Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 4, No. 3, pp. 513–525, September 2008

GENERIC WELL POSEDNESS IN LINEAR PROGRAMMING

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Abstract: We consider the following pair of linear programming problems in duality:

$$\mathcal{P}_A : \begin{cases} \inf cx \\ Ax \geq b \end{cases}$$

and

$$\mathcal{P}_A^D : \begin{cases} \sup by \\ A^t y = c, y \geq 0, \end{cases}$$

parameterized by the $m \times n$ matrix A defining the inequality constraints. The main result of the paper states that in the case $m \geq n$ the set S of well posed problems in a very strong sense is a generic subset of the set of problems having solution. Generic here means that S is an open and dense set whose complement is contained in a finite union of algebraic surfaces of dimension less than mn.

Key words: *well posedness, linear programming, genericity*

Mathematics Subject Classification: *49K40, 90C05; 90C46, 90C31*

1 Introduction

In general optimization setting, a problem is usually defined "well posed" provided it has solution (usually, but not necessarily, unique) and behaves nicely in at least one of the two following senses:

- either sequences of points approximating in value the value of the problem converge to the solution of the problem;
- or the solution does not change much if the basic data of the problem are submitted to little perturbations.

The first type is usually called well posedness of *Tykhonov type*, since the pioneering definition given by Tykhonov for (unconstrained) minimum problems. In case of constrained problems one can deal with the notions of *Levitin-Polyak* well posedness, of *strong well posedness*, and this type of ideas is underlying the definitions of well posedness for variational and quasi variational inequalities, saddle point problems, Nash equilibria ecc. The second type of good behavior is often called well posedness of *Hadamard type*, or also *continuous dependence from the data*, and it can be given several different forms, according for instance to the type of convergences required on the data and/or on the solutions (see [3, 8] for an extensive presentation of the previous issues).

More recently, new stronger notions of well posedness were defined, with the aim of capturing both the above aspects. In this framework, a suitably topologized family of problems is considered, and a problem is defined well posed if it has unique solution, toward which converge sequences of approximated solutions of approximating problems.

Let us make more precise one type of such a notion of well posedness in minimum problems (see for instance [14, 7]). Consider a metric space (M, ρ), the parameter space, and families of functions $\{f_\theta : X \to (-\infty, \infty] : \theta \in M\}$, with X a topological space.

Definition 1.1. The minimum problem engendered by $\theta \in M$ is well posed *provided:*

1. $\inf f_\theta$ *is finite and attained at a unique point* $x_0 \in X$*;*
2. *for any sequence* $\{\theta_n\}$ ρ *converging to* θ*, inf* f_{θ_n} *is finite for large* n *and for any sequence* $\{x_n\} \subset X$ *such that* $f_{\theta_n}(x_n) - \inf f_{\theta_n} \to 0$*, then* $x_n \to x_0$ *(convergence of approximating solutions);*
3. $\inf f_{\theta_n} \to \inf f_\theta$ *(continuity of the value function).*

Of course, quite often in given classes of problems not all are well posed, either because they simply do not have solutions or, even when they have exactly one solution, because approximating sequences may fail to converge. In this case, it becomes interesting to give a qualitative idea of "how many" problems are well posed in a given, suitably topologized, specific class. In literature, there are several notions to make precise the idea of "how many". For instance, a set can be considered big when it is a second category set in a Baire space. Or else, if its complement is a null measure set in a Euclidean space. A more sophisticated notion of bigness can be given by using the notion of $\sigma - porosity$, which provides stronger results with respect to both the Baire and measure type results (see [8]).

In this paper we are concerned with well posedness in the setting of classical linear programming. More precisely, we consider problems $\mathcal{P}_A$ and $\mathcal{P}_A^D$ of the form:

$$\mathcal{P}_A : \begin{cases} \inf cx \\ Ax \geq b, \end{cases}$$

together with the associated dual problem:

$$\mathcal{P}_A^D : \begin{cases} \sup by \\ A^t y = c, y \geq 0. \end{cases}$$

Here A is an $m \times n$ matrix, and b and c are vectors of suitable dimensions. We shall assume, for reasons to be explained later, that the cost vectors c and b are given and fixed, so that only the entries of the matrix A will be subject to perturbations. We now need to identify a natural (metric) space of problems. It is interesting to consider the biggest family of matrices where it is possible to prove that well posedness is generic. We cannot, in general, consider *all* matrices. For, it is simple to provide examples where all problems in a given ball (with the usual identification of the matrix space with the Euclidean space of dimension mn) are such that one problem is unfeasible (remember that a problem is unfeasible if no element of the space fulfills the constraints, and that in this case the other one is either unfeasible or unbounded). Thus not even a density result can be proved in this case. Thus we shall consider the set $\mathcal{P}$ which is the closure of the set of all matrices such that both the problem and its dual are feasible. Next, we shall provide a strong definition of well posedness, requiring essentially well posedness in the strong sense above for both the problem and its dual.

