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BI-PARAMETRIC SUPPORT SET SENSITIVITY ANALYSIS FOR PERTURBED LINEAR OPTIMIZATION

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Abstract: In bi-parametric linear optimization (LO), perturbation occurs in both the right hand side (RHS) and objective function coefficient (OFC) data that are different nonzero parameters. In this paper, the bi-parametric LO problem is considered, and we want to find the region of the parameters variation where the perturbed problem has still an optimal solution with the same positive variables for parameters values.This problem comes from managerial requirements. Managers want to find the region of the parameters that keeps the installed production lines active while only production levels would change. We are interested in identifying the regions where the support set is invariant for LO problem. These regions are referred to as invariancy regions. It is proved that invariancy regions are mesh-like area and are separated by vertical and horizontal lines. We present computable LO problems to identify the associated support set regions for LO problem.

Key words: *bi-parametric optimization, sensitivity analysis, linear optimization, interior point method, support set invariancy sensitivity analysis, invariancy region*

Mathematics Subject Classification: *90C05-90C31*

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1 Introduction

Let us consider the bi-parametric perturbed primal LO problem:

$$LP(\triangle b, \triangle c, \epsilon, \lambda) \qquad \begin{array}{ll} \min & (c+\lambda\Delta c)^T x \\ \text{s.t.} & Ax = b + \epsilon\Delta b \\ & x \geq 0, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, vectors $c, \triangle c \in \mathbb{R}^n$ and $b, \triangle b \in \mathbb{R}^m$ are fixed data, $x \in \mathbb{R}^n$ is an unknown vector and ϵ, λ are real parameters.

We refer to $\triangle b$ and $\triangle c$ as perturbation vectors. In special cases, one of the vectors $\triangle b$ and $\triangle c$ might be zero, or all but one of the components are zero. For parameter value $\epsilon = \lambda = 0$, problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ is an unperturbed primal LO problem and is denoted shortly by $LP = LP(\triangle b, \triangle c, 0, 0)$. *A standard* LO *problem* refers to the fact that the primal and dual LO problems are both in standard form.

The dual of $LP(\triangle b, \triangle c, \epsilon, \lambda)$ is defined as:

$$LD(\triangle b, \triangle c, \epsilon, \lambda) \qquad \begin{array}{ll} \max & (b+\epsilon\Delta b)^T y \\ \text{s.t.} & A^T y + s = c + \lambda\Delta c \\ & s \geq 0, \end{array}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ are unknown vectors. For the parameter values $\epsilon = 0$ and $\lambda = 0$ we denote it shortly by $LD = LD(\triangle b, \triangle c, 0, 0)$. Answering to the question "What happens to optimal solutions when such perturbation occurs in input data?" was one of the first preoccupations of optimizers soon after the simplex method was introduced. The related study area is known as *parametric programming* and *sensitivity analysis*. A classification of sensitivity analysis for LP was introduced by Koltai and Terlaky [8]. We discuss the support set sensitivity analysis for standard form LP problems containing two parameters, one in objective function and the other in the right hand side of the constraints. Any vector $x \geq 0$ satisfying the constraints of LP is called a *primal feasible solution* and any vector (y, s) with $s \geq 0$ satisfying the constraints of LD is called a *dual feasible solution*. We refer to the index set $\{1, 2, \dots, n\}$ as *variables index* set.

In this way, primal and dual feasible solutions can be denoted by x and (y, s), respectively. For any primal-dual feasible solution pair $(x; y, s)$, the *weak duality* property $b^T y \leq c^T x$ holds. If $b^T y = c^T x$ (*strong duality*), then the feasible solutions x and (y, s) are primal and dual optimal solutions of problems LP and LD, respectively. Consequently, for a primal-dual optimal solution pair $(x^*; y^*, s^*)$, we have $s^{*T} x^* = 0$. Considering the nonnegativity of variables x^* and s^*, the optimality property can be rewritten as ${s_j}^* {x_j}^* = 0$ for $j \in \{1, 2, \dots, n\}$. Clearly speaking, for a primal-dual optimal solution pair $(x^*; y^*, s^*)$, the vectors x^* and (y^*, s^*) are complementary .

The *support set* of a nonnegative vector ν is defined as $\sigma(\nu) = \{i | \nu_i > 0\}$. Considering this notation, the strong duality property implies the following equality:

$$\sigma(x^*) \cap \sigma(s^*) = \emptyset \tag{1.1}$$

where $(x^*; y^*, s^*)$ is a primal-dual optimal solution pair of problems LP and LD. A complementary (optimal) solution pair $(x^*; y^*, s^*)$ is *primal-dual strictly complementary*, if $s^{*T} x^* = 0$ with $s^* + x^* > 0$. Clearly speaking, for a strictly complementary optimal solution $(x^*; y^*, s^*)$, the following relation holds:

$$\sigma(x^*) \cup \sigma(s^*) = \{1, 2, \dots, n\}. \tag{1.2}$$

By the Goldman-Tucker Theorem [5], the existence of strictly complementary optimal solutions of problems LP and LD is guaranteed if these problems are feasible.

