Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 4, No. 3, pp. 641–649, September 2008

DISTANCE FROM A POINT TO A DOWNWARD SET IN A BANACH LATTICE

H. Mohebi and E. Naraghirad*

Abstract: Let X be a Banach lattice space with a strong unit $\mathbf{1}$. We obtain a characterization of an arbitrary downward set W in X as the level set of an increasing continuous along diagonal lines function. The main result of this paper indicates that the distance from a point $x \in X$ to a closed downward set W in X can be expressed as the upper envelope of a certain family of functions

$$\psi(x, y) = \sup\{\lambda \in \mathbb{R} : x - y \geq \lambda \mathbf{1}\}.$$

Key words: *best approximation, Banach lattice space, continuous along diagonal lines function, Downward set*

Mathematics Subject Classification: *46N10, 54D65*

---

1 Introduction

Methods of monotonic analysis (see, for example, [2]) have been used in [5] for the calculation of the distance with respect to the norm $\| \cdot \|_\infty$ to the solution set of the inequality

$$f(x) \leq c \qquad (x \in \mathbb{R}^n_+), \tag{1.1}$$

where f is a so-called increasing convex-along-rays (ICAR) function defined on the cone $\mathbb{R}^n_+$ of all n-vectors with nonnegative coordinates. In this paper, we develop a theory of monotonic analysis and best approximation in a Banach lattice space X with a strong unit $\mathbf{1}$. Recall (see [1]) that an element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 \leq \lambda \in \mathbb{R}$ such that $x \leq \lambda \mathbf{1}$. We use the following notations:
$X^+ := \{x \in X : x \geq 0\}$ and $X^{++} := \{x \in X : x > 0\}$.
For any function $f : X \longrightarrow \mathbb{R}$ the level sets of f are defined by

$$S_c(f) := \{x \in X : f(x) \leq c\}, \qquad (c \in \mathbb{R}).$$

If the function $f : X \longrightarrow \mathbb{R}$ is convex, then $S_0(f)$ is convex. For an increasing function $f : X^+ \longrightarrow \mathbb{R}$ the solution set of the inequality $f(x) \leq 0$ is normal. (The set $G \subseteq X^+$ is called normal if $x \in G,\ y \in X^+,\ y \leq x \Longrightarrow y \in G$.)

In section 3 of the present paper we consider the inequality

$$f(x) \leq 0 \qquad (x \in X), \tag{1.2}$$

*Corresponding author

with a continuous along diagonal lines function f defined on X, that is its restriction to each line of the form $\{x+\lambda\mathbf{1}\}_{x\in X}$ is continuous. We show that a downward set W in X can be expressed as the level set of this function. (The set $W \subseteq X$ is called downward if $x \in W,\ y \in X,\ y \leq x \Longrightarrow y \in W$.) We express the distance from a point $x \in X$ to a closed downward set W through the boundary points of W.
One of the well known problems of convex optimization is that of best approximation by elements of convex sets. Convexity can be also used for best approximation by complements of convex sets. Best approximation by different kinds of sets is a very complicated problem. The main result of this paper is to show that the distance from a point $x \in X$ to a closed downward set W in X can be expressed in terms of the function $\psi(x,y) = \sup\{\lambda \in \mathbb{R} : x-y \geq \lambda\mathbf{1}\}$. It is important to state clearly that the contribution of this paper in relation with the previous works (see for example [2,5]) is a technical characterization of closed downward sets in terms of the continuous along diagonal lines functions and some generalizations of results in [5] from $\mathbb{R}^n$ to a general Banach lattice with the strong unit $\mathbf{1}$ are presented.

2 Preliminaries

In the following we give definitions concerning vector lattices (see [1]).

Definition 2.1. A lattice $(L,\leq)$ is said to be conditionally complete if it satisfies one of the following equivalent conditions:
(1) Every non-empty lower bounded set admits an infimum.
(2) Every non-empty upper bounded set admits a supremum.
(3) There exists a complete lattice $\bar{L} := L \cup \{\perp, \top\}$, which we shall call the minimal completion of L, with bottom element $\perp$ and top element $\top$, such that L is a sublattice of $\bar{L}$, $\inf L = \perp$ and $\sup L = \top$.