The main result of the paper reads: in the case when $m \geq n$ the set S of the well posed problems is open, dense and its complement is contained in a finite union of algebraic surfaces of dimension less than mn. This means that the set of the well posed problems is "big" in a very strong sense. Easy examples show that the condition $m \geq n$ cannot dispensed with. To prove the main result, we shall actually show that uniqueness of the solution, for both problems, is indeed sufficient to guarantee well posedness. It is also observed that the set of problems with unique solution (without requiring uniqueness of the solution of the dual), is *not* open.

Few words about the existing literature on this topic. First of all, our research was motivated by the following result for matrix games in [1]:

Theorem 1.2. *The set of two person zero sum games having unique solution is open and dense in the space of the $m \times n$ matrices with respect to the Euclidean topology.*

It is well-known [8] that every zero-sum game can be transformed in a linear programming problem in such a manner that every solution to the linear programming problem corresponds to an optimal strategy for player one, and every solution to the dual corresponds to an optimal strategy for player two. Theorem 1.2 can be consequently read in the setting of linear programming and ensures the genericity of problems with unique solution in a fixed subset.

The key feature of the game theory approach is the fact that, due to the interpretation of the problem, only the coefficients of the matrix can be perturbed, in contrast with the known genericity results in the field of linear programming. This explains our choice of keeping fixed the cost vectors b and c. There is one more reason. Usually, when we reduce the flexibility of the possible perturbations, the results require more sophisticated techniques (see for instance [8]). And actually our arguments here are different from other ones proving similar results, but allowing perturbations also of the vectors b and c. Just to quote some of the known results, see for example [4, 6, 13, 2, 5]. In particular, in [5], the authors consider the set of problems having a strongly unique optimal solution and they show that it contains an open and dense subset of the set of solvable problems in the same class. It must also be pointed out that the above results deal with semi-infinite linear programming.

$\boxed{2}$ Statement of the Problem and Notations

Fix $m, n \in \mathbb{N}$, $b \in \mathbb{R}^m \setminus \{0\}$, $c \in \mathbb{R}^n \setminus \{0\}$. To a matrix A of size $m \times n$ we associate the following pair of linear programming problems in duality:

$$\mathcal{P}_A : \begin{cases} \inf cx \\ Ax \geq b \end{cases} \qquad\qquad \mathcal{P}_A^D : \begin{cases} \sup by \\ A^t y = c, y \geq 0. \end{cases}$$

The above inequalities between vectors in Euclidean spaces must as usual be intended coordinatewise. Denote by F_A and F_A^D the *feasible sets* of $\mathcal{P}_A$ and $\mathcal{P}_A^D$ respectively, i.e. the sets

$$F_A = \{x \in \mathbb{R}^n \,:\, Ax \geq b\}, \qquad F_A^D = \{y \in \mathbb{R}^m \,:\, A^t y = c,\ y \geq 0\}.$$

Finally, let S be the set of matrices $A \in \mathcal{P}$ such that both $\mathcal{P}_A$, $\mathcal{P}_A^D$ have unique solution.

In order to get our results we need very little about linear programming problems. For the convenience of the reader, we collect here the well known facts that we shall use in the sequel.

Remark 2.1. i) $\mathcal{P}_A$ has solution if and only if $\mathcal{P}_A^D$ has solution, if and only if the feasible sets of both problems are nonempty: this is the fundamental duality result (see [10], Theorem 13.1); furthermore, in this case there is no *duality gap*, i.e. the values of the two problems do agree;

ii) any pair of solutions (x, y) fulfills the so called complementarity condition $\langle Ax-b, y\rangle = 0$. Conversely, if two feasible vectors x, y fulfill the complementarity condition $\langle Ax - b, y\rangle = 0$, then they solve the primal and dual problem, respectively. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product in Euclidean space (for vectors u, v in a Euclidean space we shall also denote by uv their inner product).

Observe that from ii) the following holds: if $\bar{x}$ is a solution of $\mathcal{P}_A$ and $I(\bar{x}) = \{i \in \{1, \dots, m\} : a_i\bar{x} = b_i\}$ (the set of active constraints at $\bar{x}$), for any solution of $\mathcal{P}_A^D$ it must be $y_j = 0$ if $j \notin I(\bar{x})$.

