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SOME GENERALIZATIONS OF ROCKAFELLAR'S SURJECTIVITY THEOREM

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Dedicated to Michel Théra on the occasion of his 60th birthday

Abstract: We prove some generalizations of Rockafellar's surjectivity theorem and related results, which consist in replacing the duality mapping by another maximal monotone operator satisfying suitable conditions.

Key words: *Rockafellar's surjectivity theorem, maximal monotone operators, Fitzpatrick functions*

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1 Introduction and Preliminaries

This paper presents some generalizations of Rockafellar's surjectivity theorem [11, p. 78, Corollary] and related results. In our staments the duality mapping, which occurs in the classical results, is replaced by a more general maximal monotone operator satisfying suitable conditions.

Our operators are defined on a reflexive real Banach space X; we denote its dual by X^* and the duality product by $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$. We use the symbols $\|\cdot\|$ and $\|\cdot\|_*$ to represent the norms in X and X^*, respectively.

We recall that a multivalued operator $A : X \rightrightarrows X^*$ is said to be *monotone* provided that, for any $x, y \in X$, $x^* \in A(x)$ and $y^* \in A(y)$,

$$\langle x - y, x^* - y^* \rangle \geq 0; \tag{1.1}$$

if this inequality is strict whenever $x \neq y$, we say that A is *strictly monotone*. The *inverse* of A is the operator $A^{-1} : X^* \rightrightarrows X$ defined by

$$A^{-1}(x^*) := \{x \in X : x^* \in A(x)\}.$$

When for an element $x \in X$ the set $A(x)$ reduces to a singleton, we will use the same notation $A(x)$ to denote the only element in this set. The *graph* of A is the set

$$Graph(A) := \{(x, x^*) \in X \times X^* : x^* \in A(x)\};$$

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its projections onto X and X^* are called the *domain* of A and the *range* of A, respectively, and will be denoted by $D(A)$ and $R(A)$. If A is monotone and $Graph(A)$ is not properly included in the graph of any other monotone operator, one says that A is *maximal monotone.* Every monotone operator A admits a maximal monotone extension M, that is, a maximal monotone operator M whose graph contains that of A. The existence of such a maximal monotone extension is usually proved by invoking Zorn's lemma, but it has been recently proved that an explicit construction [2, Thm. 5.7] is possible. We will say that $(x, x^*) \in X \times X^*$ is *monotonically related* to $(y, y^*) \in X \times X^*$ if (1.1) holds. Clearly, a monotone operator is maximal monotone if and only if all points that are monotonically related to every point in $Graph(A)$ belong to $Graph(A)$.

We will use standard concepts and notations of convex analysis (see, e.g., the book [17]). The *domain* of a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is

$$dom\ f := \{x \in X : f(x) < +\infty\}.$$

A convex function is said to be *proper* if its domain is nonempty. The *Fenchel conjugate* of a proper convex function f is $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$, defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}.$$

From this definition the so-called *Fenchel inequality* immediately follows:

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle \quad \forall\ (x, x^*) \in X \times X^*. \tag{1.2}$$

One says that $x^* \in X^*$ is a *subgradient* of the l.s.c. proper convex function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ at $x \in dom\ f$ provided that $f(y) \geq f(x) + \langle x^*, y - x \rangle$ for all $y \in X$. The set $\partial f(x)$ of all subgradients of f at x is called the *subdifferential* of f at x; one sets $\partial f(x) := \emptyset$ for $x \in X \setminus dom\ f$. For every $(x, x^*) \in X \times X^*$, one has

$$x^* \in \partial f(x) \Longleftrightarrow f(x) + f^*(x^*) = \langle x, x^* \rangle,$$

that is, the set of points at which Fenchel inequality (1.2) holds with the equal sign is precisely $Graph(\partial f)$. The set $\partial f(x)$ reduces to a singleton if and only if f is *Gateaux differentiable* at x; this means the existence of an element of X^*, which we will denote by $\nabla f(x)$, such that, for all $d \in X$,