Let $\mathcal{LP}(\triangle b, \triangle c, \epsilon, \lambda)$ and $\mathcal{LD}(\triangle b, \triangle c, \epsilon, \lambda)$ be feasible sets of problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$, respectively. Further, let $\mathcal{LP}^*(\triangle b \ , \triangle c, \epsilon, \lambda)$ and $\mathcal{LD}^*(\triangle b, \triangle c, \epsilon, \lambda)$ denote their optimal solution sets, correspondingly. When $\epsilon = 0$ and $\lambda = 0$ we denote them shortly by $\mathcal{LP} = \mathcal{LP}(\triangle b, \triangle c, 0, 0)$ and $\mathcal{LD} = \mathcal{LD}(\triangle b, \triangle c, 0, 0)$. Analogously, we let $\mathcal{LP}^* = \mathcal{LP}^*(\triangle b, \triangle c, 0, 0)$ and $\mathcal{LD}^* = \mathcal{LD}^*(\triangle b \ , \triangle c, 0, 0)$, i.e.,

$\mathcal{LP}^* = \{x^* | x^* \text{ is an optimal solution in } \mathcal{LP}\}$

$\mathcal{LD}^* = \{(y^*, s^*) | (y^*, s^*) \text{ is an optimal solution in } \mathcal{LD}\}$.

Considering (1.1) and (1.2), one can define the following partition:

$$\begin{aligned} \mathcal{B}_V^x &= \{j : x_j^* > 0, \forall j \in \{1, 2, \dots, n\} \quad \text{for some} \quad x^* \in \mathcal{LP}^*\} \\ \mathcal{N}_V^s &= \{j : s_j^* > 0, \forall j \in \{1, 2, \dots, n\} \quad \text{for some} \quad (y^*, s^*) \in \mathcal{LD}^*\} D \end{aligned}$$

Roughly speaking, in any primal optimal solution x^*, the components with indices in $\mathcal{N}_V^s$ are always zero. We denote this partition by $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ and call it *variables optimal partition.*

The uniqueness of these partitions is a direct consequence of the convexity of optimal solution sets $\mathcal{LP}^*$ and $\mathcal{LD}^*$ (see e.g.,[3],[9]). In this paper, we survey the outlined results in support set invariancy sensitivity analysis for LP problems with two parameters.

Interior Point Methods solve LO problem in polynomial time [9]. They start from a feasible (or an infeasible) interior point of the positive orthant and generate an interior solution nearby the optimal solution. By using a simple rounding procedure [6], a strictly complementary solution of the LO problem can be obtained in strongly polynomial time and strictly complementary optimal solution of the LO problem provides the optimal partitions too.

Associated with the perturbed problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$, let ϕ denotes the *optimal value function* that is defined as:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda)=(c+\lambda\triangle c)^T x^*(\epsilon,\lambda) = (b+\epsilon\triangle b)^T y^*(\epsilon,\lambda)$$

where $(x^*(\epsilon,\lambda); y^*(\epsilon,\lambda), s^*(\epsilon,\lambda)$ is a primal-dual optimal solution pair of LP and LD problems. Further, we define:

$\phi(\triangle b, \triangle c, \epsilon, \lambda) = +\infty$ if $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon, \lambda) = \emptyset$

$\phi(\triangle b, \triangle c, \epsilon, \lambda) = -\infty$ if $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon, \lambda) = \emptyset$ and it is unbounded.

By fixing $\triangle b$ and $\triangle c$ that are nonzero vectors, ϕ is the bi-variate function of ϵ, λ. Remember that, perturbation occurs in the RHS and/or the OFC data. If perturbation in the RHS and the OFC data happens with identical parameters, the problem is referred to as *uni-parametric programming* problem and if these data vary independently, the problem is referred to as *bi- parametric programming* problem.

There are different approaches in parametric programming. One of them is the so-called *support set invariancy* sensitivity analysis. In this approach, one wants to identify the range of parameters' variation where the support set remains invariant. The first study with this point of view for optimal partition was started by Adler and Monteiro [1]. The cases when $\triangle b$ or $\triangle c$ is zero,have been studied in [9]. Further, in these cases, the range of parameter variation is an interval of the real line and is referred to as *invariancy interval* and the points that distinguish these intervals are called *transition points.* All of these studies are considered in uni-parameter. There is only a simple illustrative example in [7] that the authors have considered independently as two parameters and calculated the invariancy region.

In this paper, we consider the problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$, when $\triangle b$ and $\triangle c$ are nonzero vectors and ϵ and λ are not necessarily equal.

Definition 1.1 (Bi-parametric support set invariancy sensitivity analysis for LP problem). Consider the LP problem and let an optimal solution x^* with $\sigma(x^*) = P$ be given. In bi-parametric support set sensitivity analysis for LP, we want to find the region of parameters, where the perturbed problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ has an optimal solution $x^*(\epsilon,\lambda)$ with $\sigma(x^*(\epsilon,\lambda)) = P$.

By focusing on the support set of the reduced vector s, the following analogous definition can be considered.

Definition 1.2 (Bi-parametric active constraint set invariancy sensitivity analysis for LD problem). Consider the LD problem and let an optimal solution $(y^*\ , s^*)$ with $\sigma(s^*) = \overline{P}$ be given. In bi-parametric active constraint set sensitivity analysis for LD, we want to find the region of parameters, where the perturbed problem $LD(\triangle b, \triangle c, \epsilon, \lambda)$ has an optimal solution pair $(y^*(\epsilon,\lambda), s^*(\epsilon,\lambda))$ with $\sigma(s^*(\epsilon,\lambda)) = \overline{P}$.