Definition 2.2. A (real) vector lattice $(X,\leq,+,.)$ is a set X endowed with a partial order $\leq$ such that $(X,\leq)$ is a lattice with a binary operation $+$ and scalar product . such that $(X,+,.)$ is a vector space.

Definition 2.3. A vector lattice $(X,\leq,+,.)$ such that $(X,\leq)$ is a conditionally complete lattice is called a conditionally complete vector lattice.

Definition 2.4. A conditionally complete lattice Banach space X is a real Banach space which is also a conditionally complete vector lattice such that

$$\mid x \mid \leq \mid y \mid \Longrightarrow \parallel x \parallel \leq \parallel y \parallel \quad \forall x,y \in X.$$

where $\mid x \mid = x^+ + x^-,\ x^+ = \sup\{x,0\}$ *and* $x^- = -\inf\{x,0\}$.

Let X be a Banach lattice space. Recall (see [1]) that an element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 \leq \lambda \in \mathbb{R}$ such that $x \leq \lambda\mathbf{1}$. We assume that X contains a strong unit $\mathbf{1}$. Using $\mathbf{1}$ we can define a norm on X by

$$\parallel x \parallel = \inf\{\lambda > 0 : \mid x \mid \leq \lambda\mathbf{1}\} \quad \forall x \in X. \tag{2.1}$$

It is easy to check that

$$\mid x \mid \leq \parallel x \parallel \mathbf{1} \qquad \forall x \in X. \tag{2.2}$$

Remark 2.5. Let $x, y \in X$. Then there exist real numbers $\lambda > 0$ and $\theta < 0$ such that

$$y \leq x + \lambda \mathbf{1}, \qquad y + \theta \mathbf{1} \leq x. \tag{2.3}$$

To see this, let $\| y - x \| = r$. Then $0 \leq r < \infty$. From (2.2) it follows that

$$| y - x | \leq \| y - x \| \mathbf{1} = r\mathbf{1}.$$

This implies that $-r\mathbf{1} \leq y - x \leq r\mathbf{1}$. Let $\lambda = r$ and $\theta = -r$. Thus, we obtain (2.3).

Let Y be a Banach space. For any set K in X we shall denote by $intK$ the interior of K. The set K is called *solid* if $intK \neq \emptyset$. If $K \cap (-K) = \{0\}$, then K is called *pointed* (see for example [4]).

We now present two examples of spaces under consideration.

Example 2.6. Let $X = \mathbb{R} \times Y$, where Y is a Banach space with a norm $\| \, . \, \|$, and let $K \subset X$ be the epigraph of the norm: $K = \{(\lambda, x) : \lambda \geq \| x \|\}$. The cone K is closed solid convex and pointed. It is easy to check and well-known that $\mathbf{1} = (1, 0)$ is an interior point of K. For each $(c, y) \in X$ we have

$$p(c, y) = \inf\{\lambda \in \mathbb{R} : (c, y) \leq \lambda\mathbf{1}\} = \inf\{\lambda \in \mathbb{R} : (\lambda, 0) - (c, y) \in K\} =$$

$$\inf\{\lambda \in \mathbb{R} : (\lambda - c, -y) \in K\} = \inf\{\lambda \in \mathbb{R} : \lambda - c \geq \| -y \|\} = c + \| y \| .$$

Hence

$$\| (y, c) \| = \max\{p(c, y), p(-(c, y))\} = \max\{c + \| y \|, -c + \| y \|\} = | c | + \| y \| .$$