In order to get uniqueness of the solution of both problems for a big set of matrices A we shall assume the following standing hypotheses on the considered matrices and on the vectors b, c respectively:

$$\begin{array}{lll} m \geq n & \text{and} & \operatorname{rank}(A) = n; \\ c \neq 0 & \text{and} & b \neq 0. \end{array} \qquad (SH)$$

It is an easy matter to see that if one of these assumptions is missing, then there are open sets of matrices giving rise to problems which have more than one solution. When the rank is less than n, the problem relies on the fact that there are too few linearly independent constraints in the primal problem, while if either b or c are the null vectors, then the function to be minimized is constant, so that uniqueness of the minimizer is out of question.

A characterization of uniqueness for the primal problem can be found in [9], where it is shown that a solution of $\mathcal{P}_A$ is unique if and only if it remains a solution to all linear programs obtained from $\mathcal{P}_A$ by arbitrary and sufficiently small perturbations of the cost vector c. An easier and more geometric proof of the same result is provided in [11].

We now need to specify which set of matrices we shall consider. As already mentioned, we want to consider the biggest set of matrices for which a genericity result can be found. Thus the right choice is to consider the set $\mathcal{P}$ which is the closure of the set of matrices for which there exists solution for $\mathcal{P}_A$. Thanks to item *i*) of Remark 2.1, $\mathcal{P}$ can be written as

$$\mathcal{P} = \text{cl}\,\{A \in \mathbb{R}^{mn} : \operatorname{rank}(A) = n,\, \exists x \in \mathbb{R}^n, y \in \mathbb{R}^m : \, Ax \geq b,\, A^t y = c,\, y \geq 0\}.$$

Now, we introduce the notion of well posedness we shall use in this paper.

Definition 2.2. The problem $A \in \mathcal{P}$ is well posed *if:*

1. *the problems $\mathcal{P}_A$ and $\mathcal{P}_A^D$ both have unique solution, say $\bar{x}$ and $\bar{y}$ respectively;*
2. *for each $\{A_n\} \subset \mathcal{P}$ such that $A_n \to A$, and for each pair of solutions (x_n, y_n) of their associated problems, then $x_n \to \bar{x}$, $y_n \to \bar{y}$ (convergence of the solutions);*

3. *if $x_n, y_n, \bar{x}$ and $\bar{y}$ are as above, then $cx_n \to c\bar{x}$, $by_n \to b\bar{y}$, (continuity of the value function).*

All the above convergences are intended with respect to the Euclidean distance in the suitable (according to the right dimension) Euclidean space. We end this section by introducing some notation. Usually an element of an Euclidean space is intended as a column vector. But when considering a matrix A we shall denote by a_i its rows, and by a_i^t the associated column vector. So that a_i^t is the i-th column of the matrix A^t.

3 Properties of $\mathcal{P}$

In this section the main result deals with the topological structure of $\mathcal{P}$. First of all, observe that not all problems in $\mathcal{P}$ have solutions (i.e. the closure operator used in its definition is not redundant). This is easily seen by considering a sequence of problems for which the feasible set escapes to infinity, and so the limit problem in unfeasible. Next, we want to prove that $\mathcal{P}$ is the closure of its interior.

The following lemma will be useful to prove the main results of the paper.

Lemma 3.1. *Let b such that $b_i > 0$ for at least one i and let $\mathcal{A}$ be the set of matrices $A \in \mathcal{P}$ of size $m \times n$ such that there exists a solutions' pair $(\bar{x}, \bar{y})$ of the systems*

$$Ax \geq b \qquad \textit{and} \qquad A^t y = c,\ y \geq 0,$$

satisfying

$$|I(\bar{x})| = n \qquad \textit{and} \qquad \bar{y}_i > 0,\ \forall i \in I(\bar{x})$$

with $I(\bar{x}) = \{i : a_i\bar{x} = b\}$.
Then $\mathcal{A}$ is a dense subset of $\mathcal{P}$.

Proof. Fix $\varepsilon > 0$ and $A \in \mathcal{P}$. We must find $\tilde{A} \in \mathcal{A}$ such that $\|\tilde{A} - A\| \leq \varepsilon$. Without loss of generality we can assume that the problems $\mathcal{P}_A$ and $\mathcal{P}_A^D$ both have solutions, say $\bar{x}$ and $\bar{y}$. Thanks to Assumption (SH) we can also suppose $\bar{x}$ to be extremal (see the Fundamental Theorem of linear programming, [10], Theorem 13.2). This implies that there are $k \geq n$ active constraints, i.e. $|I(\bar{x})| = k \geq n$, and that n corresponding rows a_i are linearly independent, so that by possibly rearranging equations, we can suppose $I(\bar{x}) = \{1, \dots, k\}$ and that the rows $a_1, \dots, a_n$ are linearly independent. Now consider the system

