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DELINEATING NICE CLASSES OF NONSMOOTH FUNCTIONS

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Dedicated to Michel Théra on the occasion of his sixtieth birthday

Abstract: A whole spectrum of subdifferentiability properties is delineated in which various degrees of uniformity are present. Related properties are introduced for sets. Some characterizations in terms of monotonicity properties are displayed.

Key words: *approximately convex function, equi-subdifferentiability, monotonicity, nonsmooth analysis, normal, semi-monotonicity, subdifferential, uniform subdifferentiability*

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1 Introduction

While it is known thanks to Weierstrass that wild nonsmooth functions exist, it is of interest to select classes of nonsmooth functions which have pleasant properties. Such an appraisal can be understood in different manners: (1) one may hope to get properties close to the ones of convex or differentiable functions, (2) one may expect stability properties with respect to the usual operations, (3) one may wish that the main concepts of nonsmooth analysis coincide for such classes, (4) one may devote efforts to show that the regularization of such functions is worthwhile.

Many works have been devoted to such aims.

Central concepts are the notions of approximately convex functions ([4], [26], [28]) and paraconvex (or semiconvex) functions ([12], [13], [29], [44]-[48], [51]...). They can be given several variants. Directional variants have been studied in [18], [19], in particular in connection with integration questions. Another line of research appears in the works of Poliquin, Rockafellar, Thibault and their co-authors ([5], [6], [22], [40], [41], [42]...) . In such works, the subjet of the function is heavily present, even if it is not put to the fore. It is the purpose of the present paper to put it in clear light and to propose some variants. We keep in mind expectations (1)-(3), in particular (3) and what has been called softness in [35] and lower regularity in [25]. We refer to [2], [10], [24], [31], [43] and their references for the question of regularization. With [22], [23] and [27], the paper [31] has been our starting point.

Our main thrusts consist in putting to the fore different localizations in the subjet of the function and in clarifying the roles of different notions of uniform or equi-differentiability. Because the passages from functions to sets and from sets to functions have been dealt with in several other papers ([9], [11], [15] , [16], [22], [28]-[31], [43]...), for the sake of brevity, we do not explore all the possibilities but just present some of them.

$\boxed{2}$ Preliminaries

In the sequel X is a Banach space with dual space X^* and $\mathcal{F}(X)$ (resp. $\mathcal{S}(X)$) denotes the set of all functions (resp. lower semicontinuous (l.s.c.) functions) from X to $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ with nonempty domain. For more information about the basic concepts of nonsmooth analysis we recall in this section, we refer to the recent monographs of Borwein and Zhu ([9]), Mordukhovich [25], Rockafellar and Wets [43], Schirotzek ([49]) and Zalinescu [52].

The *lower directional derivative* (or contingent derivative or Dini-Hadamard derivative) of $f \in \mathcal{F}(X)$ finite at $\overline{x}$ is given by

$$f^D(\overline{x}, v) := \liminf_{(t,w)\to(0_+,v)} \frac{1}{t}(f(\overline{x}+tw) - f(\overline{x})).$$

The *Clarke-Rockafellar* derivative [14], or *circa-derivative* of $f \in \mathcal{S}(X)$ at $\overline{x}$ is given by

$$f^C(\overline{x}, v) := \sup_{r>0} \limsup_{\substack{(t,y)\to(0_+,\overline{x}) \\ f(y)\to f(\overline{x})}} \inf_{w\in B(v,r)} \frac{1}{t}(f(y+tw) - f(y)).$$

The *directional* (or Dini-Hadamard) *subdifferential* and the *circa* (or Clarke-Rockafellar) *subdifferential* are associated to these derivatives by

$$\partial^D f(\overline{x}) = \{\overline{x}^* \in X^* : \overline{x}^* \leq f^D(\overline{x}, \cdot)\},$$
$$\partial^C f(\overline{x}) = \{\overline{x}^* \in X^* : \overline{x}^* \leq f^C(\overline{x}, \cdot)\}.$$

The *firm (or Fréchet) subdifferential* of $f \in \mathcal{F}(X)$ at $\overline{x}$ is the set $\partial^F f(\overline{x})$ of $\overline{x}^* \in X^*$ such that for all $\varepsilon > 0$ there exists some $\delta > 0$ for which

$$f(w) - f(\overline{x}) - \langle \overline{x}^*, w - \overline{x}\rangle \geq -\varepsilon \|w - \overline{x}\| \qquad \forall w \in B(\overline{x}, \delta).$$

Given an hyper-modulus π, i.e. a function $\pi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\lim_{r\to 0_+} \pi(r)/r = 0$, $\pi(0) = 0$, $\partial^F f(\overline{x})$ contains the π-*proximal subdifferential* of f at $\overline{x}$ which is the set $\partial^\pi f(\overline{x})$ of $\overline{x}^* \in X^*$ such that

$$f(w) - f(\overline{x}) - \langle \overline{x}^*, w - \overline{x}\rangle \geq -\pi(\|w - \overline{x}\|) \qquad \forall w \in X.$$

For $\pi(t) := t^2$, $\partial^\pi f(\overline{x})$ is just called the proximal subdifferential of f at $\overline{x}$. In fact, $\partial^F f(\overline{x})$ is the union of the π-proximal subdifferentials of f at $\overline{x}$ for π in the set of hyper-modulus.

The *limiting subdifferential* of f at $\overline{x}$ is defined as $\partial^L f(\overline{x}) := \limsup_{y\to_f \overline{x}} \partial^F f(y)$, where $y \to_f \overline{x}$ means that $y \to \overline{x}$ with $f(y) \to f(\overline{x})$, the limsup being taken in a sequential way with respect to the weak* topology.

As in [35], we say that f is *soft at* $\overline{x}$ if the limiting subdifferential $\partial^L f(\overline{x})$ of f at $\overline{x}$ coincides with $\partial^F f(\overline{x})$.

We call *subdifferential* a multifunction $\partial : \mathcal{F}(X) \times X \rightrightarrows X^*$ such that for all $f \in \mathcal{F}(X)$ and all $x \in X$ one has $\partial^F f(x) \subset \partial f(x) \subset \partial^C f(x)$. These inclusions imply that for all $f \in \mathcal{F}(X)$ the domain of ∂f is included in the domain of f and that ∂f is locally bounded if f is locally Lipschitzian. The inclusion $\partial^F f \subset \partial f$ could be relaxed into $\partial^\pi f \subset \partial f$ for some hyper-modulus π, but we require it for the sake of simplicity.

Given a subdifferential ∂, recall that the ∂-*subjet* (in short *subjet)* of $f \in \mathcal{F}(X)$ is the set

$$J^\partial f := \{(x, x^*, f(x)) : x \in X,\ f(x) < +\infty,\ x^* \in \partial f(x)\}.$$

This notion generalizes the concept of one-jet classically used in differential geometry. It plays a crucial role in [21], [36], [37] and in what follows.