One may be interested in combining the goals in Definitions 1.1 and 1.2. The following statement defines this combined goal clearly.

Definition 1.3 (Bi-parametric characteristic invariancy sensitivity analysis for problem LP). Consider the LP and LD problems and let an optimal solution pair $(x^*; y^*, s^*)$

with $\sigma(x^*) = P$ and $\sigma(s^*) = \overline{P}$ be given. In bi-parametric characteristic sensitivity analysis for LP and LD, we want to find the region of parameters, where the perturbed problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$ have an optimal solution pair $(x^*(\epsilon, \lambda); y^*(\epsilon, \lambda), s^*(\epsilon, \lambda))$ with $\sigma(x^*(\epsilon, \lambda)) = P$ and $\sigma(s^*(\epsilon, \lambda)) = \overline{P}$.

We refer to this region as *invariancy region.* It will be proved that the region is a rectangle (if it is not a singleton or a line segment) and the neighboring regions are rectangles as well. It means that all invariancy regions altogether generate a mesh-like area in $\mathbb{R}^2$, constructed by vertical and horizontal (half-) line segments.

Let us refer to the lines outlined here as transition lines and the region between obtained transition lines as *Optimal(Variables)Partitions Invariancy* (OPI) region. Thus, any transition line is a proper OPI region (a singleton or a line segment). The *actual* OPI region is the one which contains the actual parameter values $\epsilon = \lambda = 0$. It should be mentioned that it might be the singleton $\{(0, 0)\}$ when $\epsilon_l = \epsilon_u = 0$ and $\lambda_l = \lambda_u = 0$.

The paper is organized as follows: Section 2 contains some necessary concepts and the convexity of invariancy regions is proved. The simultaneous perturbation case, when variation occurs in both the Right Hand Side (RHS) and the Objective Function Coefficient (OFC) data of LP, is considered and the behavior of the optimal value function on this region is studied. Auxiliary LO problems are presented in this section that allows us to identify the associated regions. The interrelation of these regions are studied as well. A simple example is presented in Section 3 to illustrate the results.

2 Invariancy Regions

Let us introduce some simplifying concepts and notations. Having a primal-dual optimal solution pair $(x^*; y^*, s^*)$, with $\sigma(x^*) = P$ and $\sigma(s^*) = \overline{P}$, partition (P, Z) of variables' index set $\{1, 2, \ldots, n\}$ and partition $(\overline{P}, \overline{Z})$ of constraints' index set $\{1, 2, \ldots, n\}$ can be defined, where $Z = \{1, 2, \ldots, n\} \setminus P$ and $\overline{Z} = \{1, 2, \ldots, n\} \setminus \overline{P}$. We refer to the partition (P, Z) as *Invariant Support Set* (ISS) partition and to the partition $(\overline{P}, \overline{Z})$ as *Invariant Active Constraint Set* (IACS) partition of LP problem. If an LP problem has an optimal solution x^* with $\sigma(x*) = P$, we say that this problem satisfies the *Invariant Support Set* (ISS) property w.r.t the ISS partition (P, Z). On the other hand, if the optimal solution pair $(x^*; y^*, s^*)$, is so that the relation $\sigma(s^*) = \overline{P}$ holds, it is said that the LP problem satisfies the *Invariant Active Constraint Set* (IACS) property w.r.t the IACS partition $(\overline{P}, \overline{Z})$. Moreover, if this problem satisfies both ISS and IACS properties, we say that it has *Invariant Characteristic* (IC) property w.r.t partitions (P, Z) and $(\overline{P}, \overline{Z})$.

Recall that in support set invariancy sensitivity analysis for primal LP problem, one aims to find the range of parameter variation, where for any parameter value in this range, say $\overline{\epsilon}$ and $\overline{\lambda}$, there is an optimal solution pair $(x^*(\overline{\epsilon}, \overline{\lambda}); y^*(\overline{\epsilon}, \overline{\lambda}), s^*(\overline{\epsilon}, \overline{\lambda})) \in \mathcal{LP}^*(\triangle b, \triangle c, \overline{\epsilon}, \overline{\lambda})$ so that $\sigma(x^*(\overline{\epsilon}, \overline{\lambda})) = \sigma(x^*) = P$. From managerial point of view, if the LO problem has been formulated for determining an optimal production plan, support set invariancy sensitivity analysis means that for any (ϵ, λ) in the associated region, *the manager neither installs new production plans nor uninstall any existent production line, but the production levels may need to be adjusted* [3]. Support set expansion as the generalized form of this concept has been used for general linear optimization with uni-parameter [11].

Recall the perturbed primal and dual problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c\ , \epsilon, \lambda)$. Let $\Upsilon(\triangle b, \triangle c, \epsilon, \lambda)$ denotes the set of values for which the primal perturbed problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ satisfies the ISS property w.r.t the ISS partition (P, Z) associated with Definition 1.1. By focusing on the Definitions 1.2 and 1.3 we can consider an analogous definition and refer to them as $\varGamma(\triangle b, \triangle c, \epsilon, \lambda)$ and $\varTheta(\triangle b, \triangle c, \epsilon, \lambda)$, respectively. Analogous notations are used when either $\triangle b$ or $\triangle c$ is a zero vector. It will be proved that these sets are regions of the real plane. Further, they may be different from the regions which are obtained by basis invariancy and optimal partition invariancy sensitivity analysis as they might be different from their counterpart regions studied in linear optimization [10].