$$\begin{cases} a_1^t y_1 + \ldots + a_k^t y_k = c \\ y \geq 0. \end{cases}$$

This system has a solution $\bar{y} = (\bar{y}_1, \dots, \bar{y}_k)$ such that $\bar{y}_i > 0$ for some i. Suppose that $a_1^t, \dots, a_k^t$ are not linearly independent, i.e. $k > n$ and, without loss of generality, $\bar{y}_i > 0$ for all $i = 1, \dots, k$. Then we claim that it is possible to write $c = a_{i_1}^t y_{i_1} + \ldots + a_{i_j}^t y_{i_j}$, with $a_{i_1}^t, \dots, a_{i_j}^t$ linearly independent and $y_{i_l} \geq 0$ for $l = 1, \dots, j$. For, let $\lambda_1 a_1^t + \ldots + \lambda_k a_k^t = 0$ and, without loss of generality, suppose at least one $\lambda_i > 0$. Set $\delta = \min\{\bar{y}_i/\lambda_i : \lambda_i > 0\}$, and $m_i = \bar{y}_i - \delta\lambda_i$. It is easy to see that $a_1^t m_1 + \ldots + a_k^t m_k = c$, that all m_i are non negative and at least one is vanishing. Thus we can actually write c as a conic combination of at most $k - 1$ vectors a_i^t. If these vectors are linearly independent we have shown the claim, otherwise we repeat the argument and get the conclusion. Suppose now we have written (by possibly renumbering the rows of A):

$$a_1^t y_1 + \ldots + a_j^t y_j = c,$$

with $a_1^t, \dots, a_j^t$ linearly independent. If $j < n$, select $n - j$ rows of A from $I(\bar{x})$, in order to have n linearly independent rows from $I(\bar{x})$. Summarizing, by possibly rearranging the rows of A, we can suppose to be in the following situation:

1. $\bar{x}$ satisfies
$$\begin{cases} a_l x &= b_l \quad \text{for } l = 1, \dots, k \\ a_l x &> b_l \quad \text{for } l = k+1, \dots, m \end{cases}$$

2. $\bar{y}$ satisfies
$$a_1^t y_1 + \dots + a_n^t y_n = c,$$
with $\bar{y}_i \geq 0$ and at least one of them is positive.

Select i such that $b_i > 0$ and consider the matrix $\bar{A}$ defined as

$$\begin{cases} \bar{a}_l &= a_l \quad \text{for } l = 1, \dots, n \\ \bar{a}_l &= a_l + \sigma a_i \quad \text{for } l = n+1, \dots, k \\ \bar{a}_l &= a_l \quad \text{for } l = k+1, \dots, m. \end{cases}$$

with σ suitably small, to be chosen later. It is quite clear that $\bar{x}$ satisfies

$$\begin{cases} \bar{a}_l x &= b_l \quad \text{for } l = 1, \dots, n \\ \bar{a}_l x &> b_l \quad \text{for } l = n+1, \dots, m. \end{cases}$$

as well as that $\bar{y}$ is a solution of the system

$$\bar{a}_1^t y_1 + \dots + \bar{a}_n^t y_n = c.$$

Since one of the $\bar{y}_i$ is positive, say $\bar{y}_1$, we can consider the following matrix $\tilde{A}$ defined by its rows as

$$\begin{cases} \tilde{a}_1 &= \bar{a}_1 - \sigma \sum_{j=1}^n \bar{a}_j, \\ \tilde{a}_l &= \bar{a}_l \quad \text{for } l = 2, \dots, m. \end{cases}$$

It is very easy to verify that the system

$$\tilde{a}_1^t y_1 + \dots + \tilde{a}_n^t y_n = c$$

admits a solution $\tilde{y}$ such that $\tilde{y}_i > 0$ for all $i = 1, \dots, n$. Moreover the system

$$\begin{cases} \tilde{a}_l x &= b_l \quad \text{for } l = 1, \dots, n \\ \tilde{a}_l x &> b_l \quad \text{for } l = n+1, \dots, m \end{cases}$$

also admits a solution, provided σ is small enough, and this completes the proof. □

Theorem 3.2. $\mathcal{P} = \operatorname{cl} \operatorname{int} \mathcal{P}$.

Proof. Given a matrix $A \in \mathcal{P}$ and $\varepsilon > 0$, we must find a matrix which is at distance less than ε to A. First of all, arguing as in the first part of the proof of Lemma 3.1, we can assume, without loss of generality, that $\mathcal{P}_A$ and $\mathcal{P}_A^D$ have solution $(\tilde{x}, \tilde{y})$, and also that $\tilde{y}_i > 0$ for all $i \in I(\tilde{x})$, with $I(\tilde{x}) \supseteq \{1, \dots, n\}$ and $a_1, \dots, a_n$ are linearly independent rows. We will directly show that with these assumptions $A \in \operatorname{int}\mathcal{P}$.