$\boxed{3}$ A Spectrum of Subdifferentiable Functions

In differential calculus, the notion of function of class C^1 is at least as important as the notion of differentiability, as it carries a form of stability. How can one extend such a notion, or capture similar properties, in the case of nonsmooth functions?

In the sequel, $f \in \mathcal{S}(X)$, i.e. $f : X \to \mathbb{R}_\infty$ is a l.s.c. function finite somewhere and $\overline{x} \in \operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. The first concept of the following definition has been introduced (in a slightly different but equivalent form) in [22] in the case ∂ is the limiting subdifferential under the name of weakly regular function (WR) at $\overline{x}$ relative to $\overline{x}^*$. The third one has been used in [1] and [38]. The fourth one is a variant of a notion introduced and studied in [31], where $\partial = \partial^F$ and a quantitative estimate is required. The last one appears in [20]. The other ones seem to be new.

Definition 3.1. (a) The function f is said to be equi-(∂)-subdifferentiable at $(\overline{x}, \overline{x}^*)$, where $\overline{x}^* \in \partial f(\overline{x})$, if there exists some $\rho > 0$ such that for any $\varepsilon > 0$ one can find some $\delta > 0$ such that

$$\langle \overline{w}^*, y - \overline{x} \rangle \leq f(y) - f(\overline{x}) + \varepsilon \|y - \overline{x}\| \tag{3.1}$$

for all $\overline{w}^* \in \partial f(\overline{x}) \cap B(\overline{x}^*, \rho)$ and all $y \in B(\overline{x}, \delta)$.

(b) The function f is said to be boundedly equi-(∂)-subdifferentiable at $\overline{x}$ if $\partial f(\overline{x})$ is nonempty and if for any bounded subset B^* of X^* and any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.1) holds for all $\overline{w}^* \in \partial f(\overline{x}) \cap B^*$ and all $y \in B(\overline{x}, \delta)$.

(c) The function f is said to be equi-(∂)-subdifferentiable at $\overline{x}$ if $\partial f(\overline{x})$ is nonempty and if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.1) holds for all $\overline{w}^* \in \partial f(\overline{x})$ and all $y \in B(\overline{x}, \delta)$.

(d) The function f is said to be uniformly ∂-subdifferentiable around $(\overline{x}, \overline{x}^*)$, where $\overline{x}^* \in \partial f(\overline{x})$, if there exists some $\rho > 0$ such that for any $\varepsilon > 0$ one can find some $\delta > 0$ such that

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\| \tag{3.2}$$

for all $(x, x^*, f(x)) \in J^\partial f \cap B((\overline{x}, \overline{x}^*, f(\overline{x})), \rho)$ and all $y \in B(x, \delta)$.

(e) The function f is said to be boundedly uniformly ∂-subdifferentiable around $\overline{x}$ if $\partial f(\overline{x})$ is nonempty and if there exists some $\rho > 0$ such that for any bounded subset B^* of X^* and for any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.2) holds for all $(x, x^*, f(x)) \in J^\partial f \cap (B(\overline{x}, \rho) \times B^* \times B(f(\overline{x}), \rho))$ and all $y \in B(x, \delta)$.

(f) The function f is said to be uniformly ∂-subdifferentiable around $(\overline{x}, f(\overline{x}))$ if $\partial f(\overline{x})$ is nonempty and if there exists some $\rho > 0$ such that for any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.2) holds for all $(x, x^*, f(x)) \in J^\partial f \cap (B(\overline{x}, \rho) \times X^* \times B(f(\overline{x}), \rho))$ and all $y \in B(x, \delta)$.

(g) Given a subset S of X, f is said to be uniformly ∂-subdifferentiable on S if $\partial f(x)$ is nonempty for all $x \in S$ and if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.2) holds for all $(x, x^*, f(x)) \in J^\partial f \cap (S \times X^* \times \mathbb{R})$ and all $y \in B(x, \delta)$.

(h) f is said to be uniformly ∂-subdifferentiable around $\overline{x}$ if there exists some neighborhood of $\overline{x}$ on which f is uniformly ∂-subdifferentiable.

Remarks. (a) When f is equi-∂-subdifferentiable at $(\overline{x}, \overline{x}^*)$ and ρ is as in assertion (a), one has $\partial f(\overline{x}) \cap B(\overline{x}^*, \rho) \subset \partial^F f(\overline{x})$. When f is boundedly equi-∂-subdifferentiable at $\overline{x}$, one has

$$\partial f(\overline{x}) = \partial^F f(\overline{x}).$$

Similar assertions hold for the other notions. For such a reason, when the mention of ∂ is omitted in the sequel, that means that $\partial = \partial^F$.

(b) The preceding assertions can be formulated by using hyper-modulus or modulus. Here, as usual, a *modulus* is a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ which is continuous at 0, with $\mu(0) = 0$; $\pi : \mathbb{R}_+ \to \mathbb{R}_\infty$ is an hypermodulus if, and only if, μ given by $\mu(t) := \pi(t)/t$ for $t > 0$, $\mu(0) = 0$ is a modulus. Thus, a quantitative form can be given to the preceding notions, as in [31].

Let us compare the preceding notions. The assertions of the following proposition are displayed for the convenience of the reader. They are either obvious or easy consequences of a compactness argument. Let us recall that f is said to be *quiet* at $\overline{x} \in \operatorname{dom} f$ if $-f$ is calm at $\overline{x}$, i.e. if there exist $c > 0$ and $r > 0$ such that $f(x) - f(\overline{x}) \leq c \|x - \overline{x}\|$ for all $x \in B(\overline{x}, r)$. We observe that $\partial^F f(\overline{x})$ is bounded when f is quiet at $\overline{x}$.

Proposition 3.2. *The preceding properties (a)-(f) and (h) are ranked in increasing strength.*

If $\partial^F f(\overline{x})$ is closed and X is finite dimensional, f is boundedly equi-(∂)-subdifferentiable at $\overline{x}$ if, and only if, it is equi-(∂)-subdifferentiable at $(\overline{x}, \overline{x}^*)$ for all $\overline{x}^* \in \partial^F f(\overline{x})$.

If f is continuous at $\overline{x}$, f is uniformly ∂-subdifferentiable around $(\overline{x}, f(\overline{x}))$ if, and only if, f is uniformly ∂-subdifferentiable around $\overline{x}$.

If f is quiet at $\overline{x}$, f is boundedly equi-(∂)-subdifferentiable at $\overline{x}$ if, and only if, it is equi-(∂)-subdifferentiable at $\overline{x}$.

If f is Lipschitzian around $\overline{x}$, f is boundedly uniformly ∂-subdifferentiable around $\overline{x}$ if, and only if, it is uniformly ∂-subdifferentiable around $(\overline{x}, f(\overline{x}))$.