Remark 2.7 (3, Remark 6.1). Let X be a vector lattice space with a strong unit $\mathbf{1}$. We define a norm on X by

$$\| x \| = \inf\{\lambda > 0 : | x | \leq \lambda\mathbf{1}\},$$

where $|x| = x^+ + x^-$, $x^+ = \sup\{x, 0\}$ and $x^- = -\inf\{x, 0\}$. It is clear that X equipped with this norm is a Banach lattice space. Also, recall that the Banach lattice space X with a strong unit is isomorphic and isometric to the space $C(Q)$ of all continuous functions defined on a compact topological space Q. Indeed, if $X = C(Q)$ and p is the function defined by

$$p(x) = \inf\{\lambda > 0 : x \leq \lambda\mathbf{1}, \ \forall x \in X\},$$

then we have:

$$p(x) = \max_{q \in Q} x(q), \qquad p(-x) = -\min_{q \in Q} x(q),$$

and

$$\| x \| = \max\{\max_{q \in Q} x(q), -\min_{q \in Q} x(q)\} \quad (x \in X).$$

Example 2.8. A well-known example of a vector lattice with a strong unit is the space $L^\infty(T, \Sigma, \mu)$ of all essentially bounded functions defined on a measure space (S, Σ, μ). If $\mathbf{1}(t) = 1$ for all $t \in T$, then

$$p(x) := \inf\{\lambda > 0 : x \leq \lambda\mathbf{1}\} = ess \sup_{t \in T} x(t),$$

and

$$\| x \| := \max\{p(x), p(-x)\} = ess \sup_{t \in T} | x(t) | \qquad (x \in L^\infty(T, \Sigma, \mu)).$$

Definition 2.9. A non-empty subset G of the cone X^+ is called normal if $x \in G,\ y \in X^+,\ y \leq x \Longrightarrow y \in G$. A set $W \subseteq X$ is called downward if $x \in W,\ y \in X,\ y \leq x \Longrightarrow y \in W$.

Let X be a Banach lattice space. For a nonempty subset W of X and $x \in X$, define

$$d(x, W) = \inf_{w \in W} \parallel x - w \parallel .$$

Theorem 2.10 (4, Theorem 2.1). *Let X be a Banach lattice space with the strong unit $\mathbf{1}$. Let I be a set of indices, $\{W_i\}_{i \in I}$ be a family of closed downward sets in X and $W := \cap_{i \in I} W_i$. Then*

$$d(x, W) = \sup_{i \in I} d(x, W_i) \qquad (x \in X).$$

For a nonempty subset A of a conditionally complete lattice Banach space X and a nonempty bounded set S in X define

$$d(S, A) := \inf_{a \in A} \sup_{s \in S} \parallel s - a \parallel .$$

Theorem 2.11 (3, Theorem 3.4). *Let X be a conditionally complete lattice Banach space with the strong unit $\mathbf{1}$. Let I be a set of indices, $\{W_i\}_{i \in I}$ be a family of closed downward sets in X and suppose the set $W := \cap_{i \in I} W_i$ be nonempty. Let S be a bounded subset of X with $S \cap W = \emptyset$. Then*

$$d(S, W) = \sup_{i \in I} d(S, W_i).$$

$\boxed{3}$ Distance from a point to a downward set in a Banach lattice space

Throughout this section X is a Banach lattice space with a strong unit $\mathbf{1}$.
We say that a certain property (P) of a function $f : X^+ \longrightarrow \mathbb{R}$ holds along-rays if the restriction of f to the ray $R_x = \{\lambda x : \lambda \geq 0\}$ starting from zero and passing through x enjoys property (P) for each $x \in X^+$. In other words, (P) holds along-rays if the function of one variable f_x defined by

$$f_x(\alpha) = f(\alpha x) \qquad (\alpha \geq 0), \tag{3.1}$$

possesses the property (P). In particular, we consider the following class of functions:

(1) lower semicontinuous-along-rays f: the function f_x is lower semicontinuous for all $x \in X^+$;

(2) continuous-along-rays functions f: the function f_x is continuous for all $x \in X^+$;

(3) convex-along-rays functions f: the function f_x is convex for all $x \in X^+$;

(4) positively homogeneous functions. A function p is called positively homogeneous (of degree one) if $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$ (that is, p is linear-along-rays);

A function $f : X^+ \longrightarrow \mathbb{R}$ is called increasing if $x \leq y$ implies that $f(x) \leq f(y)$. Let Q be either X^+ or X^{++}. A function $f : Q \longrightarrow (-\infty, +\infty]$ is called an ICAR function if the following are satisfied:
(1) f is increasing: $x \leq y$ implies $f(x) \leq f(y)$;
(2) for each $x \in Q$, the function f_x defined by (3.1) is convex.

