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OPTIMIZATION AND FEEDBACK DESIGN OF STATE-CONSTRAINED PARABOLIC SYSTEMS*

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Dedicated to Michel Théra

Abstract: The paper is devoted to optimal control and feedback design of state-constrained parabolic systems in uncertainty conditions. Problems of this type are among the most challenging and difficult in dynamic optimization for any kind of dynamical systems. We pay the main attention to considering linear multidimensional parabolic systems with Dirichlet boundary controls and pointwise state constraints, while the methods developed in this study are applicable to other kinds of boundary controls and dynamical systems of the parabolic type. The feedback design problem is formulated in the minimax sense to ensure stabilization of transients within the prescribed diapason and robust stability of the closed-loop control system under all feasible perturbations with minimizing an integral cost functional in the worst perturbation case. Exploiting certain fundamental properties of the parabolic dynamics, we single out the worst perturbations in the minimax control problem and efficiently solve the associated optimal control problems for approximating ODE and the original PDE systems with pointwise state constraints. In this way, using the transient monotonicity and turnpike asymptotic properties of the underlying parabolic dynamics on the infinite horizon, we compute optimal (in the minimax sense) parameters of the easily implemented while rigorously justified three-positional suboptimal structure of the feedback boundary controls that ensure robust stability of the closed-loop and highly nonlinear parabolic control system under consideration.

Key words: *dynamic optimization, parabolic systems, boundary controls, state constraints, uncertainty conditions, feedback minimax design, closed-loop stability*

Mathematics Subject Classification: *49K20, 49K35, 49N35, 93B50, 93D09*

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1 Problem Formulation and Initial Discussions

This paper concerns optimal control and feedback design problems for linear multidimensional parabolic systems with irregular boundary controls and uncertain distributed perturbations of the parabolic dynamics subject to pointwise state and control constraints. Problems of this type are among the most challenging and difficult in dynamic optimization and control theory for any kind of dynamical/evolution systems governed by ordinary differential, partial differential, and functional differential equations and inclusions.

The methodology developed in this paper largely involves a number of approximation techniques, which is in the mainstream direction of modern variational analysis and its applications to optimization and control; see, e.g., [13] and the references therein. Note that

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efficient techniques of variational analysis has been recently applied by Michel Théra and his collaborators [1] to evolution variational inequalities related to dynamic optimization problems for partial differential equations and their important physical applications.

The primary motivation for our study came from practical design problems of automatic control of the soil groundwater regime in irrigation engineering networks functioning under uncertain weather and environmental conditions; we refer the reader to the author's paper [12] for the description of the original problem and its simplified one-dimensional modeling.

In this paper we consider a more realistic model motivated by [12] while certainly being of independent interest for dynamic optimization, open-loop and closed-loop control, and robust stability with many other potential applications. Let us present a rigorous mathematical formulation of the problem to which we pay the main attention in this paper.

The system dynamics in the problem under consideration is given by the multidimensional linear parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t) \quad \text{a.e. in } \; Q := [0,T] \times \Omega, \\ y(0,x) = 0, \quad x \in \Omega, \\ y(t,x) = u(t), \quad (t,x) \in \Sigma := [0,T] \times \Gamma \end{cases} \tag{1.1}$$

on the time interval $[0,T]$ with controls $u(\cdot)$ acting in the Dirichlet boundary conditions and distributed perturbations $w(\cdot)$ on the right-hand side of the parabolic equation. In (1.1), A is a self-adjoint and uniformly strongly elliptic operator on $L^2(\Omega)$ defined by

$$Ay := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big) - cy, \tag{1.2}$$

where $\Omega \subset I\!R^n$ is an open bounded domain with the boundary Γ that is supposed to be a sufficiently smooth $(n-1)$-dimensional manifold in $I\!R^n$.

The sets of *admissible controls* U and *admissible perturbations* W are given by

$$U := \Big\{u \in L^\infty[0,T]\Big| -\alpha \le u(t) \le \alpha \;\text{ a.e. }\; t \in [0,T]\Big\}, \tag{1.3}$$

$$W := \Big\{w \in L^\infty[0,T]\Big| -\beta \le w(t) \le \beta \;\text{ a.e. }\; t \in [0,T]\Big\} \tag{1.4}$$

with some fixed bounds $\alpha, \beta > 0$ in the pointwise constraints (1.3) and (1.4).

As is well known, the Dirichlet boundary conditions offer the least regularity properties of the parabolic dynamics being the most challenging in control theory for parabolic systems; see, e.g., [9, 13, 17]. By [10], for any feasible pair $(u,w) \in U \times W$ there is a unique *generalized solution* $y = y(u,w) \in L^2(Q)$ to (1.1). For simplicity we omit (u,w) in the notation for the corresponding solution y to (1.1) in what follows. Fixing an arbitrary point $x_0 \in \Omega$, observe by [2, Theorem 3.9] that we can evaluate $y(t,x_0)$ for a.e. $t \in [0,T]$.

The underlying requirement on the system performance is to *stabilize* transients $y(t,x_0)$ near the initial equilibrium state $y(x,0) \equiv 0$ with a given accuracy $\eta > 0$ during the whole dynamic process. This is formalized via the *pointwise state constraints*

$$-\eta \le y(t,x_0) \le \eta \;\text{ a.e. }\; t \in [0,T]. \tag{1.5}$$

A characteristic feature of the dynamical process described by (1.1) is the *uncertainty* of perturbations $w \in W$: we can operate only with the bound β of the admissible region (1.4).

Thus we can keep the system transients $y(t, x_0)$ within the prescribed stabilization region (1.5) only by using *feedback* boundary controls $u(\cdot)$ depending on the current state position $\xi = y(t, x_0)$ for each $t \in [0, T]$.

To formalize this description, consider any function $f\colon I\!R \to I\!R$ and construct boundary controls in (1.1) via the *feedback law*

$$u(t) := f\big(y(t, x_0)\big), \quad t \in [0, T]. \tag{1.6}$$

We say that such a function f defines a *feasible feedback regulator* if it generates controls $u(t)$ by (1.6) belonging to the admissible set U from (1.3) and keeps the corresponding transients $y(t, x_0)$ of (1.1) within the prescribed constraint area (1.5) for every admissible perturbation $w \in W$ from (1.4). To estimate the quality of feasible regulators $f = f(\xi)$, impose the natural summability condition

$$\big|f\big(\gamma(t)\big)\big| \in L^1[0, T] \ \text{ whenever } \ \gamma(t) \in L^2[0, T] \tag{1.7}$$

on f and consider the (energy-type) *cost functional*

$$J(f) := \max_{w \in W} \Big\{ \int_0^T \big|f\big(y(t, x_0)\big)\big|\, dt \Big\}. \tag{1.8}$$

The *maximum* operation in (1.8) reflects the required control energy needed to neutralize the adverse effect of the *worst perturbations* from (1.4) and to keep the state performance within the prescribed area (1.5). Denote by $\mathcal{F}$ the set of all feasible feedback regulators satisfying the summability condition (1.7) and suppose that $\mathcal{F} \neq \emptyset$ referring the reader to [16], where this issue has been studied in more detail. Finally, we formulate the *minimax feedback control problem* (P) as follows:

$$\text{minimize } \ J(f) \ \text{ over } \ f \in \mathcal{F}. \tag{1.9}$$

It has been well recognized in control theory and applications that *feedback* control problems are the most challenging and important for any type of dynamical systems, while PDE systems provide additional difficulties and much less investigated in comparison with the ODE dynamics; see more discussions and references in [13]. Furthermore, significant complications come from *pointwise/hard state constraints*, which are of high nontriviality even for open-look control problems. We are not familiar with any constructive device applicable to the feedback control problem (P) under consideration among a variety of approaches and results available in the theories of differential games, H_∞-control, Riccati's feedback synthesis, and other developments in general settings; see, e.g., [3, 7, 9] with the discussions and references therein.

The constructive approach presented in this paper is initiated in [12] for the case of the one-dimensional heat equation in (1.1). Further results in this direction have been recently obtained in [14] for the case of Dirichlet boundary controls and in [15] for similar problems with boundary controls acting in the Robin/mixed boundary conditions. This paper dealing with irregular controls in the Dirichlet boundary conditions (which are essentially more difficult for the parabolic dynamics in comparison with the Neumann boundary conditions as well as the Robin/mixed ones considered in [15]) significantly improves, clarifies, and simplifies the results and arguments presented in [14] for this case and then developed in [14] for controls in mixed boundary conditions. We employ, in particular, a more efficient device to compute and justify optimal controls for open-loop approximating ODE systems with pointwise state constraints.