$$\lim_{t \to 0} \frac{f(x + td) - f(x)}{t} = \langle d, \nabla f(x) \rangle.$$

In this case $\partial f(x) = \{\nabla f(x)\}$. The operator ∂f is known to be maximal monotone [12, Thm. A]. An important function in functional analysis is $\frac{1}{2}\|\cdot\|^2$; its subdifferential is called the *duality mapping.* In our reflexive setting, the Fenchel conjugate of $\frac{1}{2}\|\cdot\|^2$ is $\frac{1}{2}\|\cdot\|_*^2$. One says that a l.s.c. proper convex $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *cofinite* if f^* is finite-valued. A sufficient condition, which is also necessary in the finite-dimensional case, for f to be cofinite is the *supercoercivity* property $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty$ (see [1, Thm. 3.4]); for an example of a cofinite function in l_2 which fails to be supercoecive, see [1, Example 7.5].

The *Fitzpatrick function* of an operator $A : X \rightrightarrows X^*$ was introduced in [6] as the function $\varphi_A : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi_A(x, x^*) := \langle x, x^* \rangle - \inf_{(y, y^*) \in Graph(A)} \langle x - y, x^* - y^* \rangle.$$

If A is maximal monotone, it satisfies $\varphi_A \geq \langle\cdot,\cdot\rangle$ and the set of points at which these two functions coincide is precisely *Graph* (A); moreover, it is the smallest convex function having these two properties. After the independent rediscovery of the Fitzpatrick function in [8] and [5], it has been used by many authors to obtain elegant convex analytic proofs of some old as well as some new results in the theory of maximal monotone operators.

Another useful convex representation of an operator $A : X \rightrightarrows X^*$ is provided by $\sigma_A : X \times X^* \longrightarrow \mathbb{R}\cup\{+\infty\}$. This function, introduced in [5], is defined as the largest l.s.c. convex function that is bounded above by the duality product on *Graph* (A). According to [7, Cor. 7], A is monotone if and only if $\sigma_A \geq \langle\cdot,\cdot\rangle$ (an equivalence that does not hold if σ_A is replaced by φ_A [7, p. 35]); if, furthermore, A is maximal monotone, it shares with φ_A the property that the set of points where it coincides with the duality product is precisely *Graph* (A) [7, p. 27 and Thm 13]. The Fenchel conjugate of σ_A is φ_A [7, (7)] and (in our reflexive setting) viceversa.

We will use the following version of *Fenchel-Rockafellar duality theorem*:

Theorem 1.1 ([10, Cor. 1]). *Suppose that $f, g : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ are l.s.c. proper convex functions. If the domain of one of these functions contains an interior point of the domain of the other, then*

$$\inf_{x\in X}\{f(x)+g(x)\} = \max_{x^*\in X^*}\{-f^*(x^*)-g^*(-x^*)\}.$$

$\boxed{2}$ Surjectivity Theorems

Our first result is a generalization of [13, Thm. 10.6].

Theorem 2.1. *For every monotone operator $A : X \rightrightarrows X^*$, the following statements are equivalent:*

(1) *A is maximal monotone.*

(2) *Graph (A) + Graph $(-B) = X \times X^*$ for every maximal monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued.*

(3) *There exists a maximal monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued, Graph (A) + Graph $(-B) = X \times X^*$, and there exists $(p,p^*) \in$ Graph (B) such that $\langle p-y, p^*-y^*\rangle > 0$ for every $(y,y^*) \in$ Graph $(B)\setminus\{(p,p^*)\}$.*

Proof. (1) $\Longrightarrow$ (2). The proof of this implication is inspired by that of [14, Thm. 1.2]. Let $(x_0, x_0^*) \in X \times X^*$, consider the operator $A' : X \rightrightarrows X^*$ such that *Graph* $(A') :=$ *Graph* $(A) - (x_0, x_0^*)$, and define $h : X \times X^* \longrightarrow \mathbb{R}\cup\{+\infty\}$ by $h(x,x^*) := \varphi_B(-x,x^*)$. Since A' is monotone and B is maximal monotone, for every $(x,x^*) \in X \times X^*$ one has

$$\sigma_{A'}(x,x^*) + h(x,x^*) \geq \langle x,x^*\rangle + \langle -x,x^*\rangle = 0.$$