Step 1. Consider the system

$$y_1a_1^t + \cdots + y_na_n^t = c - \tilde{y}_{n+1}a_{n+1}^t - \cdots - \tilde{y}_m a_m^t.$$

By assumption, it has solution $(\tilde{y}_1, \ldots, \tilde{y}_n)$, with $\tilde{y}_i > 0$ for all i. Thus, if K is sufficiently close to A, the system

$$y_1k_1^t + \cdots + y_nk_n^t = c - \tilde{y}_{n+1}k_{n+1}^t - \cdots - \tilde{y}_m k_m^t,$$

still has a non negative solution $(\hat{y}_1, \ldots, \hat{y}_n)$. It follows that $(\hat{y}_1, \ldots, \hat{y}_n, \tilde{y}_{n+1}, \ldots, \tilde{y}_m)$ is a feasible vector for P_K^D.

Step 2. Let F_K be the feasible set of problem $\mathcal{P}_K$: it remains to prove that $F_K \neq \emptyset$. Suppose $\tilde{x} = 0$. This implies $b_i \leq 0$ for all i and $F_K \neq \emptyset$ for all $K \in \mathcal{P}$.

Step 3. Suppose now $\tilde{x} \neq 0$. Thus without loss of generality we can assume, thanks to Lemma 3.1, that $|I(\tilde{x})| = n$, and that $a_1, \ldots, a_n$ are linearly independent rows. It follows that, for each matrix K sufficiently close to A, the linear equality system

$$\begin{cases} k_1x = b_1 \\ \ldots \\ k_nx = b_n \end{cases}$$

has solution x_K, which is close to $\tilde{x}$. Then x_K is feasible for $\mathcal{P}_K$, and this ends the proof.

□

4 Generic Uniqueness

The next theorem provides a characterization of those problems having the property that their solution, as well as the solution of their dual, is unique.

Theorem 4.1. *Suppose $\bar{x}$ and $\bar{y}$ are unique solutions of $\mathcal{P}_A$ and $\mathcal{P}_A^D$ and set, as usual, $I(\bar{x}) = \{i : a_i\bar{x} = b_i\}$. Then:*

1. $|I(\bar{x})| = n$;
2. $\bar{y}_i > 0 \quad \forall i \in I(\bar{x})$.

Conversely, suppose $\bar{x}$ and $\bar{y}$ are feasible for $\mathcal{P}_A$ and $\mathcal{P}_A^D$, respectively, such that

3. $|I(\bar{x})| = n$;
4. $\bar{y}_i > 0 \quad \forall i \in I(\bar{x}), \quad \bar{y}_i = 0 \quad \forall i \notin I(\bar{x})$.

Then $\bar{x}$ and $\bar{y}$ are unique solutions of $\mathcal{P}_A$ and $\mathcal{P}_A^D$.

Proof. Let us prove the first part of the statement. Since $c \neq 0$, we can suppose $a_i\bar{x} = b_i$ for $i = 1, \ldots, k$, for some $k \geq 1$. Since $\bar{x}$ is unique, then it is extremal, and therefore there are n linearly independent rows a_i such that $a_i\bar{x} = b_i$, so we can suppose, by rearranging equations, $a_1, \ldots, a_n$ linearly independent. We prove now that $\bar{y}_i > 0$ for at most n indices i There is nothing to prove if $|I(\bar{x})| = n$. So, we suppose instead $|I(\bar{x})| > n$ and, without loss of generality, $I(\bar{x}) \supseteq \{1,\dots,n+1\}$. Suppose also by sake of contradiction, $\bar{y}_1 > 0,\dots,\bar{y}_{n+1} > 0$. Consider the system:

$$y_1 a_1^t + \cdots + y_n a_n^t = c - (\bar{y}_{n+1} - \varepsilon) a_{n+1}^t - \cdots - \bar{y}_m a_m^t.$$

It has solution $(\tilde{y}_1,\dots,\tilde{y}_n)$ and $\tilde{y}_i \geq 0$, if $\varepsilon < \bar{y}_{n+1}/2$ is sufficiently small. But then $y = (\tilde{y}_1,\dots,\tilde{y}_n,\bar{y}_{n+1}-\varepsilon,0,\dots,0)$ is still a solution of $\mathcal{P}_A^D$, since it is a feasible vector fulfilling the complementarity condition $\langle A\bar{x} - b, y\rangle = 0$. A contradiction. Thus we can suppose, without loss of generality, $\{i : \bar{y}_i > 0\} \subset \{1,\dots,n\}$. Now, observe that if

$$c = y_1 a_1^t + \cdots + y_j a_j^t,$$

for some $j \leq n$ and $y_i \geq 0$, then $y = (y_1,\dots,y_j,0,\dots,0)$ satisfies the complementarity condition $\langle A\bar{x} - b, y\rangle = 0$, and thus $y = \bar{y}$. In other words, c can be written in a unique way as conic combination of the vectors $a_1^t,\dots,a_n^t$:

$$c = \bar{y}_1 a_1^t + \cdots + \bar{y}_j a_j^t,$$

$j \leq n$. Suppose now, by contradiction, $|\{i : \bar{y}_i > 0\}| < n$ and, without loss of generality, $\bar{y}_n = 0$. By Theorem 1 of [9], for all small $\varepsilon > 0$ the problems obtained by substituting c with $c - \varepsilon a_n^t$ still have solution, thus $c - \varepsilon a_n^t$ belong to the cone generated by $\{a_1^t,\dots,a_j^t\}$; thus

$$c - \varepsilon a_n^t = y_1 a_1^t + \cdots + y_j a_j^t,$$

for some $y_1 \geq 0,\dots,y_j \geq 0$. But then

$$c = y_1 a_1^t + \cdots + y_j a_j^t + \varepsilon a_n^t,$$

a contradiction to the fact that $\bar{y}_n = 0$. So, we can conclude that $\bar{y}_i > 0$ if and only if $i = 1,\dots,n$. To end the proof, let us prove that $I(\bar{x}) = \{1,\dots,n\}$, with an argument by contradiction. Without loss of generality, we can suppose $a_{n+1}\bar{x} = b_{n+1}$. Since the system

$$y_1 a_1^t + \cdots + y_n a_n^t = c$$

has unique solution $\bar{y}$ with $\bar{y}_i > 0$ for all i, then the system

$$y_1 a_1^t + \cdots + y_n a_n^t = c - \varepsilon a_{n+1}^t$$

has unique solution $\tilde{y}$ with $\tilde{y}_i \geq 0$ for all i. Thus $y = (\tilde{y}_1,\dots,\tilde{y}_n,\varepsilon,0,\dots,0)$ is a feasible vector fulfilling the complementarity condition, and this contradiction ends the proof of the first part of the statement.

Suppose now we have a pair of solutions $(\bar{x},\bar{y})$ satisfying conditions 3) and 4). We have to prove that both the primal and dual problems have unique solution. Suppose z is a solution of the primal problem. Since $\bar{y}_i > 0$ if and only if $i \in I(\bar{x})$, then, in order to satisfy the complementarity condition z must be solution of the linear system

$$\begin{cases} a_1 x = b_1 \\ \qquad \vdots \\ a_n x = b_n \end{cases}$$

which however has the unique solution $\bar{x}$; thus $z = \bar{x}$.

As far as the dual problem is concerned, suppose w is one of its solutions. From the complementary conditions it follows that $w_i = 0$ for $i = n+1, \dots, m$. Moreover, since w is a feasible vector, we have $c = w_1 a_1^t + \cdots + w_n a_n^t$ and this implies that $w_i = \bar{y}_i$ for $i = 1, \dots, n$ because the a_i's are linearly independent vectors. But this means $w = \bar{y}$ and the proof is completed. □

We want to point out that the conclusions of Theorem 4.1 can be obtained as a consequence of the more general results in [12], proved by means of different arguments, based on the use of the Balinski-Tucker simplex tableaus. Our proof, focused only on uniqueness, is simpler and more direct.

Theorem 4.2. *The set S is open in $\mathcal{P}$.*

Proof. The proof of the theorem is a straightforward consequence of Theorem 4.1. For, let A be such that $\bar{x}$ and $\bar{y}$ are unique solutions of $\mathcal{P}_A$ and $\mathcal{P}_A^D$. Then there are n linearly independent rows, say $a_1, \dots, a_n$, of A such that

1. $a_i \bar{x} = b_i \qquad$ and $\qquad |I(\bar{x})| = n;$
2. $\bar{y}_i > 0 \qquad i = 1, \dots, n.$

Consider now a matrix K and the associated systems

$$\begin{cases} k_1 x = b_1 \\ \dots \\ k_n x = b_n \end{cases}$$

and

$$y_1 k_1^t + \dots y_n k_n^t = c.$$

If the matrix K is sufficiently close to A, both systems have one and only one solution, say x^K and $y_K = (y_1^K, \dots, y_n^K)$ respectively. Moreover it follows that x^K and $y^K = (y_1^K, \dots, y_n^K, 0, \dots, 0)$ are feasible for $\mathcal{P}_K$ and $\mathcal{P}_K^D$ respectively. Furthermore, they obviously fulfill the complementarity condition

$$\langle K x^K - b, y^K \rangle = 0,$$

and thus they are solutions of $\mathcal{P}_K$ and $\mathcal{P}_K^D$ respectively. Moreover, since x^K and y^K fulfill the conditions of the second part of Theorem 4.1, they are the unique solutions of the problems, so that $K \in S$, and this ends the proof. □

The following example shows that the set of matrices A such that the primal problem has unique solution is *not* open.