The rest of the paper is organized as follows. In Section 2 we formulate and discuss the basic (rather general) assumptions on the initial data of (P) and present some preliminary material on parabolic systems broadly used in proofs of the main results.

Section 3 deals with choosing and justification of the one-sided *worst perturbations* in the feedback control problem (P) and then solving auxiliary *open-loop* optimal control problems corresponding to (P) in the case of realizing the worst perturbations of this type. It is shown, based on the fundamental Maximum Principle for the parabolic dynamics and the convolution representation of the transients that the one-sided worst perturbations can be selected, with a certain simplification, as the *extreme* ones in the admissible region (1.4). Considering further the resulting open-loop parabolic control problem under one-sided worst perturbations, we constructively approximate it by the corresponding ODE optimal control systems with state constraints and justify the possibility to determine an appropriate suboptimal control structure for the parabolic system on a *sufficiently large* time interval by studying the first-order ODE approximation. The latter state-constrained optimal control problem is *precisely solved* by using appropriate tools of ODE optimal control. Finally in this section, we optimize the justified suboptimal open-loop control structure along the parabolic dynamics with taking into account the imposed state constraints.

In Section 4 we employ the open-loop control results from Section 3 and the underlying *monotonicity property* of the parabolic dynamics to justify the structure of *three-positional feedback regulator* in the Dirichlet boundary conditions of the original parabolic system and optimize its parameters in such a way that the obtained closed-loop control system ensures the required state performance within the prescribed constraint region under *arbitrary* one-sided admissible perturbation and, furthermore, provides the best result (in the sense of *minimizing* the cost functional) when the *worst/maximum* perturbations of this type are realized. The resulting closed-loop control system is highly nonlinear and can loose *its robust stability* (stability in the large) maintaining the required state performance in a *self-vibrating regime*. Finally in Section 4, we establish efficient conditions for robust stability of the closed-loop system by developing a *variational approach* to robust stability that reduces the stability issue to a certain open-loop optimal control problem on the *infinite horizon*.

2 Basic Assumptions and Preliminaries

In this paper we consider the parabolic system (1.1), where the differential operator A in (1.2) is *self-adjoint* and *uniformly strongly elliptic* satisfying the properties:

$$\begin{aligned} &a_{ij} \in C^\infty(\operatorname{cl}\Omega), \quad a_{ij}(x) = a_{ji}(x), \quad x \in \Omega, \quad i,j = 1,\ldots,n,\\ &\sum_{i,j=1}^n a_{ij}(x) v_i v_j \ge \nu \sum_{i=1}^n v_i^2, \quad x \in \Omega, \quad (v_1,\ldots,\, v_n) \in I\!\!R^n \end{aligned} \tag{2.1}$$

with some $\nu > 0$ and an arbitrary constant $c \in I\!\!R$ in (1.2).

Consider further the homogeneous boundary value problem

$$\begin{cases} -A\phi + \lambda\phi = 0, \\ \phi|_{\partial\Omega} = 0 \end{cases} \tag{2.2}$$

and recall that the number component λ in the nontrivial pair (λ, ϕ) satisfying (2.2) is an *eigenvalue*, while ϕ is the corresponding *eigenfunction* for the operator A under the Dirichlet boundary condition. From assumptions (2.1) we have the following properties:

(a) the eigenvalues λ_i, $i = 1, 2, \ldots$, are real and form a nondecreasing sequence that accumulates only at ∞;

(b) the corresponding orthonormal system of eigenvalues $\{\phi_i(x)\} \subset C^\infty(\Omega)$ is complete in the space $L^2(\Omega)$;

(c) the first eigenvalue λ_1 is simple and has the positive eigenfunction $\phi_1(x)$;

(d) $c + \lambda > 0$ for any eigenvalue of A with an arbitrary constant $c \in I\!R$ from (1.2).

The proofs of properties (a)–(c) can be found in [4, Theorems 8.37, 8.38]; the one for (d) is given in [14, Proposition 2.1].

To establish the main results of this paper, we need to add the *only one* extra hypothesis to the standard assumptions in (2.1):

(H) The *first eigenvalue* λ_1 of the operator A is *positive*.

The general sufficient condition for the fulfillment of (H) is obtained in [14, Proposition 2.2] in terms of the initial data of A and the diameter d of the domain $\Omega \subset I\!R^n$:

$$c < \frac{2n\nu}{d^2}, \quad \text{where } d := \sup_{x_1, x_2 \in \Omega} \|x_1 - x_2\|.$$

Thus, by properties (a), (c) and assumption (H), we have

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \ldots . \tag{2.3}$$

As mentioned above, for every feasible input pair $(u, w) \in L^\infty[0,T] \times L^\infty[0,T]$ the parabolic system (1.1) admits a unique generalized solution $y \in L^2(Q)$. The next basically known result (see [14, Proposition 2.3] and its proof for more details) gives a convenient *spectral representation* of this solution via a Fourier-like series involving the eigenvalues and eigenfunctions (λ_i, ϕ_i) of the operator A.

Proposition 2.1. (spectral representation of transients). *Let* $(u, w) \in L^2[0,T] \times L^2[0,T]$ *in* (1.1) *under assumptions* (2.1) *on the strongly elliptic operator A, and let* (λ_i, ϕ_i) *be the corresponding eigenvalues and eigenfunctions of A with the weights*

$$\mu_i := \int_\Omega \phi_i(x)\, dx, \quad i = 1, 2, \ldots .$$

Then the unique solution $y \in L^2(Q)$ *to* (1.1) *admits the spectral representation*

$$y(t,x) = \sum_{i=1}^{\infty} \mu_i \Big(\int_0^t w(\theta) e^{\lambda_i \theta} d\theta + (c + \lambda_i) \int_0^t u(\theta) e^{\lambda_i \theta} d\theta \Big) e^{-\lambda_i t} \phi_i(x), \tag{2.4}$$

where the series in (2.4) *strongly converges in the space* $L^2(Q)$.

Finally in this section, we present the following *monotonicity property* of the parabolic dynamics with respect to both boundary controls and distributed perturbations established in [14, Theorem 3.1] on the base of the classical *Maximum Principle* for parabolic systems [8] and an additional *smooth approximation procedure* dealing with irregular data (u, w).

Proposition 2.2. (monotonicity property of the parabolic dynamics). *Let* (u_1, w_1) *and* (u_2, w_2) *be admissible control-perturbation pairs from* $U \times W$ *such that*

$$u_1(t) \geq u_2(t) \;\; \textit{and} \;\; w_1(t) \geq w_2(t) \;\; \textit{a.e.} \;\; t \in [0,T],$$

and let $y_1, y_2 \in L^2(Q)$ *be the corresponding generalized solutions to the parabolic system* (1.1). *Then we have the relationship*

$$y_1(t,x) \geq y_2(t,x) \;\; \textit{a.e.} \;\; (t,x) \in Q.$$

3 Optimal Control under One-Sided Worst Perturbations

A conventional approach to optimal solutions of minimax problems is to identify them with *saddle points* that consist, in the framework of problem (P), of interrelated pairs (u, w) of worst perturbations and optimal controls. However, it does not seem realistic to implement this approach in full generality in such a complicated hard-constrained setting as in the problem (P) under consideration. To develop an implementable control design, we simplify the situation splitting the problem into the two symmetric *one-sided cases*:

$$0 \le w(t) \le \beta \ \text{ and } \ -\alpha \le u(t) \le 0 \ \text{ a.e. } \ t \in [0,T]; \tag{3.1}$$

$$-\beta \le w(t) \le 0 \ \text{ and } \ 0 \le u(t) \le \alpha \ \text{ a.e. } \ t \in [0,T]. \tag{3.2}$$

The above splitting seems reasonable (although does not cover the general case) due to the *symmetricity* of the initial data in problem (P), the *monotonicity* of the parabolic dynamics, and the fact that controls of the *opposite sign* are required to keep transients within the prescribed constraint area (1.5) neutralizing the adverse impact of perturbations. We refer the reader to [12] for certain practical motivations and to [14] for some arguments that support such a symmetric splitting and the subsequent design procedure by considering the class of *odd* and *nonincreasing* feedback regulators $f \in \mathcal{F}$ in (1.7) employed also in [15].

Without loss of generality we justify the choice of worst perturbations only in the *upper* case (3.1), since the *lower* case (3.2) is fully symmetric. The following result is partly due to the *specific structures* of the cost functional and boundary controls in (P), related to each other, while it is largely based on the fundamental *Maximum Principle* for the parabolic dynamics and the related *convolution representation* of the transients; see the proof.