Example. The distinctions of Definition 3.1 may appear superfluous. However, even for differentiable functions, they exist. If f is differentiable at $\overline{x}$, it is obviously equi-subdifferentiable at $\overline{x}$ but it is not necessarily uniformly subdifferentiable around $\overline{x}$. This last property is satisfied if the derivative Df of f is uniformly continuous on some neighborhood of $\overline{x}$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \sqrt{x_+}$, where $x_+ := \max(x, 0)$. Then, as noted in [22, Example 1], f is equi-∂^F-subdifferentiable at $(0, 0)$. In fact, it is boundedly equi-∂^C-subdifferentiable at $\overline{x}$. But it is not equi-∂^F-subdifferentiable at 0, as easily seen. It is also boundedly uniformly ∂-subdifferentiable around $\overline{x} = 0$, but not uniformly ∂-subdifferentiable around $\overline{x} = 0$.

Example. Suppose f admits a lower approximation $s : X \to \mathbb{R}$ at $\overline{x}$, i.e. that there exists a modulus μ such that $f(\overline{x}+u) \geq f(\overline{x})+s(u)-\|u\|\,\mu(u)$ for all $u \in X$. Then, if s is sublinear and continuous, for every $\overline{x}^* \in \operatorname{int}(\partial s(0))$, f is equi-∂^F-subdifferentiable at $(\overline{x}, \overline{x}^*)$. Conversely, if f is equi-∂^F-subdifferentiable at $(\overline{x}, \overline{x}^*)$, then f admits a lower approximation s at $\overline{x}$ which is sublinear and continuous: one can define s by $s(u) := \sup\{\langle \overline{w}^*, u\rangle : \overline{w}^* \in \partial f(\overline{x}) \cap B(\overline{x}^*, \rho)\}$, where ρ is as in Definition 3.1 (a).

Obviously, convex functions satisfy all the conditions of Definition 3.1. In fact, property (h) of this definition corresponds to a well known class of functions.

Theorem 3.3. ***(Duda and Zajíček** [20]) Let f be continuous on some neighborhood of $\overline{x}$. Then the following assertions are equivalent:*

(a) *f is uniformly ∂-subdifferentiable around $\overline{x}$ and f is subdifferentiable around $\overline{x}$;*

(b) *there exists a neighborhood V of $\overline{x}$ such that for all $\varepsilon > 0$ one can find some $\delta > 0$ such that for all $x \in V$ one can pick some $x^* \in X^*$ satisfying (3.2) for all $y \in V \cap B(x, \delta)$;*

(c) *f is semiconvex around $\overline{x}$ in the sense that there exist some neighborhood U of $\overline{x}$ and some modulus μ such that for all $x, y \in U$, $t \in [0, 1]$*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + t(1-t)\mu(\|x-y\|)\,\|x-y\|\,.$$

On the other hand, property (a) of the preceding definition is satisfied under a weak convexity assumption.

Definition 3.4 ([27]). A function $f \in \mathcal{F}(X)$ finite at $\overline{x}$, is said to be approximately starshaped at $\overline{x}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(\overline{x}, \delta)$, $t \in [0,1]$, one has

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t) \|x - \overline{x}\| . \tag{3.3}$$

Proposition 3.5. *If f is continuous around $\overline{x}$, approximately starshaped at $\overline{x}$ and if f is ∂^D-subdifferentiable at $\overline{x}$, then f is equi-∂^D-subdifferentiable at $\overline{x}$ and $\partial^D f(\overline{x}) = \partial^F f(\overline{x})$.*

Proof. Given $\varepsilon > 0$, let $\delta \in (0,1)$ be such that relation (3.3) is satisfied for any $x \in B(\overline{x}, \delta)$, $t \in (0,1]$. Then, for all $y \in B(\overline{x}, \delta)$, dividing by t and taking the limit inferior as $(t,x) \to (0_+, y)$, one gets

$$f^D(\overline{x}, y - \overline{x}) \leq f(y) - f(\overline{x}) + \varepsilon \|y - \overline{x}\| .$$

Thus, for all $\overline{w}^* \in \partial^D f(\overline{x})$, relation (3.1) is satisfied. The equality $\partial^D f(\overline{x}) = \partial^F f(\overline{x})$ is obvious. $\square$

Remark. When f is approximately convex at $\overline{x}$ in the sense of [26], one has a property which is intermediate between properties (a) and (h) of Definition 3.1: for any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.2) holds for all $(x, x^*, f(x)) \in J^\partial f \cap (B(\overline{x}, \delta) \times X^* \times \mathbb{R})$ and all $y \in B(x, \delta)$.

Proposition 3.6. *If, for all $\overline{x}^* \in \partial f(\overline{x})$, f is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$, then $\partial f(\overline{x})$ is locally closed. If f is equi-subdifferentiable at $\overline{x}$, then $\partial f(\overline{x}) = \partial^F f(\overline{x})$ is weakly* closed.*

Proof. Given $\overline{x}^* \in \partial f(\overline{x})$ and $\rho > 0$ as in Definition 3.1 (a), the assertion follows from a passage to the limit in relation (3.1) over a sequence $(\overline{w}_n^*)$ in $\partial f(\overline{x}) \cap B(\overline{x}^*, \rho)$ converging to some $\overline{w}^*$. When f is equi-subdifferentiable at $\overline{x}$, one can take an arbitrary weakly* converging net in $\partial f(\overline{x})$. $\square$

Remark. In [22, Thm 4.2] it is stated that f is soft at $\overline{x}$ (i.e. $\partial^L f(\overline{x}) = \partial^F f(\overline{x})$) whenever f is equi-subdifferentiable at $\overline{x}$ when X is finite dimensional. The following counter-example shows that such an assertion is not true.

Example. Given a sequence $(r_n) \to 0_+$ let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(r_n) := 0$ for all $n \in \mathbb{N}$ and $f(x) := x^2$ for $x \in \mathbb{R} \backslash \{r_n : n \in \mathbb{N}\}$. Then $\partial^F f(0) = \{0\}$, so that f is equi-subdifferentiable at $\overline{x}$ but $\partial^L f(0) = \mathbb{R}$, so that f is not soft at 0.

On the other hand the implication (ii)$\Rightarrow$(i) of [22, Thm 4.2] can be given a weaker form.