Hence, by Thm. 1.1, there exists $(y^*,y) \in X^* \times X$ such that

$$\varphi_{A'}(y,y^*) + h^*(-y^*,-y) \leq 0. \tag{2.1}$$

Since

$$\begin{aligned} h^*(-y^*,-y) &= \sup_{(z,z^*)\in X\times X^*}\{\langle z,-y^*\rangle + \langle -y,z^*\rangle - h(z,z^*)\} \\ &= \sup_{(z,z^*)\in X\times X^*}\{\langle z,-y^*\rangle + \langle -y,z^*\rangle - \varphi_B(-z.z^*)\} \\ &= \sup_{(z,z^*)\in X\times X^*}\{\langle z,y^*\rangle + \langle -y,z^*\rangle - \varphi_B(z.z^*)\} \\ &= \varphi_B^*(y^*,-y) = \sigma_B(-y,y^*), \end{aligned}$$

we have

$$\varphi_{A'}(y,y^*) + h^*(-y^*,-y) = \varphi_{A'}(y,y^*) + \sigma_B(-y,y^*) \geq \langle y,y^*\rangle + \langle -y,y^*\rangle = 0;$$

so, in view of (2.1) we deduce that

$$\varphi_{A'}(y,y^*) = \langle y,y^*\rangle \text{ and } \sigma_B(-y,y^*) = \langle -y,y^*\rangle,$$

that is,

$$(y,y^*) \in Graph(A') \text{ and } (-y,y^*) \in Graph(B).$$

We thus have

$$\begin{aligned}(x_0,x_0^*) &= (x_0,x_0^*) + (y,y^*) + (-y,-y^*) \in (x_0,x_0^*) + Graph(A') + Graph(-B) \\ &= Graph(A) + Graph(-B).\end{aligned}$$

(2) $\Longrightarrow$ (3). Take $B = \partial\frac{1}{2}\|.\|^2$, the duality mapping. Then, as φ_B is the smallest convex representation of B, for every $(x,x^*) \in X \times X^*$ one has

$$\varphi_B(x,x^*) \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 < +\infty,$$

so that φ_B is finite-valued. Moreover, $(0,0) \in Graph(B)$ and, for every $(y,y^*) \in Graph(B)\setminus\{(0,0)\}$, one has

$$\langle 0-y, 0-y^*\rangle = \langle y,y^*\rangle = \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|_*^2 > 0.$$

(3) $\Longrightarrow$ (1). Let $(x,x^*) \in X \times X^*$ be monotonically related to every point in $Graph(A)$ and take (p,p^*) as in (3). Since $(x+p, x^*-p^*) \in X \times X^* = Graph(A) + Graph(-B)$, we have

$$(x+p, x^*-p^*) = (y,y^*) + (z,z^*) \tag{2.2}$$

for some $(y,y^*) \in Graph(A)$ and $(z,z^*) \in Graph(-B)$. We have $x-y = z-p$ and $x^*-y^* = z^*+p^*$; hence, by $(z,-z^*) \in Graph(B)$, we get $0 \leq \langle x-y, x^*-y^*\rangle = -\langle z-p, -z^*-p^*\rangle \leq 0$. Therefore $\langle z-p, -z^*-p^*\rangle = 0$, which implies $z = p$ and $-z^* = p^*$. Thus equality (2.2) yields $(x+p, x^*-p^*) = (y,y^*) + (p,-p^*)$, and hence $(x,x^*) = (y,y^*) \in Graph(A)$, which proves the maximality of A. □

