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COMPLEMENTS ON SUBDIFFERENTIAL CALCULUS

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Abstract: Convex subdifferential calculus obeys various rules possibly involving approximate subdifferentials. Some of these rules concern nonconvex functions like for instance the difference of convex functions. In this note we point out a formula for the subdifferential of the lower semicontinuous hull of an arbitrary extended real-valued function h. We apply the result to the case when h is the infimal convolution of functions that need not be convex. The symmetric approach in terms of minimum sets of functions is also investigate and argmin calculus rules are obtained. We derive from this approach the Hiriart-Urruty and Phelps formula on the sum of two proper lower semicontinuous convex functions and provide a remarkable topological stability property of this formula.

Key words: *subdifferential calculus, argmin calculus, convex analysis, Legendre-Fenchel conjugate Kuratowski-Painlevé convergence*

Mathematics Subject Classification: *90C25, 90C26, 49N15*

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1 Preliminary Notions

Let $\langle\ \rangle : X \times Y \to \mathbb{R}$ be a separated bilinear coupling of real vector spaces X and Y. The ϵ-subdifferential ($\epsilon \geq 0$) of an extended real-valued function $h : X \to \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$) at a point $x \in h^{-1}(\mathbb{R})$ is defined as

$$\partial_\epsilon h(x) = \{y \in Y : h(u) \geq h(x) + \langle u - x, y\rangle - \epsilon, \forall u \in X\}.$$

In the sequel we set

$$\partial_\epsilon h(x) = \emptyset \text{ if } h(x) = +\infty \text{ or } h(x) = -\infty.$$

It is worth of observing that if h takes somewhere the value $-\infty$ or if h is identically equal to $+\infty$ (one then says that h is not proper) then $\partial_\epsilon h(x)$ is empty for all $x \in X$. In all cases it is possible to express $\partial_\epsilon h(x)$ in terms of the Legendre-Fenchel conjugate h^* of h

$$h^*(y) = \sup_{x\in X}(\langle x, y\rangle - h(x)) \ \ \forall y \in Y.$$

According to the sum rule $+\infty + (-\infty) = (-\infty) + (+\infty) = +\infty$ we have, for any $x \in X$, $\epsilon \geq 0$,

$$\partial_\epsilon h(x) = \{y \in Y : h(x) + h^*(y) \leq \langle x, y\rangle + \epsilon\}. \tag{1.1}$$

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We also consider the inverse of the multivalued mapping $\partial_\epsilon h$ that we denote by $M_\epsilon h$: for any $(x, y) \in X \times Y$,

$$y \in \partial_\epsilon h(x) \quad \Leftrightarrow \quad x \in M_\epsilon h(y) = \epsilon - \text{argmin}(h - \langle ., y\rangle)$$

where for $k \in \overline{\mathbb{R}}^X$ we set

$$\epsilon - \text{argmin}(k) = \{x \in X : k(x) \in \mathbb{R} \text{ and } k(x) \leq k(u) + \epsilon, \forall u \in X\}.$$

The previous concepts can be symmetrically applied to extended real-valued functions ϕ defined on Y like for instance $\phi = h^*$. We will use the next basic fact

$$\text{if } \ h = h^{**} \ \text{ then } \ M_\epsilon h(y) = \partial_\epsilon h^*(y\ ) \ \forall \epsilon \geq 0, \forall y \in Y. \tag{1.2}$$

Here $h^{**} = (h^*)^*$ denotes the Legendre-Fenchel biconjugate of h with respect to the coupling $\langle\ \rangle : X \times Y \to \mathbb{R}$. In the sequel we equip X (resp. Y) with a topology τ (resp. ν) such that the coupling $\langle\ \rangle : X \times Y \to \mathbb{R}$ is $\tau \times \nu$-continuous. Note that τ and ν are finer than the weak topologies $\sigma(X, Y)$ and $\sigma(Y, X)$ respectively. Moreover, when ν (resp. τ) is the discrete topology, then $\langle\ \rangle$ is $\tau \times \nu$-continuous if and only if τ (resp. ν) is finer than $\sigma(X, Y)$ (resp. $\sigma(Y, X)$).