Proposition 3.7. *If f is continuous at $\overline{x}$ and boundedly uniformly subdifferentiable around $\overline{x}$, then f is soft on some neighborhood V of $\overline{x}$. If moreover X is an Asplund space, then f is firmly regular on V in the sense that $\partial^C f(x) = \partial^F f(x)$ for all $x \in V$, hence is regular in the sense of [14] that $\partial^C f(x) = \partial^D f(x)$ for all $x \in V$.*

Proof. Let $\partial := \partial^F$. Let $\rho > 0$ be such that for any bounded subset B^* of X^* and for any $\varepsilon > 0$ one can find some $\delta > 0$ such that (3.2) holds for all $(x, x^*, f(x)) \in J^\partial f \cap (B(\overline{x}, \rho) \times B^* \times B(f(\overline{x}), \rho))$ and all $y \in B(x, \delta)$. Let $x \in V := B(\overline{x}, \rho')$, $x^* \in \partial^L f(x)$, where $\rho' \in (0, \rho]$ is small enough to ensure that $f(w) \in B(f(\overline{x}), \rho)$ for all $w \in B(\overline{x}, \rho')$. Let $((x_n, x_n^*, f(x_n))$ be a sequence of $J^\partial f$ converging to $(x, x^*, f(x))$, the convergence of (x_n^*) being taken in the weak* topology. Taking for B^* a bounded subset of X^* containing $\{x_n^* : n \in \mathbb{N}\}$, for every $\varepsilon > 0$ one can find some $\delta > 0$ such that

$$\forall y \in B(x_n, \delta) \qquad \langle x_n^*, y - x_n \rangle \leq f(y) - f(x_n) + \varepsilon \|y - x_n\|$$

for n large enough to ensure $\|x_n - \overline{x}\| < \rho'$. Passing to the limit on n in this inequality one gets

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\|$$

for all $y \in B(x, \delta)$. Thus $x^* \in \partial^F f(x)$ and f is soft at x.

When X is an Asplund space, $\partial^C f(x)$ is the weak* closed convex hull of $\partial^L f(x)$. Thus, every element of $\partial^C f(x)$ satisfies the preceding relation, hence belongs to $\partial^F f(x)$. □

4 Stability Properties

We first consider the stability of the classes defined above under usual operations.

Theorem 4.1. *Let $f = \sup_{i \in I} f_i$, where $(f_i)_{i \in I}$ is a finite family of functions which are equi-∂^D-subdifferentiable at $\overline{x}$. Then f is equi-∂^D-subdifferentiable at $\overline{x}$.*

Proof. We first prove that for $I(\overline{x}) := \{i \in I : f_i(\overline{x}) = f(\overline{x})\}$ we have

$$\partial^D f(\overline{x}) = \mathrm{cl}^* \mathrm{co}(\bigcup_{i \in I(\overline{x})} \partial^D f_i(\overline{x})).$$

It suffices to show that the support functions of both sides are equal, what follows from the equality

$$f^D(\overline{x}, u) = \max_{i \in I(\overline{x})} f_i^D(\overline{x}, u),$$

which can be easily shown. Now, for every $i \in I(\overline{x})$, $\varepsilon > 0$, one can find some $\delta_i > 0$ such that

$$\forall \overline{w}^* \in \partial^D f_i(\overline{x}),\ \forall x \in B(\overline{x}, \delta_i) \qquad \langle \overline{w}^*, x - \overline{x} \rangle \leq f_i(x) - f_i(\overline{x}) + \varepsilon \|x - \overline{x}\|.$$

Setting $\delta := \min_{i \in I(\overline{x})} \delta_i$, taking convex combinations, one gets that

$$\forall \overline{w}^* \in \mathrm{co}(\bigcup_{i \in I(\overline{x})} \partial^D f_i(\overline{x})),\ \forall x \in B(\overline{x}, \delta) \qquad \langle \overline{w}^*, x - \overline{x} \rangle \leq f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|.$$

Taking a weak* converging net, one sees that the preceding inequality remains valid for all $\overline{w}^* \in \partial^D f(\overline{x})$ and all $x \in B(\overline{x}, \delta)$. Thus, f is equi-∂^D-subdifferentiable at $\overline{x}$. □

The following stability result completes [26, Thm 3.8]. It has some similarities with [31, Cor 2] but the assumptions and the conclusions are different. Note that here h is not supposed to be convex but the qualification condition is stronger than the Robinson qualification condition.

Theorem 4.2. *Let X and Y be Banach spaces, let W be an open subset of X and let $f := h \circ g$, where $g : W \to Y$ is differentiable at $\overline{x} \in W$ and $h : Y \to \mathbb{R}_\infty$ is finite at $\overline{y} := g(\overline{x})$. Suppose $A := Dg(\overline{x})$ is surjective and g is open at $\overline{x}$ with a linear rate.*

(a) *If h is equi-subdifferentiable at $(\overline{y}, \overline{y}^*)$, then f is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$, for $\overline{x}^* := \overline{y}^* \circ Dg(\overline{x})$.*

(b) *If h is boundedly equi-subdifferentiable at $\overline{y}$, then f is boundedly equi-subdifferentiable at $\overline{x}$.*

Proof. (a) Because g is open at $\overline{x}$ with a linear rate, one can easily show that

$$\partial^F f(\overline{x}) = \partial^F h(\overline{y}) \circ Dg(\overline{x}).$$

Moreover, there exists some $c > 0$ such that for every $\overline{w}^* \in \partial^F f(\overline{x})$ there exists some $\overline{z}^* \in \partial^F h(\overline{y})$ such that $\overline{w}^* = \overline{z}^* \circ Dg(\overline{x})$ and $\|\overline{z}^*\| \leq c\,\|\overline{w}^*\|$. In fact, if $c > 0$ is such that $B_Y \subset A(cB_X)$ for $A := Dg(\overline{x})$, then for every $\overline{w}^* \in A^T(Y^*)$ there exists a unique $\overline{z}^* \in Y^*$ such that $\overline{w}^* = \overline{z}^* \circ Dg(\overline{x})$ and $\|\overline{z}^*\| \leq c\,\|\overline{w}^*\|$. Uniqueness stems from the surjectivity of A; the estimate is a consequence of the fact that for every $y \in B_Y$ there exists some $x \in cB_X$ satisfying $y = A(x)$.

Let $\rho > 0$ be such that for all $\varepsilon > 0$ one can find some $\delta := \delta(\varepsilon) > 0$ such that

$$\langle \overline{z}^*, y - \overline{y} \rangle \leq h(y) - h(\overline{y}) + \varepsilon\,\|y - \overline{y}\|$$

for all $\overline{z}^* \in \partial^F h(\overline{y}) \cap B(\overline{y}^*, \rho)$ and all $y \in B(\overline{y}, \delta)$. Let $c' > \|A\| := \|Dg(\overline{x})\|$ and let $\rho' := \rho/c'$. Since g is differentiable at $\overline{x}$, given $\varepsilon \in (0, c' - \|A\|)$, there exists $\gamma := \gamma(\varepsilon) \in (0, \delta(\varepsilon)/c')$ such that $\|g(x) - g(\overline{x}) - A(x - \overline{x})\| \leq \varepsilon\,\|x - \overline{x}\|$ for all $x \in B(\overline{x}, \gamma)$. Then, for all $x \in B(\overline{x}, \gamma)$ and for all $\overline{w}^* \in \partial^F f(\overline{x}) \cap B(\overline{x}^*, \rho')$ one has $\|g(x) - g(\overline{x})\| \leq c'\,\|x - \overline{x}\| \leq c'\gamma \leq \delta(\varepsilon)$ and $\overline{w}^* := \overline{z}^* \circ A$ with $\overline{z}^* \in \partial^F h(\overline{y}) \cap B(\overline{y}^*, \rho)$, so that