Theorem 3.1 (one-sided worst perturbations). *Under the standing assumptions made, the worst perturbation $w(t)$ in (P) over all $w \in W$ satisfying* (3.1) *can be chosen as the maximal one from the admissible area, i.e.,*

$$w(t) \equiv \beta \ \textit{ for a.e. } \ t \in [0,T]. \tag{3.3}$$

Proof. By [16, Theorem 5.1], the transients $\xi(t) := y(t, x_0)$ of (1.1) generated by admissible pairs $(u, w) \in U \times W$ have the *convolution representation* for all $t \in [0,T]$:

$$\xi(t) = \varphi(t) * w(t) + \psi(t) * u(t) := \int_0^t \varphi(t-s) w(s)\, ds + \int_0^t \psi(t-s) u(s)\, ds, \tag{3.4}$$

where the functions $\psi \in L^\infty[0,T]$ and $\varphi \in L^\infty[0,T]$ are *positive* on $[0,T]$ and independent of (u, w); this is mainly due to the *Maximum Principle* for the parabolic dynamics.

To proceed further, fix a feedback function $f \in \mathcal{F}$ and, given an arbitrary perturbation $w(t)$ from (3.1), get by (1.6) the corresponding boundary control $u(t)$ in (1.1) formed via this feedback law f:

$$u(t) = f\big(y(t, x_0)\big) = f\big(\xi(t)\big) \ \text{ a.e. } \ t \in [0,T].$$

Then the cost functional (1.8) is written as

$$\int_0^T \big|f\big(y(t, x_0)\big)\big|\, dt = \int_0^T \big|u(t)\big|\, dt = -\int_0^T u(t)\, dt \tag{3.5}$$

due to the control constraints in (3.1). By the convolution representation (3.4), the state constraints (1.5) imposed on $y(t,x_0)=\xi(t)$ reduces to

$$\int_0^T \varphi(t-s)w(s)\,ds + \int_0^T \psi(t-s)u(s)\,ds \le \eta, \quad 0 \le t \le T. \tag{3.6}$$

Since both functions φ and ψ are *positive* on $[0,T]$, it follows directly from (3.6) that *the bigger magnitude of a perturbation is, the more control of the opposite sign should be applied* to neutralize this perturbation in the sense of ensuring the required state performance (3.6). Thus the *maximum value* of the cost functional in (3.5) subject to (3.1) and (3.6) corresponds to the *maximal perturbation* (3.3), which requires the *maximal control response* to keep (3.6) along the parabolic dynamics (1.1). This completes the proof of the theorem. □

By the afore-mentioned *symmetry* of (P), we conclude that in the *lower* perturbation case (3.2) the worst one-sided perturbation $w(\cdot)$ can be selected as

$$w(t) \equiv -\beta \ \text{ for a.e. } \ t \in [0,T]. \tag{3.7}$$

In the rest of this section we consider the specification of problem (P) under realizing the worst perturbations (3.3) and (3.7) in the system. By the above discussions, it is sufficient to study only the upper case (3.1) when the worst perturbation is $w \equiv \beta$. Taking into account that in this case the control systems does not involve any uncertainty, we associate with (P) as $w \equiv \beta$ the following *open-loop* optimal control problem $(\overline{P})$:

$$\text{minimize } \ \overline{J}(u) := -\int_0^T u(t)\,dt \tag{3.8}$$

along the parabolic system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = \beta \quad \text{a.e. } \ (t,x) \in Q, \\ y(0,x) = 0, \quad x \in \Omega, \\ y(t,x) = u(t) \ \text{ a.e. } \ (t,x) \in \Sigma \end{cases} \tag{3.9}$$

under the *pointwise control* and *state constraints*

$$u(\cdot) \in L^\infty[0,T] \ \text{ with } \ -\alpha \le u(t) \le 0 \ \text{ a.e. } \ t \in [0,T], \tag{3.10}$$

$$y(\cdot,x_0) \in L^2[0,T] \ \text{ with } \ y(t,x_0) \le \eta \ \text{ a.e. } \ t \in [0,T]. \tag{3.11}$$

Observe that $(\overline{P})$ is a state-constrained Dirichlet boundary control problem, which was considered in [13, 17] and the references therein in more generality with deriving rather complicated necessary optimality conditions involving *Borel measures*, which are difficult to apply to computing optimal or suboptimal controls.

In what follows we develop another strategy to solve $(\overline{P})$ that is based on its approximation by ODE state-constrained control problems and determining in this way an implementable *suboptimal* control structure to $(\overline{P})$, which is then *optimized* along the state-constrained parabolic dynamics (3.9). Let us use the spectral representation

$$y(t,x_0) = \sum_{i=1}^\infty \mu_i \Big(\frac{\beta}{\lambda_i}\big(e^{\lambda_i t} - 1\big) + (c+\lambda_i)\int_0^t u(\theta)e^{\lambda_i \theta}d\theta \Big) e^{-\lambda_i t}\phi_i(x_0) \tag{3.12}$$

of solutions to the parabolic system (3.9) at $x = x_0$, which follows from Proposition 2.1. Taking any natural $N = 1, 2, \ldots$, replace series (3.12) by the finite N-sum

$$y^N(t, x_0) = \sum_{i=1}^{N} \mu_i \Big(\frac{\beta}{\lambda_i} \big(e^{\lambda_i t} - 1 \big) + (c + \lambda_i) \int_0^t u(\theta) e^{\lambda_i \theta} d\theta \Big) e^{-\lambda_i t} \phi_i(x_0) \tag{3.13}$$

for which $y^N(t, x_0) \to y(t, x_0)$ strongly in $L^2[0, T]$. It is easy to see that $y^N(t, x_0)$ in (3.13) is represented as the sum

$$y^N(t, x_0) = \sum_{i=1}^{N} y_i(t), \quad 0 \le t \le T, \tag{3.14}$$

where each $y_i(t)$, $i = 1, \ldots, N$, satisfies the corresponding ODE:

$$\dot{y_i} = -\lambda_i y_i + \mu_i \phi_i(x_0) \big(\beta + (c + \lambda_i) u(t) \big) \ \text{ a.e. } \ t \in [0, T], \ \ y_i(0) = 0. \tag{3.15}$$

To proceed, we observe that the feedback control problem (P) as well as the open-loop control problem $(\overline{P})$ are formulated on the fixed time interval $[0, T]$. In many applications the time duration T is *sufficiently large* and can be taken conventionally as the *infinite horizon*, which allows us to involve $t \to \infty$ in the *asymptotic analysis*. If in the latter case the first term in (3.12) involving $e^{-\lambda_1 t}$ is *not zero*, it obviously *dominates* the other terms in the exponential series as $t \to \infty$ due to the basic eigenvalue relationships (2.3).

Having this in mind, our special attention is paid to the case of $N = 1$ in (3.13)–(3.15), which provides an adequate *ODE approximation* of the PDE system under consideration on sufficiently large time intervals and thus allows us to determine an appropriate suboptimal control structure in the open-loop problem $(\overline{P})$ for the parabolic dynamics (3.9). Then we optimize parameters of this suboptimal structure along the initial *parabolic* system.

By the afore-mentioned symmetry, it is sufficient to consider the following open-loop control problem $(\overline{P}_1)$ approximating the PDE dynamics at $x = x_0$: minimize the cost functional (3.8) over admissible controls $u(t)$ satisfying the constraints in (3.10) and generating absolutely continuous trajectories $y \colon [0, T] \to I\!R$ of the ODE system

$$\dot{y} = -\lambda_1 y + \mu_1 \phi_1(x_0) \big(\beta + (c + \lambda_1) u(t) \big) \ \text{ a.e. } \ t \in [0, T], \ \ y(0) = 0 \tag{3.16}$$

subject to the pointwise *state constraints*

$$y(t) \le \eta \ \text{ for all } \ t \in [0, T]. \tag{3.17}$$

Observe that the presence of the state constraints (3.17) places problem $(\overline{P}_1)$ among difficult problems for ODE control. Standard optimality conditions for such problems involve Borel measures that make them challenging for implementations and applications. In [14, 15], we developed an efficient while rather involved procedure to study $(\overline{P}_1)$ by using a *penalty-type approximation* of state constraints and then by passing to the limit in optimality conditions for penalized approximating problems.

Now we use a new device, which takes into account certain specific features of $(\overline{P}_1)$ and allows us to obtain the *exact solution* to this problem by a direct application of less conventional and less known results of ODE control theory for state-constrained systems.