The case when τ or ν (or both) is the discrete topology is of particular interest. One can also take for τ the norm topology of a normed space X and for ν the dual norm topology on the topological dual Y of X. This covers of course the case when $X = Y$ is an euclidean space or a Hilbert space and $\tau = \nu$ is the norm topology. Sequential forms of continuity of the coupling $\langle\ \rangle$ involving the weak (resp.weak*) topology in a Banach space setting are not exploited in this note. A nonstandard example is furnished by the choice of the core topology τ on a real linear space X, whose open sets are the empty set and the sets (possibly nonconvex) whose affine-hull is X and that coincide with their algebraic interior; such a topology τ is not compatible with the linear structure unless $X = \mathbb{R}$; taking for ν the discrete topology on the space Y of all linear forms on X, the coupling $(x, y) \in X \times Y \to \langle x, y\rangle = y(x)$ is then $\tau \times \nu$- continuous.

The Kuratowski-Painlevé outer limit of sets is crucial in what follows. Recall that the Kuratowski-Painlevé outer limit, also called limit superior, of a family $(N_\alpha)_{\alpha > \epsilon}$ of subsets N_α of $X \times Y$ when $\alpha \to \epsilon_+$ is defined by (see e.g.[1])

$$\tau \times \nu - \limsup_{\alpha \to \epsilon_+} N_\alpha = \bigcap_{\alpha > \epsilon} \tau \times \nu - cl \bigcup_{\epsilon < \beta < \alpha} N_\beta.$$

This concept can be equivalently expressed in terms of outer limits of the multivalued mappings

$$N_\alpha : X \rightrightarrows Y, N_\alpha(x) := \{y \in Y : (x, y) \in N_\alpha\}; \ \ M_\alpha : Y \rightrightarrows X, M_\alpha(y) := \{x \in X : (x, y) \in N_\alpha\}.$$

In fact for any $(x, y) \in X \times Y$ the following assertions are equivalent

i)
$$(x, y) \in \tau \times \nu - \limsup_{\alpha \to \epsilon_+} N_\alpha,$$

ii)
$$y \in \nu - \limsup_{\substack{\alpha \to \epsilon_+ \\ u \xrightarrow{\tau} x}} N_\alpha(u) := \bigcap_{\alpha \epsilon} \ \bigcap_{U \in \mathcal{N}_\tau(x)} \tau - cl \bigcup_{\epsilon < \beta < \alpha, u \in U} N_\beta(u),$$

iii)

$$x \in \tau - \limsup_{\substack{\alpha \to \epsilon_+ \\ v \xrightarrow{\nu} y}} M_\alpha(v) := \bigcap_{\alpha\epsilon} \bigcap_{V \in \mathcal{N}_\nu(y)} \tau - cl \bigcup_{\epsilon < \beta < \alpha, v \in V} M_\beta(v).$$

When the family $(N_\alpha)_{\alpha > \epsilon}$ is nondecreasing one has

$$\tau \times \nu - \limsup_{\alpha \to \epsilon_+} N_\alpha = \bigcap_{\alpha > \epsilon} \tau \times \nu - cl\ N_\alpha.$$

2 The ϵ-subdifferential of $\tau - cl\ h$

Let $h \in \overline{\mathbb{R}}^X$ and denote by $\overline{h}$ the τ-l.s.c. hull of h

$$\overline{h} = \tau - cl\ h.$$

Since $\langle ., y \rangle$ is τ-continuous one has, for any $(y, r) \in Y \times \mathbb{R}$,

$$\langle ., y \rangle - r \leq h \quad \Rightarrow \quad \langle ., y \rangle - r \leq \overline{h}$$

and consequently

$$h^{**} \leq \overline{h} \leq h.$$

Applying the Legendre-Fenchel transform we obtain

$$(\overline{h})^* = h^*$$

and, for any $x \in X$,

$$\partial_\epsilon \overline{h}(x) = \{y \in Y : \overline{h}(x) + h^*(y) \leq \langle x, y \rangle + \epsilon\}. \tag{2.1}$$