Recall that for $\triangle c = 0$, problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$ are reduced to the following problems, respectively:

$$LP(\triangle b, \epsilon) \qquad \min\{c^T x \mid Ax = b + \epsilon\triangle b, x \geq 0\}$$

$$LD(\triangle b, \epsilon) \qquad \max\{(b + \epsilon\triangle b)^T y \mid A^T y + s = c, s \geq 0\}.$$

Let $\varUpsilon(\triangle b, \epsilon)$ denotes the invariancy region for problem $LP(\triangle b\ , \epsilon)$. It was proved that the dual optimal solution set $\mathcal{LD}^*(\triangle b, \epsilon)$ is invariant on the invariancy region $\varUpsilon(\triangle b, \epsilon)$ (see Theorem IV.56 in [9]).The following lemma presents auxiliary LO problems to identify the end points of this region.

Lemma 2.1. *Consider the primal and dual problems LP and LD, respectively. Further, let $\varPi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variable optimal partition of problem LP and LD, respectively. Then, the optimal partition $\varPi_V$ is invariant for any $\epsilon \in (\epsilon_l^{LP}, \epsilon_u^{LP})$, where ϵ_l^{LP} and ϵ_u^{LP} are obtained by minimizing and maximizing of ϵ on the following set:*

$$\{\epsilon \mid Ax - \epsilon\triangle b = b, x_{\mathcal{B}} \geq 0, x_{\mathcal{N}} = 0\} \tag{2.1}$$

Proof. The proof is similar to Theorem IV.73 in [9] □

Now for $\triangle b = 0$, we have the following reduced primal and dual LO problems:

$$LP(\triangle c, \lambda) \qquad \min\{(c + \lambda\triangle c)^T x \mid Ax = b, x \geq 0\}.$$

$$LD(\triangle c, \lambda) \qquad \max\{b^T y \mid A^T y + s = c + \lambda\triangle c, s \geq 0\}.$$

Let $\varUpsilon(\triangle c, \lambda)$ denote the invariancy region for problem $LP(\triangle c, \lambda)$. It was proved that the dual optimal solution set $\mathcal{LP}^*(\triangle c, \lambda)$ is invariant on the invariancy region $\varUpsilon(\triangle c, \lambda)$ (see Theorem IV.60 in [9]).The following lemma presents auxiliary LO problems to identify the end points of this region.

Lemma 2.2. *Consider the primal and dual problems LP and LD, respectively. Further, let $\varPi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variable optimal partitions of problem LP and LD, respectively. Then, the optimal partition $\varPi_V$ is invariant for any $\lambda \in (\lambda_l^{LP}, \lambda_u^{LP})$, where λ_l^{LP} and λ_u^{LP} are obtained by minimizing and maximizing of λ on the following set:*

$$\{\lambda \mid A^T y + s - \lambda\triangle c = c, s_{\mathcal{N}} \geq 0, s_{\mathcal{B}} = 0\} \tag{2.2}$$

Proof. The proof is similar to Theorem IV.75 in [9] □

Remark 2.3. *Observe that the actual (region which contains the origin) invariancy interval $\varUpsilon(\triangle b, \epsilon)$ might be the singleton$\{0\}$. This situation occurs when solving two auxiliary LO problems (2.1 leads to $\epsilon_l = \epsilon_u = 0$. Moreover, if one of these problems is unbounded, then the actual invariancy interval $\varUpsilon(\triangle b, \epsilon)$ is unbounded too. Analogous argument is valid for the actual invariancy interval $\varUpsilon(\triangle c, \lambda)$. Furthermore, all auxiliary LO problems (2.1) and (2.2) can be solved in polynomial time.*

The case $\epsilon = \lambda$ has been considered in [2]. The authors proved that the invariancy interval in this case is the intersection of two invariancy intervals $\varUpsilon(\triangle b, \epsilon)$ and $\varUpsilon(\triangle c, \lambda =$

ϵ). Moreover, they proved that the optimal value function is a continuous and piecewise quadratic function when $\epsilon = \lambda$. In all these cases, the range of the optimal value function is a convex set. Also they are considered bi-parametric optimal partition invariancy sensitivity analysis for linear optimization with different nonzero parameters [10]. Also the authors surveyed tetra-parametric optimal partition invariancy sensitivity analysis for general linear optimization with different nonzero parameters [12]. And they proved that the optimal value function is a quadratic function.

2.1 Fundamental Properties

First, we study some fundamental properties of the ISS, IACS and IC sets. It is obvious that these sets are not empty, because $LP(\triangle b, \triangle c, 0, 0) = LP$ that satisfies the ISS and IACS properties, and consequently, it satisfies the IC property. Let us refer to the convexity property of these sets.

Lemma 2.4. (i) *Let the problem LP satisfies the ISS property w.r.t the ISS partition* (P, Z). *Then* $\Upsilon(\triangle b, \triangle c, \epsilon, \lambda)$ *is a convex set.*
(ii) *Let the problem LP satisfies the IACS property w.r.t the IACS partition* $(\overline{P}, \overline{Z})$. *Then* $\Gamma(\triangle b, \triangle c, \epsilon, \lambda)$ *is a convex set.*
(iii) *Let the problem LP satisfies the ICE property w.r.t the ISS partition* (P, Z) *and the IACS partition* $(\overline{P}, \overline{Z})$. *Then* $\Theta(\triangle b, \triangle c, \epsilon, \lambda)$ *is a convex set.*

Proof. It is similar to the Lemma 2.4 in [10]. □

To identify an invariancy region according to Lemma 2.4, it is enough to identify its border. Observe that the invariancy region might be unbounded.