Example 4.3. Consider the following problem:

$$\begin{cases} \inf(x+y) \\ x + y \geq 2 \\ x \geq 1 \\ y \geq 1 \end{cases} .$$

Clearly, it has unique solution, while the close by problems

$$\begin{cases} \inf(x+y) \\ \frac{1}{1+\varepsilon}x + \frac{1}{1+\varepsilon}y \geq 2 \\ x \geq 1 \\ y \geq 1 \end{cases},$$

have multiple solutions.

Theorem 4.4. *$\mathcal{P} \setminus S$ is contained in a finite union of algebraic surfaces of dimension less than mn.*

Proof. We divide the proof in two main steps.

Step 1. Let $\tilde{G}$ be the following set of $m \times n$ matrices. $A \in \tilde{G}$ if:

(i) Every subset of n rows of A is made by linearly independent vectors;

(ii) If $m > n$, given any subset of indices such that there is at least one $i \in \{i_1, \dots, i_{n+1}\}$ with $b_i \neq 0$, the matrix

$$\begin{pmatrix} a_{i_1} & b_{i_1} \\ \vdots & \vdots \\ a_{i_{n+1}} & b_{i_{n+1}} \end{pmatrix}$$

has determinant different from zero.

(iii) If H is any $n \times n$ matrix such that its rows are columns of A then the system

$$Hz = c \tag{4.1}$$

has (unique) solution y^H such that $y_i^H \neq 0 \ \forall\, i = 1, \dots, n$.

By noticing that the above conditions can be expressed in terms of the fact that some determinants are not vanishing, the following facts hold:

- The set $\tilde{G}$ is open in $\mathbb{R}^{mn}$;
- The complement of $\tilde{G}$ is a finite union of algebraic surfaces of dimension less than mn.

Therefore looking at these determinants as non null polynomials in the variables $a_{11}, \dots, a_{mn}$, the bad set is made by the zeros of these polynomials so the claim follows.

Set now $G = \tilde{G} \cap \operatorname{int}\mathcal{P}$.

Step 2. We prove now that $S \supseteq G$, and to this end is suffices to prove that given $A \in G$, both $\mathcal{P}_A$ and $\mathcal{P}_A^D$ have unique solution. But this again is a straightforward consequence of the characterization of uniqueness of solution for both problems.

Let $\bar{x} \neq 0$ be an extremal solution for $\mathcal{P}_A$. The first condition implies that $\bar{x}$ cannot fulfill more than n equalities in the system of inequalities $Ax \geq b$, thus $|I(\bar{x})| = n$. Now let

$\bar{y}$ be a solution of $\mathcal{P}_A^D$. Since $\bar{y}$ is feasible $\bar{y}_i \geq 0$. By condition (iii), $\bar{y}_i > 0$ for all $i \in I(\bar{x})$. This implies uniqueness of solution for the dual problem. A similar argument applies in the case $\bar{x} = 0$. For, if more than n equality constraints were fulfilled, then $\bar{y}$ would not be the unique solution of the dual problem (see the proof of the first part of Theorem 4.1). This concludes the proof. □

5 Continuity of the Value Function and Well Posedness

The following example shows that the value function need not to be (semi) continuous at all points where it is finite.

Example 5.1. Let $m = 3$, $n = 2$, $c = (0, -1)$, $b = (0, 0, -1)$ and let $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$. Finally, let $A_n = \begin{pmatrix} 1 & -\frac{1}{n} \\ -1 & -\frac{1}{n} \\ 0 & -1 \end{pmatrix}$. An easy calculation shows that $v(A_n) = 0$ for all n, while $v(A) = -1$. Thus v is not upper semicontinuous at A. Now, let $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$, and let $K_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{n} & -1 \end{pmatrix}$. An easy calculation shows that $v(K_n) = -\infty$ for all n, while $v(K) = -1$. Thus v is not lower semicontinuous at K.

On the other hand, uniqueness of solutions for both problems yields nice properties for the value function and for the minimizers.

