} \, dx - \\ \displaystyle C_1 \int_{\{|u-\varphi| \leq t\}} (j(x) + |\nabla u|^{p(x)-1}) \, |\nabla \varphi| \, dx \geq \frac{1}{p_+} \int_{\{|u-\varphi| \leq t\}} |\nabla u|^{p(x)} - \\ \displaystyle C \int_\Omega (|\nabla \varphi|^{p(x)} + j(x)^{p'(x)}) dx, \end{cases} \tag{3.2}$$

where C is a constant depending only on C_1 and $p(.)$.

Now, from (3.1) and (3.2), we obtain

$$\int_{\{|u-\varphi| \leq t\}} |\nabla u|^{p(x)} \, dx \leq p_+ t \|f\|_1 + C \int_\Omega (|\nabla \varphi|^{p(x)} + j(x)^{p'(x)}) dx,$$

for all $t > 0$. Replacing t with $t + \|\varphi\|_\infty$ in the last inequality, we get

$$\begin{cases} \displaystyle\int_{\{|u| \leq t\}} |\nabla u|^{p(x)} \, dx \leq \int_{\{|u-\varphi| \leq t + \|\varphi\|_\infty\}} |\nabla u|^{p(x)} \, dx \leq \\ \displaystyle C((t + \|\varphi\|_\infty) \|f\|_1 + \int_\Omega (|\nabla \varphi|^{p(x)} + j(x)^{p'(x)}) dx), \end{cases}$$

for all $t > 0$. $\square$

Lemma 3.2. *Assume (2.1)-(2.5), (2.8) and let* $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$. *If* u *is an entropy solution of the variational inequality* $(2.7)_{f,\psi}$ *then*

$$\int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}\, dx \leq M(1+t)$$

for every $t > 0$, *with* M *a positive constant. More precisely, there exists* $D > 0$ *such that*

$$meas\,\{|u| > t\} \leq D^{p_-}\frac{2+t}{t^{p_-}}.$$

Proof. By Lemma 3.1, we have

$$\begin{cases} \displaystyle\int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}\, dx = \int_{\{|u|\leq t\}} |\nabla u|^{p(x)}\, dx \leq \\ \displaystyle C((t+\|\varphi\|_\infty)\,\|f\|_1 + \int_\Omega (|\nabla\varphi|^{p(x)} + j(x)^{p'(x)})dx) \leq M(1+t), \end{cases}$$

since $f \in L^1(\Omega)$, $\varphi \in L^\infty(\Omega) \cap W_0^{1,p(.)}(\Omega)$, $j \in L^{p'(.)}(\Omega)$.

$$\int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}\, dx \leq M(1+t) \Rightarrow \int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p_-}\, dx \leq C(2+t).$$

By Poincaré inequality in constant exponent, we obtain

$$\|T_t(u)\|_{L^{p_-}(\Omega)} \leq D(2+t)^{\frac{1}{p_-}}.$$

The above inequality imply that

$$\int_\Omega |T_t(u)|^{p_-}\, dx \leq D^{p_-}(2+t),$$

from which we obtain

$$meas\,\{|u| > t\} \leq D^{p_-}\frac{2+t}{t^{p_-}}.$$

□

Lemma 3.3. *Assume (2.1)-(2.5), (2.8) and let* $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$. *If* u *is an entropy solution of the variational inequality* $(2.7)_{f,\psi}$ *and if there exists a positive constant* M *such that*

$$\int_{\{|u|>t\}} t^{q(x)} dx \leq M, \ \textit{for all } t > 0, \tag{3.3}$$

then there exists a constant C, *depending only on* C_1 *and* $p(.)$, *such that*

$$\int_{\left\{|\nabla u|^{\alpha(.)}>t\right\}} t^{q(x)} dx \leq C((1+\|\varphi\|_\infty)\,\|f\|_1 + \int_\Omega (|\nabla\varphi|^{p(x)} + j(x)^{p'(x)})dx) + M + meas(\Omega),$$

for all $t > 0$, *where* $\alpha(.) := p(.)/(q(.)+1)$.