Let us introduce the function $H : X \times Y \to \overline{\mathbb{R}}$ defined by

$$H(x, y) = h(x) + h^*(y) - \langle x, y \rangle.$$

In terms of the graph

$$\partial_\epsilon h := \{(x, y) \in X \times Y : y \in \partial_\epsilon h(x)\}$$

one has from (1.1)

$$[H \leq \epsilon] := \{(x, y) \in X \times Y : H(x, y) \leq \epsilon\} = \partial_\epsilon h. \tag{2.2}$$

The next lemma provides the $\tau \times \nu-$l.s.c. hull $\overline{H}$ of H:

Lemma 2.1. *For any $(x, y) \in X \times Y$ one has*

$$\overline{H}(x, y) = \overline{h}(x) + h^*(y) - \langle x, y \rangle. \tag{2.3}$$

Proof. If h^* is not proper the conclusion is clearly true (either $h^* = +\infty$ and then $H = \overline{H} = +\infty$, or $h^* = -\infty$ and then $h = \overline{h} = +\infty$ and so $H = \overline{H} = -\infty$). In the contrary case we have that $-\infty < h^{**} \leq \overline{h}$ and $-\infty < h^*$; in such a case we can write

$$\liminf_{\substack{u \xrightarrow{\tau} x \\ v \xrightarrow{\nu} y}} (h(u) + h^*(v)) = \liminf_{u \xrightarrow{\tau} x} h(u) + \liminf_{v \xrightarrow{\nu} y} h^*(v) = \overline{h}(x) + h^*(y)$$

because h^* is ν-l.s.c. as a supemum of affine ν-continuous functions. □

Taking into account the classical relation

$$[\overline{H} \leq \epsilon] = \bigcap_{\alpha > \epsilon} \tau \times \nu - cl[H \leq \alpha]$$

together with (2.1), (2.2), (2.3), we can state

Theorem 2.2. *Assume the coupling $\langle\ \rangle$ is $\tau \times \nu$-continuous. Then, the graph of the ϵ-subdifferential of the τ-l.s.c. hull of any function $h \in \overline{\mathbb{R}}^X$ is given by (for any $\epsilon \geq 0$)*

$$\partial_\epsilon(\tau - cl\ h) = \tau \times \nu - \limsup_{\alpha \to \epsilon_+} \partial_\alpha h. \tag{2.4}$$

According to the properties of outer limits in a product space (see the equivalence between (i) and (ii)) we get

Corollary 2.3. *For any $h \in \overline{\mathbb{R}}^X$, $\epsilon \geq 0$, $(x, y) \in X \times Y$, one has*

$$\partial_\epsilon(\tau - cl\ h)(x) = \nu - \limsup_{\substack{\alpha \to \epsilon_+ \\ u \xrightarrow{\tau} x}} \partial_\alpha h(u) \tag{2.5}$$

or, more explicitly,

$$\partial_\epsilon(\tau - cl\ h)(x) = \bigcap_{\alpha > \epsilon} \bigcap_{U \in N_\tau(x)} \nu - cl \bigcup_{u \in U} \partial_\alpha h(u),\ \forall x \in X. \tag{2.6}$$

Remark 2.4. Provided the coupling $\langle\ \rangle$ is $\tau \times \nu$-continuous, the right hand members of (2.5) and (2.6) are independant of ν. Assuming the functions $\langle ., y\rangle$, $y \in Y$, are τ-continuous and taking ν as the discrete topology we obtain

$$\partial_\epsilon(\tau - cl\ h)(x) = \bigcap_{\alpha > \epsilon} \bigcap_{U \in N_\tau(x)} \bigcup_{u \in U} \partial_\alpha h(u).$$

Remark 2.5. Assume $h \in \overline{\mathbb{R}}^X$ is τ-l.s.c. at $x \in X$. Since $h^* = \overline{h}^{\,*}$ one has $\partial_\epsilon h(x) = \partial_\epsilon \overline{h}(x)$ and by (2.5)

$$\partial_\epsilon h(x) = \nu - \limsup_{\alpha \to \epsilon_+} \partial_\alpha h(u) \supset \nu - \limsup_{u \xrightarrow{\tau} x} \partial_\epsilon h(u).$$