The finite-valuedness of the Fitzpatrick function of a maximal monotone operator, used in statements (2) and (3) of Thm. 2.1 and in some of the subsequent results in this paper, is stronger than the Brézis-Haraux condition introduced in [4], as this condition can be easily seen to be equivalent to the finite-valuedness of the Fitzpatrick function on the Cartesian product of the domain with the range. In view of a theorem due to Torralba [15] according to which, for any maximal monotone operator A, given $\alpha,\beta > 0$ and $(x,x^*) \in X \times X^*$ such that $\inf_{(y,y^*)\in Graph(A)} \langle x-y, x^*-y^*\rangle \geq -\alpha\beta$ there exists $(z,z^*) \in Graph(A)$ such that $\|z-x\| \leq \alpha$ and $\|z^*-x^*\|_* \leq \beta$, it turns out that every maximal monotone operator having a finite-valued Fitzpatrick function has a dense domain and a dense range. As a consequence of Thm. 2.1 we obtain the following stronger result:

Corollary 2.2. *Let $B : X \rightrightarrows X^*$ a maximal monotone operator. If φ_B is finite-valued then $D(B) = X$ and $R(B) = X^*$.*

Proof. The result on the range follows from implication (1) $\Longrightarrow$ (2) of Thm. 2.1 by setting A equal to the identically zero operator. Similarly, the result on the domain follows by setting A equal to the operator whose domain and range are the singleton of the origin and the whole of X^*, respectively. □

In view of the preceding corollary, it turns out that a maximal monotone operator has a finite-valued Fitzpatrick function if and only if it has full domain and full range and satisfies the Brézis-Haraux condition.

Another immediate consequence of Thm. 2.1 is the following existence result for monotone generalized variational inequalities over (non necessarily bounded) closed convex sets.

Corollary 2.3. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. If φ_T is finite-valued then for every closed convex set $K \subseteq X$ there exist $x \in K$ and $x^* \in T(x)$ such that*

$$\langle y - x, x^* \rangle \geq 0 \quad \forall\, y \in K. \tag{2.3}$$

Proof. Let $N_K : X \rightrightarrows X^*$ be the normal cone operator to K, given by

$$N_K(x) = \begin{cases} \{x^* \in X^* : \langle y - x, x^* \rangle \leq 0 \quad \forall\, y \in K\} & \text{if } x \in K \\ \emptyset & \text{if } x \notin K, \end{cases}$$

and define $B : X \rightrightarrows X^*$ by $B(x) = -T(-x)$. These two operators are maximal monotone; moreover, since $\varphi_B(x, x^*) = \varphi_T(-x, -x^*)$ for all $(x, x^*) \in X \times X^*$, φ_B is finite-valued. Hence, by implication (1) $\Longrightarrow$ (2) in Thm. 2.1, $(0,0) \in \mathit{Graph}(N_K) + \mathit{Graph}(-B)$, that is, there exists $(x, y^*) \in \mathit{Graph}(N_K)$ such that $(-x, -y^*) \in \mathit{Graph}(-B)$. Then $x \in K$ and setting $x^* = -y^*$ we have $(x, -x^*) = (x, y^*) \in \mathit{Graph}(N_K)$, which yields (2.3), and $x^* = -y^* \in -B(-x) = T(x)$. □

We will next consider the special case of Thm. 2.1 when B is the subdifferential of a l.s.c. proper convex function; to this aim we first give a characterization of those l.s.c. proper conve functions whose subdifferential operator has a finite-valued Fitzpatrick function.

Proposition 2.4. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. proper convex function. Then $\varphi_{\partial f}$ is finite-valued if and only if f is finite-valued and cofinite.*

Proof. If f is finite-valued and cofinite then, for every $(x, x^*) \in X \times X^*$, one has $\varphi_{\partial f}(x, x^*) \leq f(x) + f^*(x^*) < +\infty$. Conversely, if $\varphi_{\partial f}$ is finite-valued then, by Cor. 2.2, $D(\partial f) = X$ and $R(\partial f) = X^*$; since $D(\partial f) \subseteq \mathit{dom}\ f$ and $R(\partial f) \subseteq \mathit{dom}\ f^*$, it follows that both f and f^* are finite-valued. □