Proof. Define $\psi := T_t(u)/t$. From Lemma 3.1, we have

$$\int_\Omega t^{p(x)-1} |\nabla\psi|^{p(x)}\, dx = \frac{1}{t}\int_\Omega |\nabla T_t(u)|^{p(x)}\, dx \le M_1 + \frac{M_2}{t},$$

for all $t > 0$, where $M_1 := C\|f\|_1$ and $M_2 := C(\|\varphi\|_\infty \|f\|_1 + \int_\Omega (|\nabla\varphi|^{p(x)} + j(x)^{p'(x)})dx)$, for a constant C depending only on C_1 and $p(.)$.

Using the above inequality, the definition of $\alpha(.)$ and (3.3), we obtain

$$\begin{cases} \displaystyle\int_{\{|\nabla u|^{\alpha(.)}>t\}} t^{q(x)}dx \le \int_{\{|\nabla u|^{\alpha(.)}>t\}\cap\{|u|\le t\}} t^{q(x)}dx + \int_{\{|u|>t\}} t^{q(x)}dx \le \\ \displaystyle\int_{\{|u|\le t\}} t^{q(x)}\left(\frac{|\nabla u|^{\alpha(x)}}{t}\right)^{\frac{p(x)}{\alpha(x)}} dx + M \le \frac{1}{t}\int_{\{|u|\le t\}} |\nabla T_t(u)|^{p(x)}\, dx + M \\ \displaystyle\le C((1+\|\varphi\|_\infty)\|f\|_1 + \int_\Omega (|\nabla\varphi|^{p(x)} + j(x)^{p'(x)})dx) + M \end{cases}$$

for all $t \ge 1$, where C is a constant depending only on C_1 and $p(.)$. Noting that

$$\int_{\{|\nabla u|^{\alpha(.)}>t\}} t^{q(x)}dx \le meas(\Omega), \text{ for all } t \le 1.$$

The Proof is then complete. □

In what follows, we prove the existence and uniqueness of an entropy solution to the obstacle problem $(2.7)_{f,\psi}$. We also prove the continuous dependence of the solution with respect to the right-hand side f and the obstacle ψ. We start by proving that a sequence $\{u_n\}_n$ of entropy solutions of the obstacle problems $(2.7)_{f_n,\psi_n}$ converges in measure to a measurable function u. We also show that the sequence of weak gradients $\{\nabla u_n\}_n$ converges in measure to ∇u, the weak gradient of u.

Proposition 3.4. *Let $\{f_n, \psi_n\}_n$ be a sequence in $L^1(\Omega)\times W^{1,p(.)}(\Omega)$. Assume (2.1)-(2.5), (2.8) and that $\psi_n^+ \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, for all n. Let u_n be an entropy solution of the obstacle problem $(2.7)_{f_n,\psi_n}$. If*

$$f_n \to f \text{ in } L^1(\Omega) \text{ and } \psi_n \to \psi \text{ in } W^{1,p(.)}(\Omega), \tag{3.4}$$

then the following assertions hold:

(i) *There exists a measurable function u such that $u_n \to u$ in measure.*

(ii) *∇u_n converges in measure to ∇u, the weak gradient of u.*

(iii) *$a(x,\nabla u_n)$ converges to $a(x,\nabla u)$, strongly in $(L^1(\Omega))^N$.*