$$\begin{aligned}
\langle \overline{w}^*, x - \overline{x} \rangle = \langle \overline{z}^*, A(x - \overline{x}) \rangle &\leq \langle \overline{z}^*, g(x) - g(\overline{x}) \rangle + \|\overline{z}^*\|.\varepsilon\,\|x - \overline{x}\| \\
&\leq h(g(x)) - h(g(\overline{x})) + \varepsilon\,\|g(x) - g(\overline{x})\| + \|\overline{z}^*\|.\varepsilon\,\|x - \overline{x}\| \\
&\leq f(x) - f(\overline{x}) + \varepsilon\,(c' + c\,\|\overline{x}^*\| + \rho)\,\|x - \overline{x}\|.
\end{aligned}$$

Since ε is arbitrarily small, these inequalities show that (3.1) is satisfied.

The proof of assertion (b) is similar, taking into account the fact shown above that for every bounded subset B^* of $A^T(Y^*)$ there exists a bounded subset C^* of Y^* such that $B^* = A^T(C^*)$. $\square$

The proof of the following statement is obvious. When $\partial = \partial^L$, its second hypothesis can be ensured by assuming some compactness condition and some asymptotic subdifferential condition.

Theorem 4.3. *Let $f := f_1 + f_2$, where f_1 and f_2 are equi-∂-subdifferentiable at $\overline{x}$. Suppose $\partial f(\overline{x}) = \partial f_1(\overline{x}) + \partial f_2(\overline{x})$. Then f is equi-∂-subdifferentiable at $\overline{x}$.*

5 Subdifferential Characterizations

A number of classes of generalized convex functions on an Asplund space X have been given subdifferential characterizations ([4], [16], [18], [28], [29]...). In the present section we examine whether the same can be done for the classes we defined above.

In this section, X is an Asplund space and ∂ stands for the Fréchet subdifferential. We say that a multimap (or multifunction) $M : X \rightrightarrows X^*$ is *semimonotone* if there exists a modulus μ such that

$$\forall (x, x^*), (y, y^*) \in \mathrm{gph}(M) \quad \langle x^* - y^*, x - y \rangle \geq -\mu(\|x - y\|)\,\|x - y\|.$$

It is said to be *submonotone at $\overline{x} \in X$* if there exists a modulus μ such that

$$\forall \overline{x}^* \in M(\overline{x}), (y, y^*) \in \mathrm{gph}(M) \quad \langle \overline{x}^* - y^*, \overline{x} - y \rangle \geq -\mu(\|\overline{x} - y\|)\,\|\overline{x} - y\|.$$

We start with a qualitative reformulation of [31, Thm 4] which is a result of quantitative nature.

Theorem 5.1. *For a l.s.c. function f on X and $\overline{x}^* \in \partial^F f(\overline{x})$ the following assertions are equivalent:*

(a) *f is uniformly ∂-subdifferentiable around $(\overline{x}, \overline{x}^*)$;*

(b) *there exists $\rho > 0$ such that, for $V := B((\overline{x}, \overline{x}^*, f(\overline{x})), \rho)$, the multimap $M : X \rightrightarrows X^*$ given by*

$$\mathrm{gph}(M) := \{(x, x^*) : (x, x^*, f(x)) \in J^\partial f \cap V\},$$

is semimonotone.

The proof of the next result is much simpler. It relies on the following version of the Mean Value Theorem.

Lemma 5.2. *Let f be a l.s.c. function on X and let $x, y \in \operatorname{dom} f$. Then there exist $t \in (0, 1]$, sequences $(z_n) \to z := x + t(y - x)$, (z_n^*) such that $z_n^* \in \partial f(z_n)$ for all $n \in \mathbb{N}$ and*

$$\liminf_n \langle z_n^*, x - y\rangle \geq f(x) - f(y), \tag{5.1}$$

$$\liminf_n \langle z_n^*, z - z_n\rangle \geq 0. \tag{5.2}$$

Theorem 5.3. *For a l.s.c. function f on X and $\overline{x} \in \operatorname{dom} \partial f$, the following assertions are equivalent:*

(a) *f is uniformly ∂-subdifferentiable around $\overline{x}$;*

(b) *there exists $\rho > 0$ such that for $V := B(\overline{x}, \rho) \times X^* \times \mathbb{R}$, the multimap $M : X \rightrightarrows X^*$ given by*

$$\mathrm{gph}(M) := \{(x, x^*) : (x, x^*, f(x)) \in J^\partial f \cap V\}, \tag{5.3}$$

is semimonotone.

Proof. (a)⇒(b) Let $\rho > 0$ be such that for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for all $(x, x^*, f(x)) \in J^\partial f \cap V$, with $V := B(\overline{x}, \rho) \times X^* \times \mathbb{R}$, and all $y \in B(x, \delta)$ one has

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon \left\| y - x \right\|. \tag{5.4}$$

In particular, if $(x, x^*, f(x)) \in J^\partial f \cap V$ and $(y, y^*, f(y)) \in J^\partial f \cap V$ are such that $\|x - y\| < \delta$, one has a similar relation by interchanging (x, x^*) with (y, y^*). Adding sides by sides the two relations, one gets

$$\langle x^* - y^*, y - x\rangle \leq 2\varepsilon \left\| y - x \right\|,$$

so that M is semimonotone.

(b)⇒(a) Let $\rho > 0$, V and M be as in assertion (b), so that, for every $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ such that for all $(x, x^*), (y, y^*) \in \mathrm{gph}(M)$ satisfying $\|x - y\| < \delta$ one has

$$\langle x^* - y^*, y - x\rangle \leq \varepsilon \left\| x - y \right\|.$$

Given $\varepsilon > 0$, let $\delta' := \min(\delta(\varepsilon), \rho/2)$. Let $V' := B(\overline{x}, \rho/2) \times X^* \times \mathbb{R}$. Then, for every $(x, x^*, f(x)) \in J^\partial f \cap V'$ and $y \in B(x, \delta')$ one has $y \in B(\overline{x}, \rho)$. When $f(y) = +\infty$, relation (5.4) is trivially satisfied. Thus, we may suppose $f(y) < +\infty$ and apply the preceding lemma: there exist $t \in (0, 1]$ and sequences $(z_n) \to z := x + t(y - x)$, (z_n^*) such that $z_n^* \in \partial f(z_n)$ for all $n \in \mathbb{N}$ satisfying (5.1), (5.2). Then

$$\begin{aligned}
\langle x^*, y - x\rangle + f(x) - f(y) &\leq \langle x^*, y - x\rangle + \liminf_n \langle z_n^*, x - y\rangle \\
&\leq \langle x^*, y - x\rangle + \liminf_n (\langle x^*, x - y\rangle + \langle z_n^* - x^*, t^{-1}(x - z)\rangle) \\
&\leq \liminf_n t^{-1}(\langle z_n^* - x^*, x - z_n\rangle + \langle z_n^* - x^*, z_n - z\rangle) \\
&\leq \liminf_n t^{-1}(\varepsilon \left\| x - z_n \right\| + \langle z_n^*, z_n - z\rangle) \leq t^{-1}\varepsilon \left\| x - z \right\| = \varepsilon \left\| x - y \right\|.
\end{aligned}$$