Corollary 2.5. *Let $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \longrightarrow \mathbb{R}$ be a cofinite l.s.c. proper convex function. Then A is maximal monotone if and only if $\mathit{Graph}(A) + \mathit{Graph}(-\partial f) = X \times X^*$.*

Proof. By Prop. 2.4, $\varphi_{\partial f}$ is finite-valued. Therefore, if A is maximal monotone, by implication (1) $\Longrightarrow$ (2) in Thm. 2.1 we have $\mathit{Graph}(A) + \mathit{Graph}(-\partial f) = X \times X^*$. Conversely, assume that $\mathit{Graph}(A) + \mathit{Graph}(-\partial f) = X \times X^*$. Since f is continuous [9, Prop. 3.3] and reflexive spaces are Asplund [16, Cor. 5], that is, continuous convex functions defined on them are generically Fréchet differentiable, we can pick a point $p \in X$ at which f is Gateaux differentiable, which means that $\partial f(p)$ is a singleton $\{p^*\}$. We thus have $(p, p^*) \in \mathit{Graph}(\partial f)$, and one can easily check that the equality $\partial f(p) = \{p^*\}$ is equivalent to

$$f^*(y^*) > f^*(p^*) + \langle p, y^* - p^* \rangle \quad \forall\, y^* \in X^* \setminus \{p^*\},$$

from which it readily follows that $\langle p-y, p^*-y^*\rangle > 0$ for every $(y, y^*) \in Graph\,(\partial f) \setminus \{(p, p^*)\}$. Hence we can use implication (2) $\Longrightarrow$ (3) of Thm. 2.1 to deduce the maximality of A. □

The following lemma will be useful; we omit its easy proof.

Lemma 2.6. *If $A : X \rightrightarrows X^*$ is monotone and $B : X \rightrightarrows X^*$ is strictly monotone then $A+B$ is strictly monotone and hence $(A+B)^{-1}$ is single-valued.*

The following corollary generalizes Rockafellar surjectivity theorem [11, p. 78, Corollary].

Corollary 2.7. *Let $A : X \rightrightarrows X^*$ be a monotone operator and $B : X \rightrightarrows X^*$ be a maximal monotone operator with finite-valued Fitzpatrick function φ_B. If A is maximal monotone then $R(A+B) = X^*$. Conversely, if B is single-valued and strictly monotone and $R(A+B) = X^*$ then A is maximal monotone.*

Proof. Assume that A is maximal monotone and let $x^* \in X^*$. Define the operator $\widehat{B} : X \rightrightarrows X^*$ by $\widehat{B}(x) := -B(-x)$. Clearly, $\widehat{B}$ is maximal monotone and $\varphi_{\widehat{B}}(x, x^*) = \varphi_B(-x, -x^*)$ for every $(x, x^*) \in X \times X^*$; hence, by implication (1) $\Longrightarrow$ (2) of Thm. 2.1, $Graph\,(A) + Graph\left(-\widehat{B}\right) = X \times X^*$. Therefore we can write $(0, x^*) = (y, y^*) + (z, z^*)$ for some $(y, y^*) \in Graph\,(A)$ and $(z, z^*) \in Graph\left(-\widehat{B}\right)$. Since $y^* \in A(y)$ and $z^* \in -\widehat{B}(z) = B(-z) = B(y)$, from the equality $x^* = y^* + z^*$ it follows that $x^* \in (A+B)(y) \subseteq R(A+B)$, which proves that $R(A+B) = X^*$.

Conversely, assume that B is single-valued and strictly monotone and $R(A+B) = X^*$. Consider a maximal monotone extension M of A. Let $x \in X$ and $x^* \in M(x)$. Clearly, $x^* + B(x) \in (M+B)(x)$; on the other hand, since by the direct statement we have $R(M+B) = X^*$, it follows that $x^* + B(x) \in (A+B)(y) \subseteq (M+B)(y)$ for some y. Thus, in view of Lemma 2.6, $x = y$, and hence from $x^* + B(x) \in (A+B)(y)$ we deduce that $x^* + B(x) \in A(x) + B(x)$, that is, $x^* \in A(x)$. This proves that $M(x) \subseteq A(x)$ for every $x \in X$, which, as M is maximal monotone, shows that $A = M$. □

Another generalization of Rockafellar surjectivity theorem is provided next.