(iv) *$a(x,\nabla u) \in (L^{p'(.)}(\Omega))^N$.*

Proof. Let $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$, e.g. $\varphi = \psi^+$, and note that $\varphi_n := \varphi + (\psi_n - \varphi)^+ \in L^\infty(\Omega)$ since $\varphi \in L^\infty(\Omega)$ and ψ_n is bounded above as $\psi_n^+ \in L^\infty(\Omega)$. In particular, $\varphi_n \in \mathcal{K}_{\psi_n}\cap L^\infty(\Omega)$. Moreover, by (3.4), there exists a constant C, independent of n, such that

$$\|f_n\|_1 \le C(\|f\|_1 + 1), \quad \|\varphi_n\|_\infty \le C(\|\varphi\|_\infty + 1) \tag{3.5}$$

and

$$\int_\Omega |\nabla \varphi_n|^{p(x)}\, dx \leq C\left(\int_\Omega |\nabla \varphi|^{p(x)}\, dx + 1\right), \tag{3.6}$$

for all n.
(i) Let s, t, and ϵ be positive numbers. Noting that

$$meas\,\{|u_n - u_m| > s\} \leq meas\,\{|u_n| > t\} + meas\,\{|u_m| > t\} + meas\,\{|T_t(u_n) - T_t(u_m)| > s\}, \tag{3.7}$$

then from Lemma 3.2 and (3.5)-(3.6), we can choose $t = t(\epsilon)$ such that

$$meas\,\{|u_n| > t\} < \frac{\epsilon}{3} \text{ and } meas\,\{|u_m| > t\} < \frac{\epsilon}{3}.$$

On the other hand, from Lemma 3.1 applied to u_n and (3.5)-(3.6), we obtain

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)}\, dx \leq C((t + \|\varphi\|_\infty + 1)(\|f\|_1 + 1) + \int_\Omega (|\nabla \varphi|^{p(x)} + j(x)^{p'(x)})dx + 1),$$

for all $t > 0$, where C is a constant depending only on C_1 and $p(.)$. Therefore, by Sobolev embedding in constant exponent, we can assume that $(T_t(u_n))_n$ is a Cauchy sequence in $L^{p_-}(\Omega)$. Consequently, there exists a measurable function u such that

$$T_t(u_n) \to T_t(u), \text{ in } L^{p_-}(\Omega) \text{ and } a.e.$$

Thus,

$$meas\,\{|T_t(u_n) - T_t(u_m)| > s\} \leq \int_\Omega \left(\frac{|T_t(u_n) - T_t(u_m)|}{s}\right)^{p_-} dx \leq \frac{\epsilon}{3},$$

for all $n, m \geq n_0(s, \epsilon)$.

Finally, from (3.7), we obtain

$$meas\,\{|u_n - u_m| > s\} < \epsilon, \text{ for all } n, m \geq n_0(s, \epsilon),$$

i.e $\{u_n\}_n$ is a Cauchy sequence in measure. The assertion follows.
(ii)-(iv) The proof of these parts is entirely similar to the corresponding ones in [14, Proposition 4.10]. □

Using Proposition 3.4, we can now prove Theorem 2.4.

Proof of Theorem 2.4. Let $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ and define $\varphi_n := \varphi + (\psi_n - \varphi)^+$. Note that $\varphi_n \in \mathcal{K}_{\psi_n} \cap L^\infty(\Omega)$ and that φ_n converges strongly to φ in $W_0^{1,p(.)}(\Omega)$, due to (2.9). Taking φ_n as a test function in $(2.7)_{f_n,\psi_n}$, we obtain

$$\int_\Omega a(x, \nabla u_n).\nabla T_t(u_n - \varphi_n)dx \leq \int_\Omega f_n T_t(u_n - \varphi_n)dx.$$

Next is to pass to the limit in the previous inequality. Concerning the right-hand side, the convergence is obvious since f_n converges to f, strongly in $L^1(\Omega)$, and $T_t(u_n - \varphi_n)$ converges to $T_t(u - \varphi)$, weakly-* in $L^\infty(\Omega)$ and a.e. in Ω.