Since the inclusion

$$\partial_\epsilon h(x) \subset \nu - \limsup_{u \xrightarrow{\tau} x} \partial_\epsilon h(u)$$

always holds we get

$$\partial_\epsilon h(x) = \nu - \limsup_{u \xrightarrow{\tau} x} \partial_\epsilon h(u).$$

Remark 2.6. Let $h \in \overline{\mathbb{R}}^X$ and $x \in X$ be such that

$$\nu - \limsup_{\alpha \to 0_+, u \xrightarrow{\tau} x} \partial_\alpha h(u) \neq \emptyset.$$

Then $\overline{h}$ is subdifferentiable at x and one has

$$\emptyset \neq \partial \overline{h}(x) = \partial h^{**}(x) = \nu - \limsup_{\substack{\alpha \to 0_+ \\ u \xrightarrow{\tau} x}} \partial_\alpha h(u).$$

The interest of the above formula lies in the fact that h^{**} is often very difficult to compute ([2]).

3 The ϵ-subdifferential of the τ-l.s.c. Hull of an Infimal Convolution

We are interested in the case when h is the infimal convolution of two extended real-valued functions $f, g \in \overline{\mathbb{R}}^X$:

$$h(x) = (f \Box g)(x) = \inf_{u+z=x} f(u) + g(z).$$

Several formulas has been established for the ϵ-subdifferential of $f \Box g$ for f and g convex([3] [4] [8]...). In order to calculate the subdifferential of the l.s.c. hull of $f \Box g$ in a general setting we just need, beside Therem 2.2, the following estimation that involves the parallel sum $A \perp B$ of two multivalued mappings $A, B : X \rightrightarrows Y$:

$$A \perp B = \left(A^{-1} + B^{-1}\right)^{-1}$$

or, more explicitly,

$$(A \perp B)(x) = \bigcup_{u+z=x} A(u) \cap B(z), \ \forall x \in X.$$

Lemma 3.1. *For any* $f, g \in \overline{\mathbb{R}}^X$, $\alpha > 0$, *and* $x \in X$ *one has*

$$\partial_\alpha f \Box g(x) \subset (\partial_{2\alpha} f \perp \partial_{2\alpha} g)(x) \subset \partial_{4\alpha}(f \Box g)(x).$$

Proof. Let us prove the first inclusion, which is clearly true if $(f \Box g)(x) = \pm\infty$. Assume $(f \Box g)(x) \in \mathbb{R}$ and let $y \in \partial_\alpha (f \Box g)(x)$. Since $(f \Box g)^*(y) = f^*(y) + g^*(y)$ (see e.g. [7]), by (1.1) there exist $u, z \in X$ such that $u + z = x$ and

$$f(u) + g(z) + f^*(y) + g^*(y) - \langle x, y\rangle \leq 2\alpha$$

or equivalently

$$[f(u) + f^*(y) - \langle u, y\rangle] + [g(z) + g^*(y) - \langle z, y\rangle] \leq 2\alpha.$$

The two brakets been nonnegative (by Fenchel inequality) we get $y \in \partial_{2\alpha} f(u) \cap \partial_{2\alpha} g(z)$ and finally $y \in (\partial_{2\alpha} f \perp \partial_{2\alpha} g)(x)$.
Let us prove the second inclusion. Let $u + z = x$ and $y \in \partial_{2\alpha} f(u) \cap \partial_{2\alpha} g(z)$. One has $f^*(y) + g^*(y) = (f \Box g)^*(y) \in \mathbb{R}$ and so $f \Box g$ is proper. On the other hand $(f \Box g)(x) \leq f(u) + g(z) < +\infty$, and thus $(f \Box g)(x) \in \mathbb{R}$. Moreover, for any $v_1, v_2, v \in X$ such that $v_1 + v_2 = v$ one has

$$f(v_1)+g(v_2) \geq (f(u)+\langle v_1-u, y\rangle-2\alpha)+(g(z)+\langle v_2-z, y\rangle-2\alpha) \geq (f \Box g)(x)+\langle v-x, y\rangle-4\alpha.$$