Example 3.1. An increasing convex function defined on X^+ is ICAR.

In this paper, we shall study increasing functions with a certain property-along-rays. First, we show that the lower semicontinuity-along-rays implies the lower semicontinuity for increasing functions. (Note that the class of increasing continuous-along-rays functions are broader than the class of increasing continuous functions.) It follows directly from the definition that level sets $S_c(f) = \{x \in X^+ : f(x) \le c\}$ of an increasing function f are normal for all $c \ge f(0)$. A normal set S is called closed-along-rays if

$$\lambda_n > 0, \quad \lambda_n s \in S \quad (n = 1, 2, 3, ...) \;\; and \;\; \lambda_n \longrightarrow \lambda \quad \Longrightarrow \lambda s \in S.$$

For more details see [2, Definition 5, p. 130].

It is known that a normal closed-along-rays set is closed.

Proposition 3.2. *Let f be an increasing lower semicontinuous-along-rays function. Then f is lower semicontinuous.*

Proof. We need to prove that all the level sets $S_c(f)$ of the function f with $c \ge f(0)$ are closed. Let $c \ge f(0)$. Let $\lambda_n > 0$ and $\lambda_n \longrightarrow \lambda > 0$, let also $\lambda_n x \in S_c(f)$, that is $f(\lambda_n x) \le c$. Since f is lower semicontinuous-along-rays, it follows that $f(\lambda x) \le \liminf_{n \longrightarrow \infty} f(\lambda_n x) \le c$, so $\lambda x \in S_c(f)$. Thus the normal set $S_c(f)$ is closed-along-rays, hence closed, which completes the proof. □

In this section, we show that any downward set W in X can be expressed as the level set of a continuous along diagonal lines function and we consider the distance from a point $x \in X$ to a closed downward set W in X. We need some properties of downward sets.

We also need the following simple assertions.

Proposition 3.3. *Let W be a downward set in X and $y \notin W$. Then*

$$\psi(x, y) := \sup\{\lambda \in \mathbb{R} : x - y \ge \lambda \mathbf{1}\} < 0 \qquad for\ all\ x \in W.$$

Proof. Assume, on the contrary, that there exists $x \in W$ such that $\psi(x, y) \ge 0$. Let $\Phi : \mathbb{R} \longrightarrow X$ be the function defined by

$$\Phi(\lambda) := \lambda \mathbf{1}, \qquad\qquad (\lambda \in \mathbb{R}).$$

Let $D := \{\lambda \in \mathbb{R} : x - y \ge \lambda \mathbf{1}\}$. It is easy to check that

$$\Phi(\sup D) = \sup\{\Phi(\lambda) : \lambda \in D\}.$$

This means that

$$x - y \ge \sup\{\lambda \mathbf{1} : \lambda \in D\} = \Phi(\sup D) = \Phi(\psi(x, y)) = \psi(x, y)\mathbf{1}.$$

Thus we get $x - y \ge \psi(x, y)\mathbf{1}$. This implies that $x \ge y + \psi(x, y)\mathbf{1} \ge y$. Since W is downward it follows that $y \in W$, which contradicts our assumption. □

The boundary and interior of a downward set W in X will be denoted by bdW and $intW$, respectively.