Thus, (5.4) is satisfied. □

Now we give a variant of [22, Thm 7.1]; here we use the Fréchet subdifferential instead of the limiting subdifferential and f is not supposed to be locally Lipschitzian.

Proposition 5.4. *Let f be a l.s.c. function on X, finite at $\overline{x}$, with $\partial^F f(\overline{x})$ nonempty. If $\partial^F f$ is submonotone at $\overline{x}$, then f is equi-subdifferentiable at $\overline{x}$.*

Proof. By assumption, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\overline{x}^* \in \partial^F f(\overline{x})$, $(y, y^*) \in \mathrm{gph}(\partial^F f)$ satisfying $\|\overline{x} - y\| < \delta$ one has

$$\langle \overline{x}^* - y^*, y - \overline{x} \rangle \le \varepsilon \|\overline{x} - y\| .$$

Let us show that, for all $\overline{x}^* \in \partial^F f(\overline{x})$ and all $y \in B(\overline{x}, \delta)$,

$$\langle \overline{x}^*, y - \overline{x} \rangle \le f(y) - f(\overline{x}) + \varepsilon \|y - \overline{x}\| . \tag{5.5}$$

Given $\overline{x}^* \in \partial f(\overline{x})$ and $y \in B(\overline{x}, \delta) \cap \mathrm{dom}\, f$, Lemma 5.2 yields $t \in (0, 1]$, sequences $(z_n) \to z := \overline{x} + t(y - \overline{x})$, (z_n^*) with $z_n^* \in \partial^F f(z_n)$ for all $n \in \mathbb{N}$, such that relations (5.1), (5.2) hold. Then $z_n \in B(\overline{x}, \delta)$ for n large enough and, by these relations,

$$\begin{aligned}
\langle \overline{x}^*, y - \overline{x} \rangle &= \lim_n (1/t) \langle \overline{x}^*, z_n - \overline{x} \rangle \le \liminf_n (1/t)(\langle z_n^*, z_n - \overline{x} \rangle + \varepsilon \|z_n - \overline{x}\|) \\
&\le \liminf_n (1/t)(\langle z_n^*, z - \overline{x} \rangle + \varepsilon \|z - \overline{x}\|) \\
&\le \limsup_n (\langle z_n^*, y - \overline{x} \rangle + \varepsilon \|y - \overline{x}\|) \\
&\le f(y) - f(\overline{x}) + \varepsilon \|y - \overline{x}\| .
\end{aligned}$$

When $y \in B(\overline{x}, \delta) \setminus \mathrm{dom}\, f$, inequality (5.5) is trivial. □

6 Nice Sets and Functions

We devote the present section to the passages from functions to sets. It appears that when the subjets of the functions are involved, these passages are nicer than the corresponding ones in the case of approximate functions or paraconvex functions (see [28], [29]). We recall that the *firm or Fréchet normal cone* to a subset E of X at some point $\overline{x}$ of E is $N^F(E, \overline{x}) := \partial^F \iota_E(\overline{x})$, where the *indicator function* ι_E of E is defined by $\iota_E(x) = 0$ for $x \in E$, $\iota_E(x) := +\infty$ for $x \in X \setminus E$. The following results complete [31, Thm 2].

Theorem 6.1. *Let E be a closed subset of X and let $\overline{x} \in E$, $\overline{x}^* \in N^F(E, \overline{x}) \cap B_{X^*}$. The following assertions are equivalent:*

(a) *The set E is equi-normal at $(\overline{x}, \overline{x}^*)$ in the sense that its indicator function ι_E is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$;*

(b) *The distance function d_E is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$.*

Proof. (a)⇒(b) Assumption (a) can be expressed as: there exists some $\rho \in (0, 1)$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ for which

$$\forall x \in E \cap B(\overline{x}, \delta),\ \forall \overline{w}^* \in N^F(E, \overline{x}) \cap B(\overline{x}^*, \rho) \quad \langle \overline{w}^*, x - \overline{x} \rangle \le \varepsilon \|x - \overline{x}\| . \tag{6.1}$$

Given $\varepsilon > 0$, let $y \in B(\overline{x}, \delta/2)$ and let $\overline{w}^* \in \partial^F d_E(\overline{x}) \cap B(\overline{x}^*, \rho)$. One can find a sequence (x_n) in E such that $(\|x_n - y\|) \to d_E(y)$ and $\|x_n - y\| \le \|y - \overline{x}\|$. Then $x_n \in B(\overline{x}, \delta)$ since

$\|x_n - \overline{x}\| \leq \|x_n - y\| + \|y - \overline{x}\| \leq 2\|y - \overline{x}\|$. Since $\overline{w}^* \in \partial^F d_E(\overline{x}) = N^F(E, \overline{x}) \cap B_{X^*}$, relation (6.1) ensures that

$$\begin{aligned}\langle \overline{w}^*, y - \overline{x}\rangle &\leq \limsup_n \left(\langle \overline{w}^*, x_n - \overline{x}\rangle + \|\overline{w}^*\| . \|x_n - y\|\right)\\ &\leq \limsup_n \left(\varepsilon \|x_n - \overline{x}\| + \|x_n - y\|\right) \leq 2\varepsilon \|y - \overline{x}\| + d_E(y).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, that shows that d_E is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$.