Theorem 3.2 (exact solution to the state-constrained ODE control problem). *Assume that both conditions*

$$\mu_1 \phi_1(x_0) \beta > \lambda_1 \eta, \quad \mu_1 \phi_1(x_0) \big(\beta - \alpha(c + \lambda_1) \big) \le \lambda_1 \eta \tag{3.18}$$

are satisfied and that the time duration T is sufficiently large:

$$T > \frac{1}{\lambda_1} \ln \frac{\mu_1 \phi_1(x_0)\beta}{\mu_1 \phi_1(x_0)\beta - \lambda_1 \eta}. \tag{3.19}$$

Define the switching time $\overline{\tau} \in (0, T)$ by

$$\overline{\tau} := \frac{1}{\lambda_1} \ln \frac{\mu_1 \phi_1(x_0)\beta}{\mu_1 \phi_1(x_0)\beta - \lambda_1 \eta} \tag{3.20}$$

and consider the piecewise constant, two-positional control

$$\bar{u}(t) = \begin{cases} 0 & \text{if } t \in [0, \overline{\tau}), \\ v := \dfrac{\lambda_1 \eta - \mu_1 \phi_1(x_0)\beta}{\mu_1 \phi_1(x_0)(c + \lambda_1)} & \text{if } t \in [\overline{\tau}, T]. \end{cases} \tag{3.21}$$

Then control (3.21) *is optimal to problem* $(\overline{P}_1)$*, i.e., it is feasible with respect to both control constraints* (3.10) *and state constraints* (3.17) *and gives the minimum value to the cost functional* (3.8) *subject to these constraints along the dynamical system* (3.16)*.*

Proof. We begin with observing that if either the first condition (3.18) is not satisfied, or it is satisfied while (3.19) is not, then the trivial control $u(t) \equiv 0$ on $[0, T]$ is feasible and hence optimal to $(\overline{P}_1)$; see [14]. Furthermore, it is proved in [14] that if both above conditions are satisfied while the second one in (3.18) is not, then problem $(\overline{P}_1)$ does not have feasible controls. On the other hand, we can directly check, integrating (3.16) and using properties (c) and (d) of the operator A presented in Section 2, that control (3.21) with the switching time (3.20) is well defined and *feasible* to $(\overline{P}_1)$ under the fulfillment of (3.18) and (3.19). Let us show that control (3.21) is actually *optimal* to (P_1) under the assumptions made.

In what follows we employ optimality conditions of the *Pontryagin maximum principle* type for state-constrained control problems, which do *not* involve measures—or, more precisely, the corresponding measure multiplier reduces to a *density.* The conditions used below were initiated probably in [6], with a heuristic proof, and then were rigorously justified and developed in [11]; see also survey [5] for more details and discussions.

Applying the *necessary optimality conditions* from [5, Theorem 4.1], we observe first that the state constraint (3.17) in $(\overline{P}_1)$ is of *order one* and satisfies the *regularity condition*

$$\frac{\partial(\dot{y})}{\partial u} = \mu_1 \phi_1(x_0)(c + \lambda_1) \neq 0$$

by equation (3.16), properties (c) and (d), and assumption (H) from Section 2. In this case the corresponding measure in the necessary optimality conditions is a density $\delta(t)dt$, and hence the *Hamilton-Pontryagin function* in $(\overline{P}_1)$ is

$$H(y, p, \delta, u) = -u + p\big[-\lambda_1 y + \mu_1 \phi_1(x_0)\big(\beta + (c + \lambda_1)u\big)\big] + \delta(y - \eta), \tag{3.22}$$

where $p \in I\!R$ is the adjoint variable. By (3.22) the *adjoint equation* is

$$\dot{p} = -\frac{\partial H}{\partial y} = \lambda_1 p - \delta, \quad 0 \le t \le T, \tag{3.23}$$

with $p(T) \ge 0$, since the state constraint (3.17) is obviously active at $t = T$. Further, we form the *switching function* $s(t)$ involving of the control coefficients in (3.22) as

$$s(t) = -1 + p(t)\mu_1 \phi_1(x_0)(c + \lambda_1), \quad 0 \le t \le T. \tag{3.24}$$

By definition of the switching function and by the second assumption in (3.18) we have that the switching function vanishes

$$s(t) = 0 \quad \text{for all} \quad t \in [\tau, T] \tag{3.25}$$

on each *boundary arc*, i.e., on the part of the corresponding trajectory that entirely lies on the state constraint boundary starting with the *switching time* τ when the trajectory hits the constraint boundary $y(\tau) = \eta$. It follows from (3.24) and (3.25) that

$$p(t) = \frac{1}{\mu_1 \phi_1(x_0)(c + \lambda_1)}, \quad \tau \le t \le T. \tag{3.26}$$

Furthermore, we have by differentiating the expression (3.24) for $\delta(t)$ on $[\tau, T]$ and by taking (3.23) and (3.25) into account that

$$\dot{s}(t) = \mu_1 \phi_1(x_0)(c + \lambda_1)\dot{p}(t) = \mu_1 \phi_1(x_0)(c + \lambda_1)\big(\lambda_1 p(t) - \delta(t)\big) = 0, \quad \tau \le t \le T,$$

which yields $\lambda_1 p(t) - \delta(t) = 0$ on $[\tau, T]$ and, by (3.26), allows us to calculate $\sigma(t)$ on $[\tau, T]$:

$$\delta(t) = \frac{\lambda_1}{\mu_1 \phi_1(x_0)(c + \lambda_1)} > 0, \quad \tau \le t \le T.$$

Further, let us consider the corresponding *interior arc* of (3.16) under the state constraint (3.17), i.e., the part of $y(t)$ for which $y(t) < \eta$ as $0 \le t < \tau$. Taking into account that $\delta(t) = 0$ for $t \in [0, \tau)$ and using relationships (3.23) and (3.26), we get the Cauchy problem

$$\dot{p} = \lambda_1 p \;\text{ on }\; 0 \le t < \tau \;\text{ with }\; p(\tau) = \frac{1}{\mu_1 \phi_1(x_0)(c + \lambda_1)}$$

for the adjoint arc $p(t)$ on $[0, \tau]$, which has the unique solution

$$p(t) = \frac{1}{\mu_1 \phi_1(x_0)(c + \lambda_1)} e^{\lambda_1 (t - \tau)}, \quad 0 \le t \le \tau.$$

The latter yields, by the switching function formula (3.24), that $s(t) < 0$ for $0 \le t < \tau$. By employing the minimum condition from [5, Theorem 4.1], we conclude that any optimal control $\bar{u}(t)$ to $(\overline{P}_1)$ must satisfy the relationship

$$s(t)\bar{u}(t) = \min_{-\alpha \le u \le 0} s(t)u, \quad 0 \le t \le T, \tag{3.27}$$

which immediately implies that $\bar{u}(t) = 0$ for $0 \le t < \tau$, since the switching function is negative on this interval. Observe that the *minimum condition* (3.27) does not give any information about optimal controls on $[\tau, T]$, where the switching function $s(t)$ vanishes. This means that $[\tau, T]$ is a *singular control* interval, which $[0, \tau)$ provides a *bang-bang* mode.

It remains to find, in the framework of this procedure, optimal values of the switching time $\overline{\tau}$ and control $\bar{u}(t)$ on the singularity interval $[\overline{\tau}, T]$. It follows from the above arguments that the optimal switching time $\tau = \overline{\tau}$ can be found from the *entry condition* $y(\overline{\tau}) = \eta$, where $y(t)$ is the solution to the state equation (3.16) with $u(t) \equiv 0$. Integrating this system, we arrive at the expression (3.20) for $\overline{\tau}$ and easily check that $0 < \overline{\tau} < T$ under the assumptions made. To determine $\bar{u}(t)$ on $[\overline{\tau}, T]$, recall that it corresponds to the boundary arc $y(t) \equiv \eta$ on $[\overline{\tau}, T]$. Hence $\dot{y}(t) \equiv 0$ on this interval, and we get the *constant* value $\bar{u}(t) \equiv v$ on $[\overline{\tau}, T]$ from formula (3.21).

Thus, the above procedure allows us to compute the control function $\bar{u}(t)$ in (3.21) that satisfies the necessary optimality conditions from [5, Theorem 4.1]. Since these conditions are also *sufficient* for optimality in problem $(\overline{P}_1)$, by the linear-convex structure of this problem (see [5, Theorem 8.2]), we complete the proof of the theorem. □

Based on the discussions presented right before formulating Theorem 3.2, we can consider the optimal control found in this theorem as a *first-order approximation* of an optimal solution to the open-loop control problem $(\overline{P})$ for the state-constrained parabolic system (3.9). Moreover, this approximation is fairly *adequate*, under the basic assumption (H) made in Section 2, when the time interval $[0, T]$ is *sufficiently large*.