Proposition 3.4. *Let W be a downward set in X. A point $x \in W$ is a boundary point of W if and only if $x + \lambda \mathbf{1} \notin W$ for all $\lambda > 0$.*

Proof. If $x + \lambda\mathbf{1} \notin W$ for all $\lambda > 0$, then x is a boundary point. Assume on the contrary that $x \notin bdW$. So, $x \in intW$. Thus there exists $\epsilon > 0$ such that

$$V = \{y \in X : \| y - x \| < \epsilon\} \subset W.$$

Let $y_0 = x + \frac{\epsilon}{2}\mathbf{1}$. Then $y_0 \in V \subset W$, a contradiction.
Assume now that $x \in W$ is a point such that $x + \lambda\mathbf{1} \in W$ for some $\lambda > 0$. Since W is downward it follows that the ball $B(x, \lambda) = \{y \in X : x - \lambda\mathbf{1} \leq y \leq x + \lambda\mathbf{1}\}$ is contained in W, hence x is not a boundary point, which completes the proof. □

Definition 3.5. A function $f : X \longrightarrow \mathbb{R}$ is called continuous along diagonal lines if its restriction to each line of the form $\{x + \lambda\mathbf{1}\}_{x\in X}$ is continuous

Theorem 3.6. *A subset W of X is downward and closed if and only if there exists a continuous along diagonal lines increasing function $f : X \longrightarrow \mathbb{R}$ such that*

$$W = S_0(f) = \{x \in X : f(x) \leq 0\}.$$

Proof. Case 1. If $W = X$, then the function $f : X \longrightarrow \mathbb{R}$ defined by

$$f(x) = 0 \qquad (x \in X),$$

is a continuous along diagonal lines function and $W = S_0(f) = \{x \in X : f(x) = 0\}$.

Case 2. Let $W \subset X$ be proper. Let $f : X \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \inf\{\lambda \in \mathbb{R} : x - \lambda\mathbf{1} \in W\}, \qquad (x \in X). \tag{3.2}$$

It is obvious that $W \subseteq S_0(f)$. We now verify that $S_0(f) \subseteq W$. To see this, let $x \in S_0(f)$. Then $f(x) \leq 0$. This implies that $x \in W$. (Indeed, if $f(x) = 0$, then there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $x - \lambda_n\mathbf{1} \in W$ and $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$. Since W is closed, we conclude that $x \in W$. If $f(x) < 0$, then there exists a real number $\lambda < 0$ such that $x - \lambda\mathbf{1} \in W$. Since W is downward, it follows that $x \in W$.)
Now, we prove that $| f(x) | < \infty$ for all $x \in X$. Assume on the contrary that there exists $x \in X$ such that $| f(x) | = +\infty$. If $x \in W$, then $f(x) = -\infty$. This implies that $x - \lambda\mathbf{1} \in W$ for all $\lambda \leq 0$. Let $y \in X \setminus W$. Then, by Remark 2.5 there exists a real number $\lambda_0 \leq 0$ such that $y \leq x - \lambda_0\mathbf{1}$. Since W is downward, it follows that $y \in W$, which is a contradiction.
If $x \in X \setminus W$, then $f(x) = +\infty$. Thus we get

$$x - \lambda\mathbf{1} \notin W \qquad for\ all\ \lambda \in \mathbb{R}.$$

Assume now that $y \in W$. Then, by Remark 2.5 there exists a real number $\lambda_0 \in \mathbb{R}$ such that $x - \lambda_0\mathbf{1} \leq y$. Since W is downward, it follows that $x - \lambda_0\mathbf{1} \in W$, which is a contradiction.

Now, we show that

$$f(x + \beta\mathbf{1}) = (\beta - \alpha) + f(x + \alpha\mathbf{1}) \qquad for\ all\ \alpha, \beta \in \mathbb{R}. \tag{3.3}$$