By taking the infimum on v_1, v_2, with $v_1 + v_2 = v$, we get, for any $v \in X$,

$$(f \Box g)(v) \geq (f \Box g)(x) + \langle v - x,\ y\rangle - 4\alpha$$

that means $y \in \partial_{4\alpha}(f \Box g)(x)$. $\Box$

According to Theorem 2.2 and Lemma 3.1 we can state

Theorem 3.2. *Assume the coupling $\langle\ \rangle$ is $\tau \times \nu$-continuous. The subdifferential of the τ-l.s.c. hull $\overline{(f \square g)}$ of the infimal convolution of the extended real-valued functions $f, g \in \overline{\mathbb{R}}^X$ is given by*

$$\partial\overline{(f \square g)}(x) = \nu - \limsup_{\substack{\alpha \to 0_+ \\ u \xrightarrow{\tau} x}} (\partial_\alpha f \perp \partial_\alpha g)(u),\ \forall x \in X \tag{3.1}$$

or, explicitly,

$$\partial\overline{(f \square g)}(x) = \bigcap_{\alpha>0} \bigcap_{U \in \mathcal{N}_\tau(x)} \nu - cl \bigcup_{u+z \in U} \partial_\alpha f(u) \cap \partial_\alpha g(z). \tag{3.2}$$

Remark 3.3. Assuming the functions $\langle ., y\rangle$, $y \in Y$, are τ-continuous and taking ν as the discrete topology we get from (3.2) and for any $f, g \in \overline{\mathbb{R}}^X, x \in X$,

$$\partial\overline{(f \square g)}(x) = \bigcap_{\alpha>0} \bigcap_{U \in \mathcal{N}_\tau(x)} \bigcup_{u+z \in U} \partial_\alpha f(u) \cap \partial_\alpha g(z).$$

In the case when τ is the discrete topology and the functions $\langle x, .\rangle$, $x \in X$, are τ-continuous one has from (3.2):

$$\partial f \square g(x) = \bigcap_{\alpha>0} \nu - cl\ (\partial_\alpha f \perp \partial_\alpha g)(x).$$

Taking ν as the discrete topology we obtain a formula established in ([3]) for proper $\sigma(X, Y)$-l.s.c. convex functions

$$\partial f \square g(x) = \bigcap_{\alpha>0} (\partial_\alpha f \perp \partial_\alpha g)(x)$$

which is in fact valid for any $f, g \in \overline{\mathbb{R}}^X$, $x \in X$.

Remark 3.4. According to the properties of outer limits in a product space (cf the equivalence between ii) and iii) in Section 1) and to the definition of the parallel sum of multivalued mappings, the formula (3.1) can be reversed as follows

$$M(\tau - cl\ f \square g)(y) = \tau - \limsup_{\substack{\alpha \to 0_+ \\ v \xrightarrow{\tau} y}} M_\alpha f(v) + M_\alpha g(v) \tag{3.3}$$

for any $f, g \in \overline{\mathbb{R}}^X, y \in Y$. Taking for ν the discrete topology and assuming that the functions $\langle ., y\rangle$ are τ-continuous we obtain

$$M(\tau - cl\ f \square g)(y) = \bigcap_{\alpha>0} \tau - cl(M_\alpha f(y) + M_\alpha g(y)),\ \forall y \in Y.$$

4 Hiriart-Urruty and Phelps Formula Revisited

As usual we denote by $\Gamma_0(X)$ the set of functions $\phi : Y \to \mathbb{R} \cup \{+\infty\}$ that can be expressed as a supremum of a (nonvoid) family of X-affine functions, $\phi = \sup_{i \in I} \langle x_i, .\rangle - r_i$, $(x_i, r_i) \in X \times \mathbb{R}$ for any $i \in I$, and such that $\operatorname{dom} \phi := \{y \in Y : \phi(y) < +\infty\} \neq \emptyset$. It is wellknown that $\Gamma_0(Y)$ is the set of extended real-valued functions on Y which are proper and coincide with their biconjugate, or the set of proper convex functions which are l.s.c. with respect to any