We can do even better in what follows. Let us accept a *two-positional* control structure of the exact solution to problem (P_1) founded in Theorem 3.2 as a *suboptimal structure* of feasible controls to problem (P) and *optimize* its parameters along the original *parabolic dynamics*. This leads us to the following *parametric dynamic optimization* problem $(\widehat{P})$ for the state-constrained parabolic system under consideration:

$$\text{minimize} \;\; J(v,\tau) := -\int_0^T u(t)\,dt \tag{3.28}$$

over admissible Dirichlet boundary controls of the form

$$u(t) = \begin{cases} 0 & \text{if } \; t \in [0,\tau), \\ -v & \text{if } \; t \in [\tau, T] \end{cases} \tag{3.29}$$

subject to the constraints on control recourses v and switching control times τ given by

$$0 \le v \le \alpha, \quad 0 < \tau \le T \tag{3.30}$$

and the pointwise state constraints (3.11) on trajectories of the parabolic system (3.9).

The next theorem gives *exact solutions* to problem $(\widehat{P})$ in the case when the time interval $[0, T]$ is *sufficiently large* and also in the asymptotic case of this problem when $T = \infty$, i.e., when $(\widehat{P})$ is considered on the *infinite horizon* $[0, \infty]$.

Theorem 3.3 (optimal parameters of open-loop suboptimal control for the state-constrained parabolic system). *In addition to the standing assumptions of Section* 2*, impose the following conditions:*

$$0 < \gamma\beta - \eta \le \min\Big\{\alpha(1 + c\gamma),\; \frac{\beta(1 + c\gamma)}{c + \lambda_1}\Big\}, \tag{3.31}$$

where γ is the aggregate spectral parameter of the elliptic operator A from (1.2) *defined by*

$$\gamma := \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i}.$$

Let $T_0 > 0$ be a unique solution to the equation

$$\eta = \beta\Big(\gamma - \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i t}\Big), \tag{3.32}$$

which exists under the assumptions made. Then for all $T > T_0$ the equation

$$\sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i T} \Big[(c + \lambda_i)(\gamma\beta - \eta)e^{\lambda_i \tau} - \beta(1 + c\gamma)\Big] = 0 \tag{3.33}$$

has a unique solution $\tau = \overline{\tau}(T) \in (0,T)$ *and the boundary control*

$$u_\tau(t) = \begin{cases} 0 & \text{if } t \in [0,\tau), \\ \dfrac{\eta - \gamma\beta}{1+c\gamma} := -\bar{v} & \text{if } t \in [\tau, T] \end{cases} \tag{3.34}$$

is feasible to $(\widehat{P})$ *whenever* $0 < \tau \le \overline{\tau}(T)$. *When* $\tau = \overline{\tau}(T)$, *the control* $\bar{u}(t) := u_{\overline{\tau}(T)}(t)$ *is optimal to problem* $(\widehat{P})$ *among any controls of type* (3.34). *Furthermore,* $\overline{\tau}(T) \downarrow \overline{\tau}^\infty$ *as* $T \to \infty$, *where the asymptotically optimal switching time* $\overline{\tau}^\infty$ *is computed by*

$$\overline{\tau}^\infty := \frac{1}{\lambda_1} \ln \frac{\beta(1+c\gamma)}{(c+\lambda_1)(\gamma\beta - \eta)} \tag{3.35}$$

and is maximal among the switching times $\tau \ge 0$ *generating controls* (3.34) *with* $T = \infty$, *which all are admissible by the state constraints* (3.11) *along the parabolic system* (3.9) *on the infinite horizon* $[0,\infty)$.

Proof. Let $u(t)$ be a boundary control of form (3.29) satisfying (3.30), and let $y(t,x)$ be the corresponding trajectory of (3.9) generated by this control. Denoting

$$y(t) := y(t,x_0) \;\text{ on }\; [0,\tau] \;\text{ and }\; y(t;\tau) := y(t,x_0) \;\text{ on } [\tau,T]$$

and employing Proposition 2.1, we have the representations:

$$y(t) = \beta\Big(\gamma - \sum_{i=1}^\infty \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i t}\Big), \quad t \in [0,\tau], \tag{3.36}$$

$$y(t;\tau) = \gamma\beta - (1+c\gamma)v + \sum_{i=1}^\infty \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i t}\Big[(c+\lambda_i)ve^{\lambda_i \tau} - \beta\Big], \quad t \in [\tau,T], \tag{3.37}$$

with $y(\tau) = y(\tau;\tau)$. By $\gamma\beta - \eta > 0$, the equation $y(t) = \eta$ in (3.32) has a solution $t = T_0$, which is *unique* by the *monotonicity* of $y(t)$ in (3.36). Taking now $T > T_0$ and using the monotonicity of $y(t;\tau)$ with respect to τ that follows from Proposition 2.2, we conclude that the equation $y(T;\tau(T)) = \eta$, which reduces to (3.33) due to (3.37), has a unique solution $\overline{\tau}(T) \in (0,T)$. Furthermore, the same monotonicity property and the explicit representation of $y(t;\tau)$ in (3.10) imply under the assumptions made that $y(t;\tau) \le \eta$ whenever $t \in [0,T]$ for every transient generated by the controls $u_\tau(t)$ from (3.34) with $\tau \le \overline{\tau}(T)$. Thus $\tau = \overline{\tau}(T)$ is *maximal* among all τ generating feasible controls to $(\widehat{P})$ with the control resource $\bar{v}$. It follows directly from structure (3.28) of the cost functional in $(\widehat{P})$ that the Dirichlet boundary control $\bar{u}(t) = u_{\overline{\tau}}(t)$ is indeed *optimal* to problem $(\widehat{P})$ among any feasible controls (3.34) whenever $T > T_0$.

It remains to consider the *asymptotic case* of problem $(\widehat{P})$ as $T \to \infty$ and its behavior on the infinite horizon. Similarly to the prof of Proposition 2.2 based on the Maximum Principle for the parabolic dynamics, we conclude that the optimal switching time $\overline{\tau}(T)$ in $(\widehat{P})$ is strictly decreasing in T and it is obviously bounded from below. Thus $\overline{\tau}(T)$ converges as $T \to \infty$, and its limit $\overline{\tau}^\infty$ reduces to that computed in (3.35) due to the eigenvalue properties (2.3), which reflect the *first eigenvalue dominance*.

Further, we observe directly from (3.37) that the control $u_\tau(t)$ from (3.34) with $T = \infty$ and $\tau = \overline{\tau}^\infty$ preserves the state constraints (3.11) for the corresponding transient in (3.36) and (3.37) whenever $t \ge 0$, i.e., this control is *feasible* to problem $(\widehat{P})$ on the *infinite horizon* $[0,\infty)$. Furthermore, $\overline{\tau}^\infty$ is the *maximal* switching time τ in (3.34) satisfying this property. First of all, we easily check that the state constraints (3.11) are satisfied on $[0,\infty)$ if $\tau < \overline{\tau}^\infty$ in (3.34) with $T = \infty$. Considering now any $\tau > \overline{\tau}^\infty$, apply the *Fermat stationary rule* to (3.37) on the open interval (τ,∞) by differentiating $y(t;\tau)$ in t. We check in this way that the *maximum* of $y(t;\tau)$ over (τ,∞) is *bigger* than η whenever $\tau > \overline{\tau}^\infty$. This completes the proof of the theorem. □

Observe that the *asymptotically optimal* switching time $\overline{\tau}^\infty$ in (3.35) can be computed directly from the condition of *vanishing the first term* in the series of (3.37):

$$(c+\lambda_1)ve^{\lambda_1\tau} - \beta = 0$$

with $v = \bar{v}$ computed in (3.34). This justifies the simple and convenient *first term rule* to deal with the parabolic dynamics under the basic assumption (H) as $t \to \infty$.

The results obtained above describe the best possible reaction of the control system to keep the required state constraints under the realization of the *upper/maximal* case of the *worst perturbations* $w(t) \equiv \beta$ on $[0,T]$. Due to the *full symmetry* of the initial problem (P) discussed in Section 2, the *lower case* $w(t) \equiv -\beta$ of the worst perturbations on $[0,T]$ can be considered similarly by using *open-loop Dirichlet boundary controls*

$$u(t) = \begin{cases} 0 & \text{if } t \in [0,\tau), \\ v \in [0,\alpha] & \text{if } t \in [\tau,T] \end{cases} \tag{3.38}$$

for the linear parabolic system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = -\beta \quad \text{a.e. } (t,x) \in Q, \\ y(0,x) = 0, \quad x \in \Omega, \\ y(t,x) = u(t) \quad \text{a.e. } (t,x) \in \Sigma \end{cases} \tag{3.39}$$

subject to the pointwise state constraints

$$y(\cdot,x_0) \in L^2[0,T] \ \text{ with } \ y(t,x_0) \ge -\eta \ \text{ a.e. } \ t \in [0,T]. \tag{3.40}$$

Then, taking into account the sign changes in (3.38)–(3.40), we have *exactly the same formulas* for computing optimal parameters $(\bar{v},\overline{\tau}(T),\overline{\tau}^\infty)$ in the open-loop problem

$$\text{minimize} \ \int_0^T u(t)\,dt \quad \text{over constraints (3.38)–(3.40)} \tag{3.41}$$

and its asymptotic infinite horizon version.