(b)⇒(a). Let $\rho \in (0,1)$ be such that for all $\varepsilon > 0$ there exists $\delta > 0$ for which

$$\forall y \in B(\overline{x}, \delta),\ \forall \overline{z}^* \in \partial^F d_E(\overline{x}) \cap B(\overline{x}^*, \rho) \quad \langle \overline{z}^*, y - \overline{x}\rangle \leq d_E(y) + \varepsilon \|y - \overline{x}\|. \tag{6.2}$$

Then, for all $x \in E \cap B(\overline{x}, \delta)$ and for all $\overline{w}^* \in N^F(E, \overline{x}) \cap B(\overline{x}^*, \rho^2)$, we have

$$\begin{aligned}\overline{z}^* := (1+\rho)^{-1}\overline{w}^* \in N^F(E, \overline{x}) \cap (1+\rho)^{-1}(1+\rho^2)B_{X^*} \subset \partial^F d_E(\overline{x}),\\ \|\overline{z}^* - \overline{x}^*\| \leq (1+\rho)^{-1}\|\overline{w}^* - \overline{x}^*\| + \rho(1+\rho)^{-1}\|\overline{x}^*\| \leq (1+\rho)^{-1}\rho^2 + \rho(1+\rho)^{-1} = \rho,\end{aligned}$$

so that, by (6.2), $\langle \overline{z}^*, x - \overline{x}\rangle \leq \varepsilon \|x - \overline{x}\|$ and $\langle \overline{w}^*, x - \overline{x}\rangle \leq (1+\rho)\varepsilon \|x - \overline{x}\| \leq 2\varepsilon \|x - \overline{x}\|$ so that (6.1) holds with ρ replaced with ρ^2 and ε replaced by 2ε. □

The following consequence stems from the fact that for $\overline{x}^* \in N^F(E, \overline{x})\backslash\{0\}$, E is equi-normal at $(\overline{x}, \overline{x}^*)$ if, and only if, E is equi-normal at $(\overline{x}, \overline{u}^*)$ where $\overline{u}^* := \|\overline{x}^*\|^{-1}\overline{x}^*$.

Corollary 6.2. *For $\overline{x}$ in a closed subset E of X the following assertions are equivalent:*

(a) *The indicator function ι_E of E is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$ for every $\overline{x}^* \in N^F(E, \overline{x})$;*

(b) *The distance function d_E is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$ for every $\overline{x}^* \in \partial^F d_E(\overline{x})$.*

The proofs of the next equivalences are simpler than the preceding one since it suffices to use the relation $\partial^F d_E(\overline{x}) = N^F(E, \overline{x}) \cap B_{X^*}$ and the fact that d_E is continuous.

Theorem 6.3. *Let E be a closed subset of X and let $\overline{x} \in E$. The following assertions are equivalent:*

(a) *The set E is boundely equi-normal at $\overline{x}$ in the sense that its indicator function ι_E is boundedly equi-subdifferentiable at $\overline{x}$;*

(b) *The distance function d_E is equi-subdifferentiable at $\overline{x}$.*

Theorem 6.4. *Let E be a closed subset of X and let $\overline{x} \in E$. The following assertions are equivalent:*

(a) *The set E is boundedly uniformly normal around $\overline{x}$ in the sense that its indicator function ι_E is boundedly uniformly subdifferentiable around $\overline{x}$;*

(b) *The distance function d_E is uniformly subdifferentiable on E around $\overline{x}$.*

Now let us turn to the links between a function f and its epigraph E_f. For $\overline{x} \in \operatorname{dom} f$ we set $\overline{x}_f := (\overline{x}, f(\overline{x}))$. We endow $X \times \mathbb{R}$ with a *product norm*, i.e. a norm for which the projections and the insertions $x \mapsto (x, 0)$ and $r \mapsto (0, r)$ are nonexpansive.

Theorem 6.5. *For f lower semicontinuous and $\overline{x} \in \operatorname{dom} f$, $\overline{x}^* \in \partial f(\overline{x})$, the following assertions are equivalent:*

(a) *The function f is equi-subdifferentiable at $(\overline{x}, \overline{x}^*)$;*

(b) *The epigraph E_f of f is equi-normal at $(\overline{x}_f, (\overline{x}^*, -1))$.*

Proof. (a)⇒(b) Let $\rho \in (0,1)$ be such that for all $\varepsilon > 0$ one can find some $\delta > 0$ such that

$$\langle \overline{w}^*, y - \overline{x} \rangle \leq f(y) - f(\overline{x}) + \varepsilon \|y - \overline{x}\| \tag{6.3}$$

for all $\overline{w}^* \in \partial f(\overline{x}) \cap B(\overline{x}^*, \rho)$ and all $y \in B(\overline{x}, \delta)$. Let $\sigma \in (0, \rho/2(1 + \|\overline{x}^*\|))$ and let $(\overline{w}^*, -\overline{r}) \in N^F(E_f, \overline{x}_f) \cap B((\overline{x}^*, -1), \sigma)$. Then one has $\overline{r} > 1/2$ and $\overline{w}^*/\overline{r} \in \partial^F f(\overline{x})$ with

$$\|\overline{w}^*/\overline{r} - \overline{x}^*\| \leq (1/\overline{r}) \|\overline{w}^* - \overline{x}^*\| + (1/\overline{r}) |\overline{r} - 1| \|\overline{x}^*\| \leq 2\sigma(1 + \|\overline{x}^*\|) \leq \rho.$$

Thus, for all $(y, r) \in E_f \cap B(\overline{x}_f, \delta)$, relation (6.3) yields

$$\begin{aligned} \langle (\overline{w}^*, -\overline{r}), (y, r) - (\overline{x}, f(\overline{x})) \rangle &\leq \overline{r} \left(\langle \overline{w}^*/\overline{r}, y - \overline{x} \rangle - (f(y) - f(\overline{x})) \right) \\ &\leq 2\varepsilon \|y - \overline{x}\| \leq 2\varepsilon \|(y, r) - (\overline{x}, f(\overline{x}))\|, \end{aligned}$$

so that E_f is equi-normal at $(\overline{x}_f, (\overline{x}^*, -1))$.

(b)⇒(a) Let $\rho \in (0,1)$ be such that for all $\varepsilon > 0$ there exists $\gamma > 0$ for which

$$\langle \overline{w}^*, y - \overline{x} \rangle - \overline{r}(r - f(\overline{x})) \leq \varepsilon \|(y, r) - (\overline{x}, f(\overline{x}))\| \tag{6.4}$$

whenever $(y, r) \in B((\overline{x}, f(\overline{x})), \gamma) \cap E_f$ and $(\overline{w}^*, -\overline{r}) \in N^F(E_f, \overline{x}_f) \cap B((\overline{x}^*, -1), \rho)$. Let $\overline{w}^* \in \partial^F f(\overline{x}) \cap B(\overline{x}^*, \rho)$, so that $(\overline{w}^*, -1) \in N^F(E_f, \overline{x}_f) \cap B((\overline{x}^*, -1), \rho)$. Let $c := \|\overline{x}^*\| + 1$ and let $\delta \in (0, \gamma/(c+1))$. Let $y \in B(\overline{x}, \delta)$. When $r := f(\overline{x}) + c \|y - \overline{x}\| \geq f(y)$ we have $(y, r) \in B((\overline{x}, f(\overline{x})), \gamma) \cap E_f$, hence

$$\begin{aligned} \langle \overline{w}^*, y - \overline{x} \rangle - (f(y) - f(\overline{x})) &\leq \varepsilon \|(y, r) - (\overline{x}, f(\overline{x}))\| \\ &\leq \varepsilon \|y - \overline{x}\| + \varepsilon |r - f(\overline{x})| = \varepsilon(c+1) \|y - \overline{x}\|. \end{aligned}$$

When $f(\overline{x}) + c \|y - \overline{x}\| < f(y)$ we have

$$\langle \overline{w}^*, y - \overline{x} \rangle - (f(y) - f(\overline{x})) < \langle \overline{w}^*, y - \overline{x} \rangle - c \|y - \overline{x}\| \leq 0.$$

In both cases relation (6.3) is satisfied with ε changed into $\varepsilon(c+1)$. □

The proof of the next result is similar.