2.2 Identifying the Invariancy Regions

Now, we present a fundamental theorem that talks about a relationship between the actual invariancy region $\Upsilon(\triangle b, \triangle c, \epsilon, \lambda)$ and two actual invariancy regions $\Upsilon(\triangle b, \epsilon)$ and $\Upsilon(\triangle c, \lambda)$. This relationship plays a significant role in identifying the actual invariancy region $\Upsilon(\triangle b, \triangle c, \epsilon, \lambda)$ and speaks of the fact that this identification can be done in polynomial time. To identify all possible invariancy regions, we can use an analogous statement. The proof is a direct generalization of the proof of Theorem 2.5 in [10] and it is omitted.

Theorem 2.5. *Consider the bi-parametric LO problem* $LP(\triangle b, \triangle c, \epsilon, \lambda)$. *Let* $\Upsilon(\triangle b, \epsilon)$ *be the invariancy interval of problems* $LP(\triangle b, \epsilon)$ *and* $LD(\triangle b, \epsilon)$. *Moreover, let* $\Upsilon(\triangle c, \lambda)$ *be the actual invariancy interval of problems* $LP(\triangle c, \lambda)$ *and* $LD(\triangle c, \lambda)$. *Then,*

$$\Upsilon(\triangle b, \triangle c, \epsilon, \lambda) = \Upsilon(\triangle b, \epsilon) \times \Upsilon(\triangle c, \lambda).$$

Corollary 2.6. *Consider the bi-parametric LO problem* $LP(\triangle b, \triangle c, \epsilon, \lambda)$. *Let* $\Upsilon(\triangle b, 0)$ *be the actual invariancy interval of problems* $LP(\triangle b, 0)$ *and* $LD(\triangle b, 0)$. *Moreover, let* $\Upsilon(\triangle c, 0)$ *be the actual invariancy interval of problems* $LP(\triangle c, 0)$ *and* $LD(\triangle c, 0)$. *Then,*

$$\Upsilon(\triangle b, \triangle c, 0, 0) = \Upsilon(\triangle b, 0) \times \Upsilon(\triangle c, 0).$$

2.3 Optimal Value Function on an Invariancy Region

In this subsection, we investigate the behavior of the optimal value function on invariancy regions.

Theorem 2.7. *The optimal value function $\phi(\triangle b, \triangle c, \epsilon, \lambda)$ is a bivariate quadratic function on actual invariancy region $\Upsilon(\triangle b, \triangle c, \epsilon, \lambda)$.*

Proof. When the actual invariancy region is the $\{(0,0)\}$, there is nothing to prove. Let the actual invariancy region be a nontrivial one containing the origin. Further, let (ϵ_1, λ_1), (ϵ_2, λ_2) and (ϵ_3, λ_3) be three arbitrary points in the actual invariancy region . Let $(x_1; y_1, s_1)$, $(x_2; y_2, s_2)$ and $(x_3; y_3, s_3)$ be primal-dual optimal solutions for these three points, respectively. Let (ϵ, λ) be a point in the interior of the triangle made of these three points as vertices. Therefore, there are $\theta_1, \theta_2 \in (0,1)$ with $0 < \theta_1 + \theta_2 < 1$ such that

$$\epsilon = \epsilon_3 - \theta_1(\Delta\epsilon_1 + \Delta\epsilon_2) - \theta_2\Delta\epsilon_2 \tag{2.3}$$

$$\lambda = \lambda_3 - \theta_1(\Delta\lambda_1 + \Delta\lambda_2) - \theta_2\Delta\lambda_2 \tag{2.4}$$

where $\Delta\epsilon_1 = \epsilon_2 - \epsilon_1$, $\Delta\epsilon_2 = \epsilon_3 - \epsilon_2$, $\Delta\lambda_1 = \lambda_2 - \lambda_1$ and $\Delta\lambda_2 = \lambda_3 - \lambda_2$. Let us define:

$$x^*(\epsilon, \lambda) = x_3 - \theta_1(\Delta x_1 + \Delta x_2) - \theta_2\Delta x_2 \tag{2.5}$$

$$y^*(\epsilon, \lambda) = y_3 - \theta_1(\Delta y_1 + \Delta y_2) - \theta_2\Delta y_2 \tag{2.6}$$

$$s^*(\epsilon, \lambda) = s_3 - \theta_1(\Delta s_1 + \Delta s_2) - \theta_2\Delta s_2 \tag{2.7}$$

where $\Delta x_j = x_{j+1} - x_j, \Delta y_j = y_{j+1} - y_j$, and $\Delta s_j = s_{j+1} - s_j$, with $j = 1, 2$. It is easy to verify that $(x^*(\epsilon), y^*(\epsilon), s^*(\epsilon))$ is a primal-dual optimal solution of problems $LP(\triangle b, \triangle c, \epsilon, \lambda)$ and $LD(\triangle b, \triangle c, \epsilon, \lambda)$. Replacing (2.3) and (2.4) and (2.6) in