Finally in this section, we conclude from the results of Theorem 3.3 and their counterparts for the lower case of the worst perturbations that the *passage to the infinite horizon* allows us to significantly simplify optimal solutions to the open-loop control problems under consideration and to arrive at the convenient analytic formulas for computing their optimal parameters. The discovered phenomenon reveals a certain *turnpike property*, which happens to be a characteristic feature of such *state-constrained* control problems governed by the *parabolic dynamics.*

4 Feedback Control Design and Robust Stability of the Closed-Loop Parabolic System

The last section of the paper is devoted to the construction and justification of a *suboptimal feedback regulator* for the original minimax feedback control problem (P) and then computing the range of parameters of this regulator, which ensures *robust stability* of the designed closed-loop control system.

Recall that the purpose of feedback controls in (P) is to keep transients within the given state constraint region (1.5) for *all* uncertain perturbations $w \in W$ from (1.4) subject to the imposed constrains on controls in such a way that the cost functional (1.8) is *minimized* under the realization of the (one-sided) *worst perturbations* determined in (3.3) and (3.7). The results obtained above for computing optimal *open-loop* controls in the case of the worst perturbations allow us to justify the following *suboptimal structure* $f = f(\xi)$ of *feedback controls* (1.6) acting in the Dirichlet boundary conditions of the parabolic system (1.1):

$$f(\xi) = \begin{cases} -v & \text{if } \xi \geq \sigma, \\ 0 & \text{if } -\sigma < \xi < \sigma, \\ v & \text{if } \xi \leq -\sigma \end{cases} \tag{4.1}$$

describing a *three-positional feedback regulator* with the "dead region" $(-\sigma, \sigma)$. Observe that the feedback control law $f(\xi)$ in (4.1) is given by an *odd* and *nonincreasing* function satisfying all the requirements of Theorem 3.1.

The feedback control design, in the minimax sense of problem (P), reduces therefore to computing appropriate parameters (v, σ) in (4.1) such that the resulting closed-loop control system keeps the state position $\xi = y(t, x_0)$ under observation within the admissible state constraint area (1.5) for all uncertain perturbation $w \in W$ and then ensures the minimum value of the cost functional (1.8) under the realization of the worst perturbations. As in Section 3, we split the problem into the two symmetric *one-sided cases* (3.1) and (3.2) considering both feasibility and optimality issues for these cases separately.

The next theorem provides in the exact calculation of the *optimal value* $\sigma(T)$ on the given time interval $[0, T]$ and fully describes its limiting/asymptotic behavior as $T \to \infty$, which corresponds to problem (P) on the *infinite horizon.*

Theorem 4.1 (optimal parameters of the three-positional regulator in the minimax feedback control problem for the parabolic system). *Let the feedback boundary control regulator $f(\xi)$ in* (1.6) *and* (1.1) *have the suboptimal three-positional structure* (4.1), *let T_0 be a unique solution to equation* (3.32), *and let the control resource $\bar{v}$ be computed in* (3.34). *The following assertions hold, where the feasibility and optimality are understood in the afore-mentioned one-side sense:*

(i) *For any $T > T_0$ the feedback control* (4.1) *is feasible to* (P) *on the time interval* $[0, T]$ *whenever $0 < \sigma \leq \overline{\sigma}(T)$ with*

$$\overline{\sigma}(T) := \beta\Big(\gamma - \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i \overline{\tau}(T)}\Big), \tag{4.2}$$

where $\overline{\tau}(T)$ is a unique solution to the transcendental equation (3.33). *Moreover, the dead region $\big(-\overline{\sigma}(T), \overline{\sigma}(T)\big)$ is optimal to problem (P) with the feedback control structure* (4.1).

(ii) *We have $\sigma(T) \downarrow \overline{\sigma}$ as $T \to \infty$, where the number $\overline{\sigma}^{\infty} \geq 0$ is computed by*

$$\overline{\sigma}^{\infty} := \beta\Big(\gamma - \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} \Big[\frac{(c + \lambda_1)(\gamma\beta - \eta)}{\beta(1 + c\gamma)}\Big]^{\frac{\lambda_i}{\lambda_1}}\Big) \tag{4.3}$$

being in fact positive under the condition

$$\gamma\beta - \eta < \frac{\beta(1 + c\gamma)}{c + \lambda_1}. \tag{4.4}$$

In this case the three-positional regulator (4.1) *with the resource* $v \in (0, \alpha]$ *computed in* (3.34) *is feasible to* (P) *on* $[0, \infty)$ *whenever* $0 < \sigma \le \overline{\sigma}^\infty$*, while the dead region* $\big(-\overline{\sigma}^\infty, \overline{\sigma}^\infty\big)$ *is optimal to* (P) *with the feedback control structure* (4.1) *on the infinite horizon* $[0, \infty)$.

Proof. Due to Proposition 2.2 on the *monotonicity* of transients with respect to *controls* (and thus with respect to *switching times* τ) and also due to the time-monotonicity of $y(t)$ in (3.36), the feasible and optimal values of σ asserted in the theorem directly relate, concerning the one-sided worst perturbations, to the corresponding values of $y(t)$ at T_0 and $\overline{\tau}(T)$ determined in Theorem 3.3 and its proof. On the other hand, these values of σ found for the case of the worst perturbations occur to be appropriate for *any* one-sided perturbations from the admissible area (1.4) with the symmetric splitting (3.1) and (3.2) due to the extremality of the worst perturbations by Theorem 3.1 and due to the *monotonicity* of transients with respect to *perturbations* by Proposition 2.2. In this way we arrive at all the conclusions of assertion (i).

Regarding assertion (ii) of the theorem, we can observe that the value of $\overline{\sigma}^\infty$ in (4.3) corresponds to $\overline{\sigma}^\infty = y(\overline{\tau}^\infty)$ with $y(t)$ from (3.36) and the asymptotically optimal switching time $\overline{\tau}^\infty$ computed by (3.35) due to the above arguments based on the *monotonicity* results of Proposition 2.2. The limiting conclusion $\overline{\sigma}(T) \downarrow \overline{\sigma}^\infty$ as $T \to \infty$ and the other conclusions in (ii) can be checked directly employing by the transient monotonicity. □

For further simplifications and developments of the results obtained, impose the following additional assumption:

$$\sum_{i=2}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} \theta^{-\frac{\lambda_i}{\lambda_1}} < 0 \quad \text{whenever} \quad \theta > 1, \tag{4.5}$$

which surely holds for various standard parabolic equations in the presence of *symmetry*, in particular, for the *multidimensional heat equation* defined on rectangulars, balls, etc.; see, e.g., [4, 8] and the references therein.

Taking now the *first term*

$$\overline{\sigma}_1 := \beta\Big(\gamma - \frac{\mu_1 \phi_1(x_0)(c + \lambda_1)(\gamma\beta - \eta)}{\lambda_1 \beta(1 + c\gamma)}\Big) \tag{4.6}$$

in the series (4.3), we have under assumptions (4.4) and (4.5) that $0 < \overline{\sigma}_1 < \overline{\sigma}^\infty$ and thus conclude from Theorem 4.1(ii) that the three-positional feedback regulator (4.1) with $v = \bar{v}$ and $\sigma = \overline{\sigma}_1$ is *feasible* in the above sense to problem (P) on the *infinite horizon.*

Let us consider next the *closed-loop* control system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t), & x \in \Omega, \quad t \ge 0, \\ y(0, x) = 0, & x \in \Omega, \\ y(t, x) = f\big(y(t, x_0)\big), & x \in \partial\Omega, \quad t \ge 0, \end{cases} \tag{4.7}$$

where $f = f(\xi)$ is a (discontinuous) *three-positional* feedback regulator with parameters (v, σ) given in (4.1). Our goal in what follows is to derive efficient conditions ensuring the *robust stability* of system (4.7) and (4.1) in the sense precisely defined below and then to combine these conditions with the relationships on (v, σ) established above from the viewpoint of minimax optimality in the feedback control problem (P) for the parabolic system (4.7) subject to the control and state constraints. In this way we arrive at the *reliable* feedback control design ensuring the required suboptimal performance of the closed-loop control system in a stable regime acceptable for applications.