Theorem 6.6. *Let f be lower semicontinuous and let $\overline{x} \in \operatorname{dom} f$. The following assertion (b) implies assertion (a). If f is quiet at $\overline{x}$, both assertions are equivalent:*

(a) *The function f is boundedly equi-subdifferentiable at $\overline{x}$;*

(b) *The epigraph E_f of f is boundedly equi-normal at $\overline{x}_f$.*

Proof. We just mention the necessary changes.

(b)⇒(a) Given a bounded subset B^* of X^* and $\varepsilon > 0$ one can find $\gamma > 0$ such that

$$\langle \overline{w}^*, y - \overline{x} \rangle - \overline{r}(r - f(\overline{x})) \leq \varepsilon \|(y, r) - (\overline{x}, f(\overline{x}))\|$$

whenever $(y, r) \in B((\overline{x}, f(\overline{x})), \gamma) \cap E_f$ and $(\overline{w}^*, -\overline{r}) \in N^F(E_f, \overline{x}_f) \cap (B^* \times \{-1\})$ which is bounded. Let $c := \sup\{\|b^*\| : b^* \in B^*\} + 1$ and let $\delta := \gamma/(c+1)$. Then, as in the preceding proof, for every $\overline{w}^* \in \partial^F f(\overline{x}) \cap B^*$ and for every $y \in B(\overline{x}, \delta)$, we get that relation (6.3) is satisfied with ε changed into $\varepsilon(c+1)$.

(a)⇒(b) when f is quiet at $\overline{x}$. Let $c > 0$ and $q > 0$ be such that $f(y) - f(\overline{x}) \leq c \|y - \overline{x}\|$ for all $y \in B(\overline{x}, q)$. Then the cone $Q := \{(u, r) \in X \times \mathbb{R} : \ r \geq c \|u\|\}$ is contained in the tangent cone to E_f at $\overline{x}_f$. It follows that $N^F(E_f, \overline{x}_f)$ is contained in the polar cone of Q which is

$$Q^0 = \{(u^*, -r^*) \in X^* \times \mathbb{R} : \ r^* \geq (1/c) \|u^*\|\}.$$

Let A^* be a subset of $N^F(E_f, \overline{x}_f)$ contained in the ball with center $(0,0)$ and radius $a > 0$. Since A^* is contained in Q^0, for every $(\overline{u}^*, -r^*) \in A^* \backslash \{(0,0)\}$ we have $r^* > 0$ and $\overline{x}^* := \overline{u}^*/r^* \in \partial^F f(\overline{x}) \subset cB_{X^*}$. Thus, for all $\varepsilon > 0$, one can find some $\delta > 0$ such that

$$\langle \overline{u}^*/r^*, y - \overline{x} \rangle \leq f(y) - f(\overline{x}) + \varepsilon \|y - \overline{x}\|$$

for all $(\overline{u}^*, -r^*) \in A^* \backslash \{(0,0)\}$ and all $y \in B(\overline{x}, \delta)$. It follows that

$$\langle \overline{u}^*, y - \overline{x} \rangle - r^*(r - f(\overline{x})) \leq r^* \varepsilon \|y - \overline{x}\| \leq a\varepsilon \|(y,r) - (\overline{x}, f(\overline{x}))\|$$

for all $(\overline{u}^*, -r^*) \in A^*$, all $y \in B(\overline{x}, \delta)$ and all $r \geq f(y)$. $\square$

The proof of the next result is not as simple; we refer to [31, Thm 3].

Theorem 6.7. *([31, Thm 3]) For f l.s.c. and $\overline{x} \in \operatorname{dom} f$, $\overline{x}^* \in \partial f(\overline{x})$, the following assertions are equivalent:*

(a) *The function f is uniformly subdifferentiable around $(\overline{x}, \overline{x}^*)$;*

(b) *The epigraph E_f of f is uniformly normal around $(\overline{x}_f, (\overline{x}^*, -1))$ in the sense that its indicator function is uniformly subdifferentiable around $(\overline{x}_f, (\overline{x}^*, -1))$.*

Now let us consider the case of sublevel sets.

Proposition 6.8. *Let X be an Asplund space and let $f : X \to \mathbb{R}$ be a continuous function. Let $S := \{x \in X : f(x) \leq 0\}$ and let $\overline{x} \in X$ be such that $f(\overline{x}) = 0$. Suppose f is equi-∂^D-subdifferentiable at $\overline{x}$ and $\liminf_{x \to \overline{x}, x \in X \backslash S} d(0, \partial^F f(x)) > 0$. Then S is equi-normal at $\overline{x}$.*

Proof. We can pick $c > 0$, $r > 0$ such that $\|x^*\| \geq c$ for all $x \in (X \backslash S) \cap B(\overline{x}, r)$ and all $x^* \in \partial^F f(x)$. Then, by [33, Thm 9.1] with $\varphi = c$, (see also, among several other contributions, [17], [30], [53], with various assumptions on X) we have $f_+(x) \geq c d_S(x)$ for $x \in V$, where $f_+ := \max(f, 0)$ and $V := B(\overline{x}, r/2)$. Given $\varepsilon > 0$ one can find $\delta \in (0, r/2)$ such that

$$\forall y \in B(\overline{x}, \delta), \ \overline{x}^* \in \partial^D f(\overline{x}) \qquad \langle \overline{x}^*, y - \overline{x} \rangle \leq f(y) - f(\overline{x}) + c\varepsilon \|y - \overline{x}\|.$$

Elementary calculus rules yield

$$c \partial^D d_S(\overline{x}) \subset \partial^D f_+(\overline{x}) = \overline{\text{co}}^* (\partial^D f(\overline{x}) \cup \{0\}).$$

Given $y \in S \cap B(\overline{x}, \delta)$, $\overline{x}^* \in \partial^D d_S(\overline{x})$, by the preceding inclusion and inequality and a passage to the convex hull and the closure, we get $\langle c\overline{x}^*, y - \overline{x} \rangle \leq c\varepsilon \|y - \overline{x}\|$. Thus, d_S is equi-∂^D-subdifferentiable at $\overline{x}$, hence S is equi-normal at $\overline{x}$ by Theorem 6.6. $\square$

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