Theorem 2.8. *Let $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \longrightarrow \mathbb{R}$ be a l.s.c. proper convex function. If f is Gateaux differentiable everywhere, $R(A+\nabla f) = X^*$, and ∂f^* is single-valued on its domain then A is maximal monotone. Conversely, if A is maximal monotone and f is cofinite then $R(A+\partial f) = X^*$.*

Proof. Let us first assume that f is Gateaux differentiable everywhere, $R(A+\nabla f) = X^*$ and ∂f^* is single-valued on its domain, and let $(x, x^*) \in X \times X^*$ be monotonically related to every point in $Graph\,(A)$. Since $x^* + \nabla f(x) \in X^* = R(A+\nabla f)$, there exists $(a, a^*) \in Graph\,(A)$ such that

$$x^* + \nabla f(x) = a^* + \nabla f(a). \tag{2.4}$$

As $\nabla f(a) - \nabla f(x) = x^* - a^*$, it follows that $0 \le \langle a-x, \nabla f(a) - \nabla f(x)\rangle = -\langle a-x, a^*-x^*\rangle \le 0$; therefore

$$\begin{aligned} 0 &= \langle a-x, \nabla f(a) - \nabla f(x)\rangle \\ &= \langle a, \nabla f(a)\rangle - \langle a, \nabla f(x)\rangle - \langle x, \nabla f(a)\rangle + \langle x, \nabla f(x)\rangle \\ &= f(a) + f^*(\nabla f(a)) - \langle a, \nabla f(x)\rangle - \langle x, \nabla f(a)\rangle + f(x) + f^*(\nabla f(x)). \end{aligned}$$

From this equality and the fact that $f(a)+f^*(\nabla f(x))-\langle a,\nabla f(x)\rangle$ and $f(x)+f^*(\nabla f(a))-\langle x,\nabla f(a)\rangle$ are nonnegative we deduce that these two numbers are equal to 0, which means that $\nabla f(x)=\nabla f(a)$. Hence, by (2.4), $x^*=a^*$; moreover, as $(\nabla f)^{-1}=\partial f^*$ is single-valued on its domain it follows that $x=a$. We thus have $(x,x^*)=(a,a^*)\in \mathit{Graph}(A)$, which proves the maximality of A.

The converse is an immediate consequence of Cor. 2.7, since, as observed at the beginning of the proof of Cor. 2.5, the finite-valuedness of f together with its cofiniteness implies that φ_B is finite-valued. □

We remark that the direct statement in Thm. 2.8 is not a consequence of Cor. 2.7, since $\varphi_{\partial f}$ is not necessarily finite-valued as f^* may take the value $+\infty$. On the other hand, as observed in [13, p. 44] (in the particular case when $f=\frac{1}{2}\|\cdot\|^2$, in which ∂f is the duality mapping), the maximality of A does not necessarily hold without the assumption that f is Gateaux differentiable and ∂f^* is single-valued on its domain. To avoid this assumption one has to strengthen the surjectivity condition; this is done in the next proposition, which generalizes [3, Prop. 4].