Note that the minimax design results developed above establish relationships between parameters of the parabolic dynamics, feedback boundary controls, perturbations, and imposed constraints under which the closed-loop control system (4.7) allows us to keep the transients at the point of observation within the prescribed state constraint area for *any one-sided* admissible uncertain perturbations with the *optimal effect* in the *worst* perturbation case. However, the minimax control design procedure developed above does not address *stability issues* for the resulting closed-loop control system that are of crucial importance for practical applications. Let us indicate the following *two major sources* that can cause possible *instability* of the closed-loop control system given by (4.7) and (4.1):

(1) The closed-loop control system described by (4.7) and (4.1) is *highly nonlinear*, despite the linearity of its parabolic dynamics. Of course, the source of nonlinearity is the *discontinuous* three-positional regulator (4.1) in the Dirichlet boundary conditions of (4.7).

(2) The parabolic dynamical system (4.7) is of *distributed parameters*. The most visible manifestation of the distributed parameter nature in (4.7) is that the control acts in the *boundary conditions* while the feedback is formed by observing the current state position $\xi = y(t, x_0)$ at the *intermediate point* $x_0 \in \Omega$ of the space domain. The latter generates *delay* in the closed-loop control parabolic system that significantly affects stability.

We can easily see that if the current state position $\xi = y(t, x_0)$ lies *inside* the dead region $(-\sigma, \sigma)$ after terminating all the perturbations, then the closed-loop control system (4.7) with the three-positional regulator (4.1) maintains the starting *stationary equilibrium regime* $y \equiv 0$ as $t \to \infty$. This signifies *stability in the small* of the initial equilibrium state $y \equiv 0$ in this system for any dead region $(-\sigma, \sigma)$ as $\sigma > 0$. However, the latter property is *not sufficient* for the acceptable functioning of the nonlinear control system given by (4.7) and (4.1) with distributed parameters. We need in fact *robust stability*, or *stability in the large*, of the equilibrium state $y \equiv 0$ for the closed-loop system under consideration, which in our case means that $y(t, x_0) \to 0$ as $t \to \infty$ even if the current state ξ of (4.7) is *outside* the dead region of (4.1) *after terminating* all the perturbations. The presence of perturbations $w(t)$ on some finite interval $[0, T]$ is clearly *irrelevant* to this stability issue, which is an *internal property* of the parabolic dynamics generated by the elliptic operator A from (1.2) on the *infinite horizon* and the three-positional feedback regulator (4.1) in the Dirichlet boundary conditions of (4.7).

It has been well recognized in the literature that stability in the large (or robust stability) issues are among the *most challenging* in stability theory for nonlinear dynamics, even in the case of finite-dimensional control systems governed by ordinary differential equations. We are not familiar with *any results* in this direction for the parabolic systems studied in this paper. To derive efficient conditions for stability in the large of the equilibrium state $y \equiv 0$ in the closed-loop control system (4.7) with the three-positional feedback regulator (4.1), we develop a *variational approach* to such robust stability, which is largely based on the *monotonicity* properties of the parabolic dynamics discussed above and reduces the stability issue to solving an open-loop *optimal control* problem for the initial system (1.1) on the *infinite horizon*.

To proceed, observe from the structure of the closed-loop control system under consideration that the required robust stability of its stationary equilibrium state $y \equiv 0$ can be *lost* if the dead region in (4.1) is *not sufficiently wide*. Indeed, in such cases the transients $\xi = y(t, x_0)$ would move back and forth between the dead region boundaries under switching control positions in (4.1) with *no external perturbations*, just by *inertia* of the control system that relates to a certain *time-delay*. This means that the closed-loop control system given by (4.7) and (4.1) may start functioning in a non-acceptable *self-vibrating regime* as $t \to \infty$ thus signifying *instability in the large* of the initial equilibrium stare $y = 0$. We intend to find efficient and verifiable conditions that exclude such instability.

It follows from the above discussions that the unstable self-vibrating regime will *not occur* if the transient $y(t, x_0)$ starting at one boundary of the dead region *does not reach* the other boundary whenever $t > 0$ under the control switching in (4.1) with no external perturbations. Moreover, the *limiting stability resource* of the system relates to the *minimal width* of the dead region ensuring the afore-mentioned property. This allows us to derive efficient stability conditions by solving an open-loop optimal control problem for (1.1) on the *infinite horizon* as is done in the proof of the next theorem.

Theorem 4.2 (robust stability of the closed-loop parabolic control system). *Let* (4.7) *be a closed-loop parabolic system under the standing assumptions of Section* 2*, and let* (4.1) *be a three-positional feedback regulator in the boundary conditions of* (4.7) *with arbitrary parameters* $v > 0$ *and* $\sigma > 0$*. Then the closed-loop control system given by* (4.7) *and* (4.1) *exhibits robust stability in the above sense if its parameters satisfy the relationship*

$$\sigma \geq -\frac{v(1 + c\gamma)}{2} + \frac{v + \sigma}{2} \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)(c + \lambda_i)}{\lambda_i} \Big(\frac{v}{v + \sigma}\Big)^{\frac{\lambda_i}{\lambda_1}}, \tag{4.8}$$

where the right-hand side is always positive. Furthermore, if the additional assumption (4.5) *is satisfied, then the stability condition can be simplified as*

$$\sigma \geq \frac{v}{2\lambda_1} \Big[\mu_1 \phi_1(x_0)(c + \lambda_1) - \lambda_1(1 + c\gamma)\Big], \tag{4.9}$$

where the right-hand side in (4.9) *is always greater than the one in* (4.8) *whenever* $v, \sigma > 0$.

Proof. Developing a *variational approach* to robust stability, we consider the following *open-loop* control system on the *infinite horizon*:

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = 0, \quad x \in \Omega, \quad t > 0, \\ y(0, x) = 0, \quad x \in \Omega, \\ y(t, x) = u(t), \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{4.10}$$

with piecewise constant Dirichlet boundary controls given by

$$u(t) = \begin{cases} h + \Delta h & \text{if } \ 0 \leq t \leq \tau, \\ h & \text{if } \ t > \tau, \end{cases} \tag{4.11}$$

where h and Δh are some *positive* numbers (to be specified later) while τ is a control switching time to be determined. Employing Proposition 2.1 on the spectral representation of the trajectories $y_\tau(t, x)$ for system (4.10) generated by controls (4.11) and taking into account the relationships

$$\sum_{i=1}^{\infty} \mu_i \phi_i(x) = 1 \ \text{ in } \ L^2(0, T) \ \text{ and } \ \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} = \gamma,$$

we get the explicit formulas for $y_\tau(t,x_0)$ used in what follows:

$$\begin{aligned} y_\tau(t,x_0) &= \sum_{i=1}^{\infty}\mu_i\phi_i(x_0)(c+\lambda_i)e^{-\lambda_i t}\int_0^t u(\theta)e^{\lambda_i\theta}\,d\theta \\ &= \sum_{i=1}^{\infty}\mu_i\phi_i(x_0)(c+\lambda_i)e^{-\lambda_i t}\Big(\int_0^\tau (h+\Delta h)e^{\lambda_i\theta}\,d\theta+\int_\tau^t he^{\lambda_i\theta}\,d\theta\Big) \\ &= (1+\gamma c)h+\sum_{i=1}^{\infty}\frac{\mu_i\phi_i(x_0)(c+\lambda_i)}{\lambda_i}\Big[\Delta he^{\lambda_i\tau}-(h+\Delta h)\Big]e^{-\lambda_i t}. \end{aligned} \tag{4.12}$$

It is easy to see from (4.12) that

$$y_\tau(t,x_0)\to(1+c\gamma)h \ \text{ as } \ t\to\infty \ \text{ whenever } \ \tau>0. \tag{4.13}$$

However, the transient $y(t,x_0)$ may intersect the stabilization level (4.13) if the switching time τ is not properly chosen. We intend to find efficient conditions under which the latter situation does *not occur*. These conditions, being of certain *interest for their own sake*, ensure the required *robust stability* of the closed-loop system (4.7), (4.1) when the control levels h and Δh in (4.11) are appropriately specified.