locally convex vector space topology on Y compatible with the coupling $\langle\ \rangle : X \times Y \to \mathbb{R}$ (that means for which the topological dual of Y coincides with $\{\langle x, .\rangle : x \in X\}$) like for instance the weak topology $\sigma(Y, X)$.
Let us consider $\phi, \psi \in \Gamma_0(Y)$ such that

$$\text{dom}\ \phi \cap \text{dom}\ \psi \neq \emptyset \tag{4.1}$$

and let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be convex and such that

$$f^* = \phi,\ g^* = \psi. \tag{4.2}$$

One can take $f = \phi^*$, $g = \psi^*$ but other choices are possible. In any case one has from (4.2)

$$\partial_\epsilon \phi = M_\epsilon f^{**},\ \partial_\epsilon \psi = M_\epsilon g^{**},\ \forall \epsilon \geq 0. \tag{4.3}$$

Let us now assume that τ is a locally convex vector space topology on Y which is compatible with the coupling $\langle\ \rangle : X \times Y \to \mathbb{R}$. According to (4.1) and (4.2), it is a classical fact that the common τ-l.s.c.-hull of $f \Box g$ and $f^{**} \Box\, g^{**}$ coincides with the biconjugate of $f \Box g$ (see e.g. [6], [8]); in a word

$$(\phi + \psi)^* = (f \Box g)^{**} = \tau - cl\ f \Box g = \tau - cl\ f^{**} \Box\, g^{**}.$$

We therefore get by (1.2)

$$\partial(\phi + \psi)(y) = M(\tau - cl\ f \Box g)(y) = M(\tau - cl\ f^{**} \Box\, g^{**})(y),\ \forall y \in Y. \tag{4.4}$$

Appying (3.3) twice in (4.4) and using (4.3) we obtain

Corollary 4.1. *Assume that τ is a locally convex vector space topology on X compatible with the coupling $\langle\ \rangle : X \times Y \to \mathbb{R}$ and let ν be a topology on Y such that $\langle\ \rangle$ is $\tau \times \nu$-continuous. For any $\phi, \psi \in \Gamma_0(Y)$ satisfying (4.1), any convex functions $f, g : X \to \mathbb{R} \cup +\{\infty\}$ satisfying (4.2), and any $y \in Y$, one has*

$$\partial(\phi + \psi)(y) = \tau - \limsup_{\substack{\alpha \to 0_+ \\ v \xrightarrow{\nu} y}}\ M_\alpha f(v) + M_\alpha g(v) \tag{4.5}$$

$$\partial(\phi + \psi)(y) = \tau - \limsup_{\substack{\alpha \to 0_+ \\ v \xrightarrow{\nu} y}}\ \partial_\alpha \phi(v) + \partial_\alpha \psi(v). \tag{4.6}$$

Remark 4.2. The choice of the discrete topology ν on Y in (4.6) leads to the Hiriart-Urruty and Phelps formula

$$\partial(\phi + \psi)(y) = \bigcap_{\alpha > 0} \tau - cl\ (\partial_\alpha \phi(y) + \partial_\alpha \psi(y))$$

while (4.5) gives

$$\partial(\phi + \psi)(y) = \bigcap_{\alpha > 0} \tau - cl\ (M_\alpha f(y) + M_\alpha g(y)).$$

Also, Corollary 4.1 says that Hiriart-Urruty and Phelps formula benefits a remarkable topological stability property around each point.

Remark 4.3. Formula (4.5) can be reversed as follows

$$M(\phi+\psi)(x) = \nu - \limsup_{\substack{\alpha\to 0+ \\ u \overset{\tau}{\to} x}} (\partial_\alpha f \perp \partial_\alpha g)(u).$$

In the same way by reversing (4.6) we get

$$M(\phi+\psi)(x) = \bigcap_{\alpha>0} \bigcap_{U\in\mathcal{N}_\tau(x)} \nu - cl \bigcup_{u+z\in U} M_\alpha\phi(u) \cap M_\alpha\psi(z)$$

a formula for the argmin of the sum of $\Gamma_0(Y)$ functions without additional condition (compare with [7] Corollary 1.16).

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