To see this, let $\alpha, \beta \in \mathbb{R}$. By the definition of f we conclude that

$$f(x + \beta\mathbf{1}) = \inf\{\lambda \in \mathbb{R} : x + \beta\mathbf{1} - \lambda\mathbf{1} \in W\}$$

$$= (\beta - \alpha) + (\alpha - \beta) + \inf\{\lambda \in \mathbb{R} : x + \beta\mathbf{1} + \alpha\mathbf{1} - \alpha\mathbf{1} - \lambda\mathbf{1} \in W\}$$
$$= (\beta - \alpha) + \inf\{\alpha - \beta + \lambda : \lambda \in \mathbb{R}; x + \alpha\mathbf{1} - (\alpha - \beta + \lambda)\mathbf{1} \in W\}$$
$$= (\beta - \alpha) + \inf\{\theta \in \mathbb{R} : x + \alpha\mathbf{1} - \theta\mathbf{1} \in W\} = (\beta - \alpha) + f(x + \alpha\mathbf{1}).$$

Finally, we prove that f is a continuous along diagonal lines function. To this end, Let λ_n and λ be such that $\lambda_n \longrightarrow \lambda,\ (n \longrightarrow \infty)$. From (3.3) we conclude that

$$f(x + \lambda_n\mathbf{1}) = (\lambda_n - \lambda) + f(x + \lambda\mathbf{1}) \qquad \textit{for all } n \in \mathbb{N}.$$

This implies that

$$\lim_{n \longrightarrow \infty} f(x + \lambda_n\mathbf{1}) = f(x + \lambda\mathbf{1}),$$

which completes the proof. □

Let the function $\psi : X \times X \longrightarrow \mathbb{R}$ be defined by

$$\psi(x, y) := \sup\{\lambda \in \mathbb{R} : x - y \geq \lambda\mathbf{1}\}, \qquad (x,\ y \in X).$$

For $y \in X$ consider the set

$$W_y = \{x \in X : \psi(x, y) \leq 0\}. \tag{3.4}$$

For any $y \in X$, the set W_y defined by (3.4) is a closed downward set in X. An illustrative example of the set W_y is given in Figure 1. We now prove the following assertion.

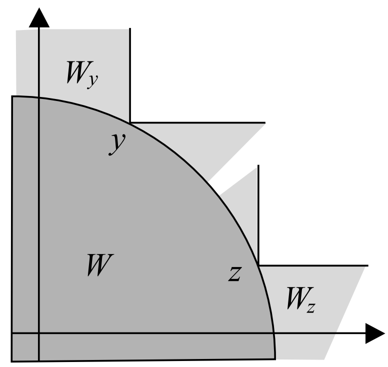

Figure 1: An example of the set W_y in $(R \times R)$ with the strong unit (1,1)

Lemma 3.7. *Let W be a nonempty closed downward set in X. Then*

$$W = \cap\{W_y : y \in bd\ W\}, \tag{3.5}$$

where W_y is defined by (3.4). (It is assumed that the intersection over the empty set coincides with X.)

Proof. First we assume that bdW is empty. Let $y \in X$. Then the set $B_y = \{\lambda \in \mathbb{R} : y + \lambda\mathbf{1} \in W\}$ is nonempty. Indeed, by Remark 2.5, for each $z \in W$ there exists $\lambda < 0$ such that $y + \lambda\mathbf{1} \leq z$. Since W is downward it follows that $y + \lambda\mathbf{1} \in W$. Clearly $(\lambda \in B_y,\ \lambda' \leq \lambda) \Longrightarrow \lambda' \in B_y$. Thus B_y is a segment. Let $b = \sup B_y$. If $b < +\infty$, then $y + b\mathbf{1} \in W$ (it follows from continuity along diagonal lines) and $y + b\mathbf{1} + \lambda\mathbf{1} \notin W$, for all $\lambda > 0$, so $y + b\mathbf{1} \in bdW$,