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = (b + \epsilon\triangle b)^T y^*(\epsilon, \lambda)$$

implies:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = a_0 - a_1\theta_1 - a_2\theta_2 - a_3\theta_1\theta_2 - a_4\theta_1^2 - a_5\theta_2^2 \tag{2.8}$$

where

$$\begin{aligned}
a_0 &= (b + \epsilon_3\Delta b)^T y_3 \\
a_1 &= (b + \epsilon_3\Delta b)^T(\Delta y_1 + \Delta y_2) + (\Delta\epsilon_1 + \Delta\epsilon_2)\Delta b^T y_3 \\
a_2 &= (b + \epsilon_3\Delta b)^T\Delta y_2 + (\Delta\epsilon_2\Delta b)^T y_3 \\
a_3 &= (\Delta\epsilon_1 + \Delta\epsilon_2)\Delta b^T\Delta y_2 + \Delta\epsilon_2\Delta b^T(\Delta y_1 + \Delta y_2) \\
a_4 &= (\Delta\epsilon_1 + \Delta\epsilon_2)\Delta b^T(\Delta y_1 + \Delta y_2) \\
a_5 &= \Delta\epsilon_2\Delta b^T\Delta y_2 + \Delta\epsilon_2\Delta b^T\Delta y_2
\end{aligned} \tag{2.9}$$

On the other hand, solving equations (2.3) and (2.4) for θ_1 and θ_2 gives:

$$\theta_1 = \alpha_1 + \beta_1\epsilon + \gamma_1\lambda \tag{2.10}$$

$$\theta_2 = \alpha_2 + \beta_2\epsilon + \gamma_2\lambda \tag{2.11}$$

where

$$\alpha_1 = \frac{\epsilon_3\Delta\lambda_2 - \lambda_3\Delta\epsilon_2}{\Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1},\ \beta_1 = -\frac{\Delta\lambda_2}{\Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1},\ \gamma_1 = \frac{\Delta\epsilon_2}{\Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1}$$

$$\alpha_2 = \frac{\lambda_3(\Delta\epsilon_1 + \Delta\epsilon_2) - \epsilon_3(\Delta\lambda_1 + \Delta\lambda_2)}{\Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1},\ \beta_2 = \frac{\Delta\lambda_1 + \Delta\lambda_2}{\Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1},$$

$$\gamma_2 = -\frac{\Delta\epsilon_1 + \Delta\epsilon_2}{\Delta\epsilon_1\Delta\lambda_2 - \Delta\epsilon_2\Delta\lambda_1}.$$

Replacing (2.9)-(2.11) in (2.8) gives the following representation of the optimal value function:

$$\phi(\triangle b, \triangle c, \epsilon, \lambda) = b_0 + b_1\epsilon + b_2\lambda + b_3\epsilon\lambda + b_4\epsilon^2 + b_5\lambda^2 \quad (2.12)$$

where

$$\begin{aligned}
b_0 &= a_0 - a_1\alpha_1 - a_2\alpha_2 + a_3\alpha_1\alpha_2 + a_4\alpha_1^2 + a_5\alpha_2^2,\\
b_1 &= -a_1\beta_1 - a_2\beta_2 + a_3\alpha_2\beta_1 + a_3\alpha_1\beta_2 + 2a_4\alpha_1\beta_1 + 2a_5\alpha_2\beta_2,\\
b_2 &= -a_1\gamma_1 - a_2\gamma_2 + a_3\alpha_2\gamma_1 + a_3\alpha_1\gamma_2 + 2a_4\alpha_1\gamma_1 + 2a_5\alpha_2\gamma_2,\\
b_3 &= a_3\beta_1\gamma_2 + a_3\beta_2\gamma_1 + 2a_4\beta_1\gamma_1 + 2a_5\beta_2\gamma_2,\\
b_4 &= a_3\beta_1\beta_2 + a_4{\beta_1}^2 + a_5{\beta_2}^2,\\
b_5 &= a_3\gamma_1\gamma_2 + a_4{\gamma_1}^2 + a_5{\gamma_2}^2
\end{aligned}$$

that is a quadratic function of ϵ and λ. The proof is complete. □

2.4 Identifying the ISS, IACS and IC Intervals

In this subsection we present auxiliary LO problems that enable us to identify the ISS, IACS and IC regions. First we mention a trivial observation. Let Φ_{LP} denotes the solution set of the following equations system:

$$Ax - \epsilon\triangle b = b$$

where $x \geq 0$. Analogously, let Φ_{LD} denotes the solution set of the following equations system:

$$A^T y + s - \lambda\triangle c = c$$

where $s \geq 0$. Further, combining Φ_{LP} and Φ_{LD}, let Φ be the solution set of the following equations system:

$$\begin{aligned}
Ax - \epsilon\triangle b &= b\\
A^T y + s - \lambda\triangle c &= c
\end{aligned}$$

where $x \geq 0$ and $s \geq 0$. It should be mentioned that when the LP is in canonical form, the set Φ is reduced to the following set:

$$\begin{aligned}
Ax - r - \epsilon\triangle b &= b\\
A^T y + s - \lambda\triangle c &= c
\end{aligned}$$

where $x \geq 0$, $r \geq 0$, $y \geq 0$ and $s \geq 0$ and we denote it by Φ_C. Observe that in the feasible solution set $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon, \lambda)$, the parameters ϵ and λ are considered to be fixed parameters values and this set contains all vectors $x(\epsilon)$ that satisfy the constraints of problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$. Meanwhile, in the solution set Φ, ϵ is considered as unknown and its smallest and biggest value (if it exists) denotes the domain of the optimal value function $\phi(\triangle b, \triangle c, \epsilon, \lambda)$. Analogous discussion is valid for solution sets Φ_{LP} and Φ_{LD}.