Theorem 5.2. *The problem $A \in \mathcal{P}$ is well posed if and only if the problems $\mathcal{P}_A$ and $\mathcal{P}_A^D$ both have unique solution.*

Proof. The proof readily follows from Theorem 4.1. For, if $\bar{x}$ and $\bar{y}$ are solutions of $\mathcal{P}_A$ and $\mathcal{P}_A^D$ respectively, then they are (unique) solution of a particular linear system. Matrices close to A give raise to problems having unique solution, again characterized by the fact that they solve a slightly perturbed linear problem with respect to that one solved by $\bar{x}$ and $\bar{y}$. This actually entails a locally lipschitz behavior of the solution, with respect to perturbations of the matrix A. From this, continuity of the value function easily follows. □

6 Conclusions

In this paper we have considered linear programming problems, in duality, of the form:

$$\mathcal{P}_A : \begin{cases} \inf cx \\ Ax \geq b \end{cases}$$

and:

$$\mathcal{P}_A^D : \begin{cases} \sup by \\ A^t y = c, y \geq 0. \end{cases}$$

We have shown, in particular, that the set of matrices such that the initial problem, as well as its dual, have unique solution, is an open set in the space of matrices (restricted to the closure of the set of those matrices for which a solution exists). Moreover, the complement of this set is contained in a finite union of algebraic surfaces, and thus the set of well behaved problems is big in any sense (Baire category, measure, σ-porosity). Also, we have proved that uniqueness of solution for the primal and the dual problem at the same time, actually implies a strong form of well posedness.

There are other interesting linear programming problems in duality. For instance, one could consider non negativity constraints also in the primal problem. Thus one gets the two following types of problems in duality:

$$\mathcal{P}_A : \begin{cases} \inf cx \\ Ax \geq b \\ x \geq 0 \end{cases}$$

and:

$$\mathcal{P}_A^D : \begin{cases} \sup by \\ A^t y \leq c, y \geq 0. \end{cases}$$

Also, one could explicitly consider equality constraint, and this give raise to another formulation of problems in duality.

Though there are standard ways to switch from a formulation to another one in an equivalent way (for instance, non negativity conditions in the above formulation could be incorporated in the matrix A), it could be observed that our results do *not* directly apply to the family of the problems introduced above, unless we allow to perturb the non negativity constraint, which however is not interesting. Even more, they are clearly *false*, at least without further assumptions (for instance on the vectors c and b):

Example 6.1. Consider the following problem:

$$\begin{cases} \inf x \\ x + y \geq 1 \\ -x - y \geq -2 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

It is obvious that all close by problems will have multiple solutions.

Conditions sufficient to guarantee that uniqueness is generic will be provided in another paper.

References

[1] H.F. Bohnenblust, S. Karlin and L.S. Shapley, Solutions of discrete, two-person games, in *Contributions to the Theory of Games, Annals of Mathematics Studies*, no. 24, Princeton University Press, Princeton, N. J., 1950, pp. 51–72.

[2] G. Christov and M. Todorov, Semi-infinite optimization. Existence and uniqueness of the solution, *Math. Balkanica (N.S.)* 2 (1988) 182–191.

[3] A.L. Dontchev and T. Zolezzi, *Well-posed Optimization Problems*, Lecture Notes in Mathematics, vol. 1543, Springer-Verlag, Berlin, 1993.

[4] M.A. Goberna, M.A. López and M.I. Todorov, Extended active constraints in linear optimization with applications, *SIAM J. Optim.* 14 (2003) 608–619 (electronic).

[5] M.A. Goberna, M.A. López and M.I. Todorov, A generic result in linear semi-infinite optimization, *Appl. Math. Optim.* 48 (2003) 181–193.

[6] S. Helbig and M. I. Todorov, Unicity results for general linear semi-infinite optimization problems using a new concept of active constraints, *Appl. Math. Optim.* 38 (1998) 21–43.

[7] A.D. Ioffe and A.J. Zaslavski, Variational principles and well-posedness in optimization and calculus of variations, *SIAM J. Control Optim.* 38 (2000) 566–581 (electronic).

[8] R. Lucchetti, *Convexity and Well-posed Problems*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 22, Springer, New York, 2006.

[9] O.L. Mangasarian, Uniqueness of solution in linear programming, *Linear Algebra Appl.* 25 (1979) 151–162.

[10] J. Nocedal and S.J. Wright, *Numerical Optimization*, Springer Series in Operations Research, Springer-Verlag, New York, 1999.

[11] C.R. Seshan, K.K. Achary and V.G. Tikekar, A simple proof of a basic result on uniqueness of solution in linear programming, *Internat. J. Math. Ed. Sci. Tech.* 11 (1980) 591–593.

[12] G.A. Tijssen and G. Sierksma, Balinski-Tucker simplex tableaus: dimensions, degeneracy degrees, and interior points of optimal faces, *Math. Program.* Ser. A 81 (1998) 349–372.

[13] M.I. Todorov, Generic existence and uniqueness of the solution to linear semi-infinite optimization problems, *Numer. Funct. Anal. Optim.* 8 (1985/86) 541–556.

[14] T. Zolezzi, Well-posedness criteria in optimization with application to the calculus of variations, *Nonlinear Anal.* 25 (1995) 437–453.

Manuscript received 5 December 2007
revised 16 February 2008
accepted for publication 15 April 2008

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