Proposition 2.9. *Let $A:X\rightrightarrows X^*$ be monotone and $f:X\longrightarrow \mathbb{R}\cup\{+\infty\}$ be a l.s.c. proper convex function. If $R(A(\cdot+w)+\partial f)=X^*$ for all $w\in X$ and both f and f^* attain their unique global minima at the origin then A is maximal monotone. Conversely, if A is maximal monotone and f is finite-valued and cofinite then $R(A(\cdot+w)+\partial f)=X^*$ for all $w\in X$.*

Proof. Assume first that $R(A(\cdot+w)+\partial f)=X^*$ for all $w\in X$ and both f and f^* attain their unique global minima at the origin, and let $(x,x^*)\in X\times X^*$ be monotonically related to every point in $\mathit{Graph}(A)$. Since $x^*\in X^*=R(A(\cdot+x)+\partial f)$, we have $x^*\in A(a+x)+\partial f(a)$ for some $a\in X$. We can therefore write $x^*=a^*+s^*$ for some $a^*\in A(a+x)$ and $s^*\in\partial f(a)$. Using that $a^*-x^*=-s^*$, we obtain

$$\begin{aligned} 0 &\le \langle a+x-x,a^*-x^*\rangle=\langle a,a^*-x^*\rangle=-\langle a,s^*\rangle=-(f(a)+f^*(s^*))\\ &\le -(f(0)+f^*(0))\le 0; \end{aligned}$$

hence $f(a)+f^*(s^*)=f(0)+f^*(0)$, which, by the assumption on f and f^*, implies that $a=0$ and $s^*=0$. We thus conclude that $x^*=a^*\in A(x)$, thus proving the maximality of A.

The converse follows immediately from Thm. 2.8, taking into account that $A(\cdot+w)$ is maximal monotone if A is. □

To conclude, we observe that some of our results, namely implication (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) in Thm. 2.1, Lemma 2.6, the "only if" statement in Prop. 2.4, the converse statement in Cor. 2.7 and the direct statements in Thm. 2.8 and Prop. 2.9 do not require the space to be reflexive.

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References

[1] H.H. Bauschke, J.M. Borwein and P.L. Combettes, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, *Commun. Contemp. Math.* 3 (2001) 615–647.

[2] H.H. Bauschke and X. Wang, The kernel average for two convex functions and its application to the extension and representation of monotone operators, Preprint available at http://www.optimization-online.org/DB_HTML/2007/05/1658.html (2007).

[3] J.M. Borwein, Maximality of sums of two maximal monotone operators, *Proc. Am. Math. Soc.* 134 (2006) 2951–2955.

[4] H. Brézis and A. Haraux, Image d'une somme d'opérateurs monotones et applications, *Isr. J. Math.* 23 (1976) 165–186.

[5] R.S. Burachik and B.F. Svaiter, Maximal monotone operators, convex functions and a special family of enlargements, *Set-Valued Anal.* 10 (2002) 297–316.

[6] S. Fitzpatrick, Representing monotone operators by convex functions, *Proc. Cent. Math. Anal. Aust. Natl. Univ.* 20 (1988) 59–65.

[7] J.-E. Martínez-Legaz and B.-F. Svaiter, Monotone operators representable by l.s.c. convex functions, *Set-Valued Anal.* 13 (2005) 21–46.

[8] J.-E. Martínez-Legaz and M. Théra, A convex representation of maximal monotone operators, *J. Nonlinear Convex Anal.* 2 (2001) 243–247.

[9] R.R. Phelps, *Convex functions, monotone operators and differentiability*, Springer, Berlin, 1989.

[10] R.T. Rockafellar, Extension of Fenchel's duality theorem for convex functions*, Duke Math. J.* 33 (1966) 81–89.

[11] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators*, Trans. Am. Math. Soc.* 149 (1970) 75–88.

[12] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pac. J. Math.* 33 (1970) 209–216.

[13] S. Simons, *Minimax and Monotonicity*, Springer, Berlin, 1998.

[14] S. Simons. and C. Zalinescu, A new proof for Rockafellar's characterization of maximal monotone operators, *Proc. Am. Math. Soc.* 132 (2004) 2969–2972.

[15] D. Torralba, *Convergence épigraphique et changements d'échelle en analyse variationnelle et optimisation*, Thèse de Doctorat, Université de Montpellier II, 1996.

[16] S.L. Troyanski, On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, *Stud. Math.* 37 (1970) 173–180.

[17] C. Zalinescu, *Convex analysis in general vector spaces*, World Scientific, Singapore, 2002.

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