2.4.1 Identifying the ISS Interval

The following theorem presents two computable auxiliary LO problems that lead to identify the ISS region:

$$\Upsilon(\triangle b, \triangle c, \epsilon, \lambda) = \Upsilon(\triangle b, \epsilon) \times \Upsilon(\triangle c, \lambda).$$

The proof is analogous to the proof of Theorem 3 in [4] and it is omitted.

Theorem 2.8. *Let x be a primal optimal solution of problem LP, where $\sigma(x) = P$ and (P, Z) be the ISS partition and let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variables' optimal partitions of LP problem. The ϵ_l and ϵ_u, the end points of $\overline{\Upsilon}(\triangle b, \epsilon)$ can be obtained by solving the following two auxiliary LO problems, respectively:*

$$\epsilon_l = \min\{\epsilon \mid x \in \Phi_{LP}, \sigma(s) \subseteq \mathcal{N}_V^s, \sigma(x) \subseteq P\}, \tag{2.13}$$

$$\epsilon_u = \max\{\epsilon \mid x \in \Phi_{LP}, \sigma(s) \subseteq \mathcal{N}_V^s, \sigma(x) \subseteq P\}. \tag{2.14}$$

Then, $\overline{\Upsilon}(\triangle b, \epsilon) = [\epsilon_l, \epsilon_u]$. λ_l and λ_u, the end points of $\overline{\Upsilon}(\triangle c, \lambda)$ can be obtained by solving the following two auxiliary LO problems, respectively:

$$\lambda_l = \min\{\lambda \mid (y, s) \in \Phi_{LD}, \sigma(x) \subseteq \mathcal{B}_V^x, \sigma(s) \subseteq Z\}, \tag{2.15}$$

$$\lambda_u = \max\{\lambda \mid (y, s) \in \Phi_{LD}, \sigma(x) \subseteq \mathcal{B}_V^x, \sigma(s) \subseteq Z\}. \tag{2.16}$$

Then, $\overline{\Upsilon}(\triangle c, \lambda) = [\lambda_l, \lambda_u]$.

As in the proof of Theorem 2.8, if $\epsilon_l = \epsilon_u = \lambda_l = \lambda_u = 0$, then there is no possibility to perturb the RHS and OFC data of the problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ in the perturbing direction Δb and Δc while maintaining the ISS property of this problem. In this case, the ISS region $\Upsilon(\triangle b, \triangle c, \epsilon, \lambda)$ is the singleton $\{(0, 0)\}$.

2.4.2 Identifying the IACS Interval

The following theorem presents two computable auxiliary LO problems that lead to identify the IACS region:

$$\Gamma(\triangle b, \triangle c, \epsilon, \lambda) = \Gamma(\triangle b, \epsilon) \times \Gamma(\triangle c, \lambda).$$

The proof is analogous to the proof of Theorem 5 in [4] and it is omitted.

Theorem 2.9. *Let (y, s) be a dual optimal solution of LD problem, where $\sigma(s) = \overline{P}$ and $(\overline{P}, \overline{Z})$ be the IACS partition and let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variables' optimal partitions of LP problem. The ϵ_l and ϵ_u, the end points of $\overline{\Gamma}(\triangle b, \epsilon)$ can be obtained by solving the following two auxiliary LO problems, respectively:*

$$\epsilon_l = \min\{\epsilon \mid (y, s) \in \Phi_{LD}, \sigma(x) \subseteq \mathcal{B}_V^x, \sigma(s) \subseteq \overline{P}\}, \tag{2.17}$$

$$\epsilon_u = \max\{\epsilon \mid (y, s) \in \Phi_{LD}, \sigma(x) \subseteq \mathcal{B}_V^x, \sigma(s) \subseteq \overline{P}\}. \tag{2.18}$$

Then, $\overline{\Gamma}(\triangle b, \epsilon) = [\epsilon_l, \epsilon_u]$. λ_l and λ_u, the end points of $\overline{\Gamma}(\triangle c, \lambda)$ can be obtained by solving the following two auxiliary LO problems, respectively:

$$\lambda_l = \min\{\lambda \mid x \in \Phi_{LP}, \sigma(s) \subseteq \mathcal{N}_V^s, \sigma(x) \subseteq \overline{Z}\}, \tag{2.19}$$

$$\lambda_u = \max\{\lambda \mid x \in \Phi_{LP}, \sigma(s) \subseteq \mathcal{N}_V^s, \sigma(x) \subseteq \overline{Z}\}. \tag{2.20}$$

Then, $\overline{\Gamma}(\triangle c, \lambda) = [\lambda_l, \lambda_u]$.

As in the proof of Theorem 2.9, if $\epsilon_l = \epsilon_u = \lambda_l = \lambda_u = 0$, then there is no possibility to perturb the RHS and OFC data of problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ in the perturbing direction Δb and Δc while maintaining the IACS property of this problem. In this case, the IACS region $\Gamma(\triangle b, \triangle c, \epsilon, \lambda)$ is the singleton $\{(0, 0)\}$.