For the left-hand side, we write it as

$$\int_{\{|u_n - \varphi_n| \leq t\}} a(x, \nabla u_n).\nabla u_n dx - \int_{\{|u_n - \varphi_n| \leq t\}} a(x, \nabla u_n).\nabla \varphi_n dx. \tag{3.8}$$

Note that $\{|u_n - \varphi_n| \leq t\}$ is a subset of $\{|u_n| \leq t + C(\|\varphi\|_\infty + 1)\}$ where C is a constant which does not depend on n due to (3.5). Hence, taking $s = t + C(\|\varphi\|_\infty + 1)$, we rewrite the second integral in (3.8) as

$$\int_{\{|u_n-\varphi_n|\leq t\}} a(x, \nabla T_s(u_n)).\nabla\varphi_n dx.$$

Since $a(x, \nabla T_s(u_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$ (by assumptions (2.3) and Lemma 3.1), it converges weakly to $a(x, \nabla T_s(u))$ in $(L^{p'(\cdot)}(\Omega))^N$, due to Proposition 3.4 (ii). Therefore the last integral converges to

$$\int_{\{|u-\varphi|\leq t\}} a(x, \nabla u).\nabla\varphi dx.$$

The first integral (3.8) is nonnegative by (2.5), and it converges a.e. by Proposition 3.4. It follows from Fatou's Lemma that

$$\int_{\{|u-\varphi|\leq t\}} a(x, \nabla u).\nabla u dx \leq \liminf_{n\to+\infty} \int_{\{|u_n-\varphi_n|\leq t\}} a(x, \nabla u_n).\nabla u_n dx.$$

Then gathering results, we obtain

$$\int_\Omega a(x, \nabla u).\nabla T_t(u - \varphi)dx \leq \int_\Omega f T_t(u - \varphi)dx,$$

b.e., u is an entropy solution of $(2.7)_{f,\psi}$. □

The proof of Theorem 2.3 is an application of Theorem 2.4.

Proof of Theorem 2.3. * **Existence.** Let (f_n) be a sequence of bounded functions, strongly converging to $f \in L^1(\Omega)$. As $f_n \in L^\infty(\Omega)$, we know by [15, Theorem 3.2](cf. [10, 13] for the link) that there exists a unique weak energy solution $u_n \in W_0^{1,p(.)}(\Omega)$ of the obstacle prroblem $(2.7)_{f_n,\psi}$. As a weak energy solution is also an entropy solution, we may apply Theorem 2.4 to obtain that u_n converges to a measurable function u which is an entropy solution of the limit obstacle problem $(2.7)_{f,\psi}$.

* **Uniqueness.** Let u and v be two entropy solutions of $(2.7)_{f,\psi}$. Since $\psi^+ \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $\psi \leq \|\psi^+\|_\infty$, then $T_h u$ and $T_h v$ belong to the convex set $\mathcal{K}_\psi$ for $h > 0$ large enough. We write the variational inequality $(2.7)_{f,\psi}$ corresponding to the solution u, with $T_h v$ as test function, and to the solution v, with $T_h u$ as test function. Upon addition we get

$$\begin{cases} \displaystyle\int_{\{|u-T_h v|\leq t\}} a(x, \nabla u).\nabla(u - T_h v)dx + \int_{\{|v-T_h u|\leq t\}} a(x, \nabla v).\nabla(v - T_h u)dx \\ \displaystyle\leq \int_\Omega f(T_t(u - T_h v) + T_t(v - T_h u))dx. \end{cases} \tag{3.9}$$

Define

$$E_1 := \{|u - v| \leq t, |v| \leq h\}, \ E_2 := E_1 \cap \{|u| \leq h\}, \ E_3 := E_1 \cap \{|u| > h\}.$$