To proceed, consider the following auxiliary *dynamic optimization problem* for the parabolic system (4.10) on the *infinite horizon*:

$$\begin{cases} \text{minimize } \ J(\tau):=(1+c\gamma)h-y_\tau(\tau,x_0) \\ \text{subject to } \ (4.10),(4.11), \ \text{ and the state constraint} \\ y_\tau(t,x_0)<(1+c\gamma)h \ \text{ for all } \ t>0. \end{cases} \tag{4.14}$$

The meaning of this problem is to find an *optimal switching time* $\tau=\underline{\tau}>0$ in (4.11) such that the corresponding trajectory $y_{\underline{\tau}}(t,x_0)$ of system (4.10) lies strictly below the stabilization level (4.13) for all $t>0$ and that the distance between the stabilization level (4.13) and the underlying switching level

$$y(\underline{\tau},x_0):=y_{\underline{\tau}}(\underline{\tau},x_0)$$

is *minimal* in comparison with any other switching time τ satisfying all the constraints in (4.14). According to the above discussions on robust stability, solving the optimal control problem (4.14) leads us to required robust stability conditions.

It follows from the *monotonicity property* of Proposition 2.2 that

$$y_{\tau_1}(t,x_0)\le y_{\tau_2}(t,x_0) \ \text{ whenever } \ t>0 \ \text{ and } \ \tau_1\le\tau_2$$

for the transients $y_\tau(t,x_0)$ generated in (4.12) by the switching controls (4.11). This implies that the optimal switching time $\underline{\tau}$ to (4.14) is the *largest* one under which the corresponding transient $y_\tau(t,x_0)$ does not intersect the stabilization level $(1+c\gamma)h$ for all $t>0$. The *exact solution* to the open-loop control problem (4.14) on the infinite horizon is given in Theorem 3.3. It is provided by the *first term rule*, i.e., by vanishing the first term in the last series of (4.12). By this result we have the rigorously justified formula for the *optimal switching time* to (4.14):

$$\underline{\tau}=\frac{1}{\lambda_1}\ln\Big(\frac{h+\Delta h}{\Delta h}\Big)>0 \ \text{ whenever } \ v,\sigma>0,$$

and hence the *exact optimal value* of the cost functional in this problem is computed by:

$$\begin{aligned}\underline{\vartheta}: \quad &= J(\underline{\tau}) = (1+c\gamma)h - y(\underline{\tau}, x_0)\\ &= -\Delta h(1+c\gamma) + (h+\Delta h)\sum_{i=1}^{\infty}\frac{\mu_i\phi_i(x_0)(c+\lambda_i)}{\lambda_i}\Big(\frac{\Delta h}{h+\Delta h}\Big)^{\frac{\lambda_i}{\lambda_1}} > 0.\end{aligned} \tag{4.15}$$

Imposing the additional assumption (4.5), we get the feasible *first-order approximation*

$$\underline{\vartheta}_1 := \Delta h\Big[\frac{\mu_1\phi_1(x_0)(c+\lambda_1)}{\lambda_1} - (1+c\gamma)\Big] > \underline{\vartheta} > 0 \tag{4.16}$$

to (4.15), which occurs to be independent of the control level h in (4.11).

According to the description of the instability phenomenon given before the formulation of Theorem 4.2, robust stability of the closed-loop control system given by (4.7) and (4.1) is ensured if the width of the dead region 2σ is *not smaller* than the value $\underline{\vartheta}$ in (4.15) with $h = \sigma$ and $\Delta h = v$. Substituting these data into (4.15), we arrive at the stability condition (4.8) of the theorem. The explicit first-order approximation condition (4.9) corresponds to substituting the above values of h and Δh into formula (4.16) for $\underline{\vartheta}_1$ via the sufficient stability requirement $2\sigma \geq \underline{\vartheta}_1$. This completes the proof of the theorem. □

Finally, we combine the feedback control results derived in Section 6 from the viewpoint of controllability and minimax optimality with the robust stability conditions obtained in this section; thus we establish *optimal relationships* between all the parameters of the feedback control system that ensure its required behavior from the viewpoints of feasibility, minimax (sub)optimality, and robust stability.

Theorem 4.3 (optimal parameters of the stable feedback control design). *Consider the closed-loop control parabolic system* (4.7) *with uncertain perturbations* $w \in W$ *from* (1.4) *and with the there-positional feedback regulator* (4.1) *in the Dirichlet boundary conditions. In addition to the standing assumptions of Section* 2*, suppose that*

$$0 < \gamma\beta - \eta < \min\Big\{\alpha(1+c\gamma),\ \frac{\beta(1+c\gamma)}{c+\lambda_1}\Big\} \tag{4.17}$$

and that $\sigma^0 \leq \overline{\sigma}^\infty$*, where* $\overline{\sigma}^\infty > 0$ *is computed by* (4.3) *and where*

$$\begin{aligned}\sigma^0: \quad &= \frac{\gamma\beta-\eta+\eta(1+c\gamma)}{2(1+c\gamma)}\sum_{i=1}^{\infty}\frac{\mu_i\phi_i(x_0)(c+\lambda_i)}{\lambda_i}\Big(\frac{\gamma\beta-\eta}{\gamma\beta-\eta+\eta(1+c\gamma)}\Big)^{\frac{\lambda_i}{\lambda_1}}\\ &+\frac{\eta-\gamma\beta}{2} > 0.\end{aligned} \tag{4.18}$$

Then the feedback control system given by (4.7) *and* (4.1) *with the control resource* $v = \bar{v}$ *computed by*

$$\bar{v} = \frac{\gamma\beta-\eta}{1+c\gamma} \tag{4.19}$$

and the dead region parameter $\sigma > 0$ *belonging to the nonempty interval*

$$\sigma^0 \leq \sigma \leq \overline{\sigma}^\infty \tag{4.20}$$

is reliable on the infinite horizon in the sense that it is feasible by all the constraints in (P) on $[0,\infty)$ *for any one-sided perturbations* $w \in W$ *enjoying simultaneously robust stability.*

If in addition the first-order approximation assumption (4.5) *and the inequality*

$$3\mu_1\phi_1(x_0)(c+\lambda_1)(\gamma\beta-\eta) \le \lambda_1(3\gamma\beta-\eta)(1+c\gamma) \tag{4.21}$$

are satisfied, then we have

$$\sigma^0 < \sigma_1^0 := \frac{\gamma\beta-\eta}{2\lambda_1(1+c\gamma)}\Big[\mu_1\phi_1(x_0)(c+\lambda_1)-\lambda_1(1+c\gamma)\Big] \le \overline{\sigma}_1 \tag{4.22}$$

with $\overline{\sigma}_1 \in (0,\overline{\sigma})$ *computed in* (4.6), *and the feedback control system given by* (4.7) *and* (4.1) *with the control resource* $v=\bar{v}$ *from* (4.19) *and the dead region parameter* $\sigma > 0$ *satisfying*

$$\sigma_1^0 \le \sigma \le \overline{\sigma}_1 \tag{4.23}$$

is reliable on the infinite horizon in the sense described above.

Proof. To a large extend, this theorem unifies and summarizes the feedback control design and robust stability results derived above.

Indeed, Theorem 4.2 ensures robust stability of system (4.7) with the three-positional regulator (4.1), where v is computed by (4.19) and where $\sigma \ge \sigma^0$ with σ^0 computed by (4.18). This follows from the observation that the value of σ^0 in (4.18) is in fact obtained by substituting $\bar{v}$ from (4.19) into the right-hand side of (4.8) and by replacing σ with η therein. Further, we easily conclude that σ^0 satisfies inequality (4.8) whenever $0 < \sigma \le \eta$ in the right-hand side of it, which is the case under consideration. The other statements in the first part of the theorem follow directly from Theorem 4.1.

To justify the last part of the theorem, under conditions (4.5) and (4.21), we first observe that the value of σ_1^0 in (4.22) is obtained by substituting $\bar{v}$ from (4.19) into the right-hand of (4.9). Furthermore, condition (4.21) easily follows from $\sigma_1^0 \le \overline{\sigma}_1$ by substituting there $\overline{\sigma}_1$ from (4.6) and σ_1^0 from (4.22). Thus the one-sided feasibility of the three-positional regulator (4.1) with $v=\bar{v}$ from (4.19) and σ from (4.23) follows from Theorem 4.1 due to

$$\sigma_1^0 \le \sigma \le \overline{\sigma}_1,$$

while the corresponding robust stability of the closed-loop system given by (4.7) and (4.1) with σ from (4.23) follows from the last part of Theorem 4.2. □

Observe finally the first order reliability condition (4.21) can be surprisingly rewritten in the very simple form $\overline{\sigma}_1 \ge \eta/3$ via just the first-order suboptimality value $\overline{\sigma}_1$ computed by (4.6). Note that the equality therein can be used as an additional equation for *shape optimization* to determine, e.g., the *optimal parameters of the domain* Ω ensuring a stable feedback control design under the other given data of the minimax problem (P) and the feedback regulator (4.1).

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