$\boxed{2.4.3}$ Identifying the IC Interval

The following theorem presents two computable auxiliary LO problems that lead to identify the IC region:

$$\Theta(\triangle b, \triangle c, \epsilon, \lambda) = \Theta(\triangle b, \epsilon) \times \Theta(\triangle c, \lambda).$$

The proof is analogous to the proof of Theorem 7 in [4] and it is omitted.

Theorem 2.10. *Let (x, y, s) be a primal-dual optimal solution of problem LP and LD where $\sigma(x) = P$, $\sigma(x) = P$ and (P, Z),$(\overline{P}, \overline{Z})$ be the Ic partitions and let $\Pi_V = (\mathcal{B}_V^x, \mathcal{N}_V^s)$ be the variables' optimal partitions of problem LP and LD. The ϵ_l and ϵ_u, the end points of $\overline{\Theta}(\triangle b, \epsilon)$ can be obtained by solving the following two auxiliary LO problems, respectively:*

$$\epsilon_l = \min\{\epsilon \mid (x, y, s) \in \Phi, \sigma(x) \subseteq P, \sigma(s) \subseteq \overline{P}\}, \tag{2.21}$$

$$\epsilon_u = \max\{\epsilon \mid (x, y, s) \in \Phi, \sigma(x) \subseteq P, \sigma(s) \subseteq \overline{P}\}. \tag{2.22}$$

Then, $\overline{\Theta}(\triangle b, \epsilon) = [\epsilon_l, \epsilon_u]$. λ_l and λ_u, the end points of $\overline{\Theta}(\triangle c, \lambda)$ can be obtained by solving the following two auxiliary LO problems, respectively:

$$\lambda_l = \min\{\lambda \mid (x, y, s) \in \Phi, \sigma(x) \subseteq P, \sigma(s) \subseteq \overline{P}\}, \tag{2.23}$$

$$\lambda_u = \max\{\lambda \mid (x, y, s) \in \Phi, \sigma(x) \subseteq P, \sigma(s) \subseteq \overline{P}\}. \tag{2.24}$$

Then, $\overline{\Theta}(\triangle c, \lambda) = [\lambda_l, \lambda_u]$.

As in the proof of Theorem 2.10, if $\epsilon_l = \epsilon_u = \lambda_l = \lambda_u = 0$, then there is no possibility to perturb the RHS and OFC data of the problem $LP(\triangle b, \triangle c, \epsilon, \lambda)$ in the perturbing direction Δb and Δc while maintaining the IC property of this problem. In this case, the IC region $\Theta(\triangle b, \triangle c, \epsilon, \lambda)$ is the singleton $\{(0, 0)\}$.

$\boxed{3}$ Illustrative Examples

In this section, we apply the results of the previous sections and express an example to illustrate the general LO problems.

Example 3.1. Consider the following LP problem in the form:

$$\begin{array}{llll} \min & x_1 & + x_2 & \\ s.t & x_1 & + x_2 & \geq 4 \\ & x_1 & & \geq 2 \\ & x_1 & , x_2 & \geq 0 \end{array}$$

Its dual is:

$$\begin{array}{llll} \max & 4y_1 & + 2y_2 & \\ s.t & y_1 & + y_2 & \leq 1 \\ & y_1 & & \leq 1 \\ & y_1 & , y_2 & \geq 0 \end{array}$$

It is easy to verify that the primal problem has multiple optimal solutions, while its dual problem has a unique solution (y^*, s^*), where $y^* = (1, 0)^T$ and $s^* = (0, 0)^T$. It is easy to verify that the optimal partition of the index set $\{1, 2\}$ is $\Pi = (\mathcal{B}, \mathcal{N}) = \{\{1, 2\}, \phi\}$.

A strictly complementary optimal solution of (LP) and (LD) are:

$$x^* = (2,2), \quad r^* = (0,0), \quad y^* = (1,0) \quad and \quad s^* = (0,0).$$

Let $\Delta b = (-1,1)$ and $\Delta c = (-1,1)$ be the perturbing directions. We have then $\Upsilon(\triangle b,\epsilon) = [0,1]$ and $\Upsilon(\triangle c,\lambda) = [-\infty,1]$. Thus

$$\Upsilon(\triangle b,\triangle c,\epsilon,\lambda) = [0,1]\times(-\infty,1]$$

for $\mathcal{B}_y = \{1\}$ and $\mathcal{N}_y = \{2\}$. Also$\Upsilon(\triangle b,\epsilon) = [-1,1]$ and $\Upsilon(\triangle c,\lambda) = [-2,\infty)$, thus

$$\Upsilon(\triangle b,\triangle c,\epsilon,\lambda) = [-1,1]\times(-2,\infty)$$

for $\mathcal{B}_y = \{2\}$ and $\mathcal{N}_y = \{1\}$ and also$\Upsilon(\triangle b,\epsilon) = [-1,0]$ and $\Upsilon(\triangle c,\lambda) = [-2,1]$, thus

$$\Upsilon(\triangle b,\triangle c,\epsilon,\lambda) = [-1,0]\times[-2,1]$$

for $\mathcal{B}_y = \{1,2\}$ and $\mathcal{N}_y = \phi$.

4 Conclusion

In this paper we introduced the concept of bi-parametric support set sensitivity analysis for LO. We presented auxiliary LO problems that enable us to identify associated regions. We are interested in developing the results of this study to the bi-(tetra)parametric expansion support set sensitivity analysis for general LO problems and CQO, as well.

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