We start with the first integral in (3.9). By (2.5), we have

$$\begin{cases} \int_{\{|u-T_hv|\leq t\}} a(x,\nabla u).\nabla(u-T_hv)dx = \\ \int_{\{|u-T_hv|\leq t\}\cap(\{|v|\leq h\}\cup\{|v|>h\})} a(x,\nabla u).\nabla(u-T_hv)dx = \\ \int_{\{|u-T_hv|\leq t,|v|\leq h\}} a(x,\nabla u).\nabla(u-T_hv)dx + \int_{\{|u-T_hv|\leq t,|v|>h\}} a(x,\nabla u).\nabla(u-T_hv)dx \\ = \int_{\{|u-v|\leq t,|v|\leq h\}} a(x,\nabla u).\nabla(u-v)dx + \int_{\{|u-h|\leq t,|v|>h\}} a(x,\nabla u).\nabla u dx \\ \geq \int_{\{|u-v|\leq t,|v|\leq h\}} a(x,\nabla u).\nabla(u-v)dx = \int_{E_1} a(x,\nabla u).\nabla(u-v)dx \\ = \int_{E_1\cap(\{|u|\leq h\}\cup\{|u|>h\})} a(x,\nabla u).\nabla(u-v)dx = \int_{E_2} a(x,\nabla u).\nabla(u-v)dx \\ + \int_{E_3} a(x,\nabla u).\nabla(u-v)dx = \int_{E_2} a(x,\nabla u).\nabla(u-v)dx + \\ \int_{E_3} a(x,\nabla u).\nabla u dx - \int_{E_3} a(x,\nabla u).\nabla v dx \geq \int_{E_2} a(x,\nabla u).\nabla(u-v)dx \\ - \int_{E_3} a(x,\nabla u).\nabla v dx. \end{cases} \tag{3.10}$$

Using (2.3) and (2.6), we estimate the last integral in (3.10) as follow

$$\begin{cases} \left|\int_{E_3} a(x,\nabla u).\nabla v dx\right| \leq C_1 \int_{E_3} \left(j(x) + |\nabla u|^{p(x)-1}\right)|\nabla v|\, dx \\ \leq C\left(|j|_{p'(.)} + \left||\nabla u|^{p(x)-1}\right|_{p'(.),\{h<|u|\leq h+t\}}\right)|\nabla v|_{p(.),\{h-t<|v|\leq h\}}. \end{cases} \tag{3.11}$$

The quantity $C\left(|j|_{p'(.)} + \left||\nabla u|^{p(x)-1}\right|_{p'(.),\{h<|u|\leq h+t\}}\right)$ is finite, since $u \in W_0^{1,p(.)}(\Omega)$ and $j \in L^{p'(.)}(\Omega)$; Then by Lemma 3.2, the last expression converges to zero as h tends to infinity. Therefore, from (3.10) and (3.11), we obtain

$$\int_{\{|u-T_hv|\leq t\}} a(x,\nabla u).\nabla(u-T_hv)dx \geq I_h + \int_{E_2} a(x,\nabla u).\nabla(u-v)dx, \tag{3.12}$$

where I_h converges to zero as h tends to infinity. We may adopt the same procedure to treat the second term in (3.9) and we obtain

$$\int_{\{|v-T_hu|\leq t\}} a(x,\nabla v).\nabla(v-T_hu)dx \geq J_h - \int_{E_2} a(x,\nabla v).\nabla(u-v)dx, \tag{3.13}$$

where J_h converges to zero as h tends to infinity.

Next, consider the right-hand side of inequality (3.9). Noting that

$$T_t(u - T_h v) + T_t(v - T_h u) = 0 \text{ in } \{|u| \leq h, |v| \leq h\}\,;$$

we obtain

$$\begin{aligned}
\left|\int_\Omega f(x)(T_t(u - T_h v) + T_t(v - T_h u))dx\right| &= \left|\int_{\{|u|>h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u))dx\right. \\
&\quad \left. + \int_{\{|u|\leq h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u))dx\right| \\
&= \left|\int_{\{|u|>h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u))dx\right. \\
&\quad \left. + \int_{\{|u|\leq h, |v|>h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u))dx\right| \\
&\leq 2t\left(\int_{\{|u|>h\}} |f|\,dx + \int_{\{|v|>h\}} |f|\,dx\right).
\end{aligned}$$