which contradicts our assumption. Thus $b = +\infty$. Since W is downward, it follows that $B_y = (-\infty, +\infty)$. We proved that W contains lines $\{y + \lambda\mathbf{1} : \lambda \in \mathbb{R}\}$ for all $y \in X$, so $W = X$. Thus (3.5) is valid in the case under consideration.
Now let us assume that bdW is nonempty and let $y \in bdW$. Then $y + \epsilon\mathbf{1} \notin W$ for all $\epsilon > 0$. Due to Proposition 3.3, we have $\psi(x, y + \epsilon\mathbf{1}) = \sup\{\lambda \in \mathbb{R} : x - y - \epsilon\mathbf{1} \geq \lambda\mathbf{1}\} < 0$ for all $x \in W$ and $\epsilon > 0$, so $\sup\{\lambda \in \mathbb{R} : x - y \geq \lambda\mathbf{1}\} \leq 0$ for all $x \in W$. Thus each $x \in W$ belongs to the set W_y with arbitrary $y \in bdW$.
Let $x \notin W$, that is, $f(x) > 0$, where f is defined by (3.2). Since W is downward, by Theorem 3.6, it follows that $f(x + \alpha\mathbf{1}) > 0$ for all $\alpha > 0$. Since W is nonempty and downward, by Theorem 3.6, it follows that there exists a number $\alpha_0 < 0$ such that $f(x + \alpha_0\mathbf{1}) \leq 0$. Since the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(\alpha) = f(x + \alpha\mathbf{1}) \qquad (\alpha \in \mathbb{R}),$$

is increasing and continuous, it follows that the set $M := \{\alpha \in \mathbb{R} : f(x + \alpha\mathbf{1}) = 0\}$ is nonempty and coincides either with a segment $[\nu, \mu]$ or with a segment $(-\infty, \mu]$ with $\mu < 0$. Let $y_0 = x + \mu\mathbf{1}$. It follows from the definition of μ that $y_0 \in bdW$. Indeed, for any $\lambda > 0$ we have $\mu + \lambda \notin M$. This implies that $f(x + \mu\mathbf{1} + \lambda\mathbf{1}) > 0$, that is $y_0 + \lambda\mathbf{1} \notin W$ for all $\lambda > 0$. We have $\psi(x, y_0) = \sup\{\lambda \in \mathbb{R} : x - y_0 \geq \lambda\mathbf{1}\} = -\mu > 0$, hence $x \notin W_{y_0}$ and $x \notin \cap\{W_y : y \in bdW\}$, which completes the proof. □

Theorem 3.8. *Let W be a nonempty closed downward set in X and $x \in X$. Then*

$$d(x, W) = \sup_{y \in bdW} d(x, W_y),$$

where W_y is defined by (3.4).

Proof. This is an immediate consequence of Theorem 2.10 and Lemma 3.7. □

Theorem 3.9. *Let X be a conditionally complete lattice Banach space with the strong unit $\mathbf{1}$. Let W be a nonempty closed downward set in X and S be a non-empty bounded subset of X with $S \cap W = \emptyset$. Then*

$$d(S, W) = \sup_{y \in bdW} d(S, W_y),$$

where W_y is defined by (3.4).

Proof. This is an immediate consequence of Theorem 2.11 and Lemma 3.7. □

References

[1] G. Birkhoff, Lattice theory, *Colloquium Publications*, American Mathematical Society, Vol. 25, 1967.

[2] J.E. Martinez-Legaz, A.M. Rubinov and I. Singer, Downward sets and their separation and approximation properties, *J. Global Optim.* 23 (2002) 111–137.

[3] H. Mohebi, Downward sets and their best simultaneous approximation properties with applications, *J. Numer. Func. Anal. Optim.* 25 (2004) 685–705.

[4] H. Mohebi and A.M. Rubinov, Best approximation by downward sets with applications, *J. Anal.Theory Appl.* 22 (2006) 1–22.

[5] A.M. Rubinov and I. Singer, Distance to the intersection of normal sets and applications, *J. Numer. Func. Anal. Optim.* 21 (2001) 521–536.

Manuscript received
revised
accepted for publication

H. Mohebi
Mahani Mathematical Research Center, University of Kerman, Kerman, Iran
E-mail address: hmohebi@mail.uk.ac.ir

E. Naraghirad
Mahani Mathematical Research Center, University of Kerman, Kerman, Iran
E-mail address: eskandarrad@gmail.com