According to Lemma 3.2, both $meas\,\{|u| > h\}$ and $meas\,\{|v| > h\}$ tend to zero as h goes to infinity, then by the inequality above, the right-hand side of inequality (3.9) tends to zero as h goes to infinity. From this assertion, (3.9), (3.12) and (3.13), we obtain, letting $h \to +\infty$,

$$\int_{\{|u-v|\leq t\}} (a(x, \nabla u) - a(x, \nabla v)).\nabla(u - v)dx \leq 0, \text{ for all } t > 0.$$

By assertion (2.4), we conclude that $\nabla u = \nabla v$, a.e. in Ω.

Finally, from Poincaré inequality in constant exponent, we have

$$\int_\Omega |T_t(u - v)|^{p_-}\,dx \leq C\int_\Omega |\nabla(T_t(u - v))|^{p_-}\,dx = 0, \text{ for all } t > 0;$$

and hence $u = v$, a.e. in Ω. □

4 Lewy-Stampacchia Inequalities and Stability Results of the Coincidence Set

As $E := W_0^{1,p(.)}(\Omega)$ is a reflexive Banach space even under assumption (2.1) on $p(.)$, we show the Lewy-Stampacchia inequalities in the same way as in [16].

Proof of Theorem 2.5. Consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ such that f_n converges strongly in $L^1(\Omega)$ to f. Let $u_n \in W_0^{1,p(.)}(\Omega)$ be the unique weak energy solution of the obstacle problem

$$u_n \in \mathcal{K}_\psi : \langle \mathcal{A}u_n - f_n, v - u_n\rangle \geq 0,\ \forall v \in \mathcal{K}_\psi.$$

Since E is a reflexive Banach space and $\mathcal{A} : E \to E'$ ($E' := W^{-1,p'(.)}(\Omega)$) is strictly monotone, it follows from the abstract theory developed in [14] that

$$f_n \leq \mathcal{A}u_n \leq f_n + (\mathcal{A}\psi - f_n)^+ \text{ in } E'.$$

Note that, in particular, the above inequalities holds in the sense of distribution.

Let $0 \leq \varphi \in D(\Omega)$, then

$$\int_\Omega f_n \varphi dx \leq \int_\Omega a(x, \nabla u_n).\nabla \varphi dx \leq \int_\Omega \left[f_n + (\mathcal{A}\psi - f_n)^+ \right] \varphi dx.$$

As $a(x, \nabla u_n) \to a(x, \nabla u)$ in $\left(L^1(\Omega)\right)^N$ (cf. (iii)-Proposition 3.4) and $f_n \to f$ strongly in $L^1(\Omega)$, we obtain after passing to the limit in the inequalities above

$$f \leq \mathcal{A}u \leq f + (\mathcal{A}\psi - f)^+ \text{ in } D'(\Omega).$$

Since f and $f + (\mathcal{A}\psi - f)^+$ are in $L^1(\Omega)$, we conclude that $\mathcal{A}u \in L^1(\Omega)$. □

Remark 4.1. As the Lewy-Stampacchia inequalities holds true in the context of assumption (2.1), then the proof of Theorems 2.6, 2.7 and 2.8 can be found in [16].

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Stanislas Ouaro
Laboratoire d'Analyse Mathématique des Equations (LAME)
UFR. Sciences Exactes et Appliquées, Université de Ouagadougou
03 BP 7021 Ouaga 03
Ouagadougou, Burkina Faso
E-mail address: souaro@univ-ouaga.bf

Sado Traore
Laboratoire d'Analyse Mathématique des Equations (LAME)
Institut des Sciences Exactes et Appliquées, Université de Bobo Dioulasso
01 BP 1091 Bobo-Dioulasso 01
Bobo Dioulasso, Burkina Faso
E-mail address: sado@univ-ouaga.bf