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NECESSARY AND SUFFICIENT CONDITIONS FOR GLOBALLY STABLE CONVEX MINIMAX THEOREMS

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Dedicated to Professor Michel Théra on his 60^{th} *Birthday.*

Abstract: In this paper, we establish a necessary and sufficient condition for a globally stable minimax equality, where the minimax equality holds for each convex perturbation of the convex-concave bi-function involved. The necessary and sufficient condition is expressed as a closedness condition using conjugate functions of the bi-function. As an application, we obtain a necessary and sufficient condition for a globally stable Lagrangian duality theorem, and also a constraint qualification which completely characterizes the strong Lagrangian duality theorem for convex minimization problems. As a consequence of these results, we obtain a globally stable Farkas' lemma for cone-convex systems.

Key words: *stable minimax theorems, closedness condition, globally stable duality theorem and Farkas lemma, convex optimization*

Mathematics Subject Classification: *46A55, 49K35, 90C25*

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1 Introduction

The minimax theorem is the basis to the remarkable theory of zero-sum games and it has played a significant role in optimization, operations research and economics. The inequality,

$$\sup_{y\in B}\inf_{x\in A} h(x,y) \leq \inf_{x\in A}\sup_{y\in B} h(x,y), \tag{1.1}$$

always holds for a bi-function $h(.,.)$ on a product set $A\times B$. The development of conditions, ensuring the equality in (1.1) has been the subject of research for over many years, and the equality is commonly known as "minimax theorem". For such developments, the reader is referred to [1, 3, 11, 12]. The classic convex minimax theorem for a convex-concave bi-function $h(\cdot,\cdot)$ on a closed convex set $A \times B$ states that

$$\max_{y\in B}\inf_{x\in A} h(x,y) \;=\; \inf_{x\in A}\sup_{y\in B} h(x,y)\,,$$

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where at the left hand side the sup is attained and thus replaced by max. On the other hand, for each choice of function f on A,

$$\sup_{y\in B}\inf_{x\in A}[f(x)+h(x,y)] \le \inf_{x\in A}\sup_{y\in B}[f(x)+h(x,y)]. \tag{1.2}$$

Conditions ensuring the equality in (1.2) where sup can be replaced by max are of great interest in optimization as then the equality readily applies to constrained optimization through the Lagrangian function [2, 8, 9], covers the conventional minimax theorem, [1, 12] and leads to solvability of systems of inequalities of Farkas' type [2, 5, 6, 7].

The purpose of this paper is to establish conditions on the convex-concave bi-function $h(.,.)$, characterizing the minimax equality

$$\inf_{x\in A}\sup_{y\in B}\Big[f(x)+h(x,y)\Big] \;=\; \max_{y\in B}\inf_{x\in A}\Big[f(x)+h(x,y)\Big], \tag{1.3}$$

which holds for each choice of continuous convex function $f(\cdot)$. We refer this equality as the globally stable convex minimax theorem. We show that a simple closedness condition on a convex epigraph set involving conjugate functions is necessary and sufficient for the globally stable minimax theorem. The same closedness condition has also been used in [10] for characterizing the minimax equality (1.3) in the sense that (1.3) holds for each choice of continuous linear function $f(\cdot)$. For related results on stable minimax theorems, see [3, 4]. As an application, we obtain a necessary and sufficient condition for a globally *stable Lagrangian duality theorem*, and a constraint qualification which characterizes *strong Lagrangian duality theorem* for convex minimization problems. As a consequence of these results, we also establish a globally stable Farkas' lemma for cone-convex systems.

2 Preliminaries

We recall in this section some notations and basic results that will be used later in the paper. Let X and X' be linear spaces in duality with respect to the bilinear form $\langle\cdot,\cdot\rangle$ such that X' is endowed with the w^*-topology $\sigma(X',X)$. Then X' is locally convex and its topological dual is X. Let Y be another linear space. Let $h : X \to \mathbb{R}\cup\{-\infty,+\infty\}$ be a convex function. The conjugate function of h, $h^* : X^* \to \mathbb{R}\cup\{+\infty\}$, is defined by

$$h^*(v) := \sup\{\langle v,x\rangle - h(x) \mid x \in \text{dom } h\},$$

where $\text{dom } h := \{x \in X \mid h(x) < +\infty\}$ is the effective domain of h. The function h is said to be proper if h does not take on the value $-\infty$ and $\text{dom } h \neq \emptyset$. The epigraph of h is defined by

$$\text{epi } h := \{(x,r) \in X\times\mathbb{R} \mid x \in \text{dom } h, h(x) \le r\}.$$

For a closed convex subset D of X, the indicator function δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$.

For proper lower semicontinuous convex functions $g, h : X \to \mathbb{R}\cup\{+\infty\}$, the infimal convolution of g with h, denoted $g\Box h$, is defined by

$$(g\Box h)(x) \;=\; \inf_{x_1+x_2=x}\{g(x_1)+h(x_2)\}.$$

The lower semicontinuous envelope and lower semicontinuous convex hull of a function g : $X \to \mathbb{R}\cup\{-\infty,+\infty\}$ are denoted respectively by $\text{cl}g$ and $\text{clco}g$. That is, $\text{epi}(\text{cl}g) = \text{cl}(\text{epi } g)$

and epi(cl co$g$) = cl co(epi g). For details, see [13]. Let $g,\ h$ and $g_i,\ i \in I$ (where I is an arbitrary index set) be proper lower semicontinuous convex functions. It is well known from the conjugate operation (see [13]) that if dom$\gamma \cap$ dom $h \neq \emptyset$, then

$$(g \Box h)^* = g^* + h^*,\ (g+h)^* = \mathrm{cl}(g^* \Box h^*)$$

and if $\sup_{i\in I} g_i$ is proper, then

$$(\sup_{i\in I} g_i)^* = \mathrm{cl\ co}(\inf_{i\in I} g_i^*).$$

Thus we can check that

$$\mathrm{epi}(g+h)^* = \mathrm{cl}(\mathrm{epi} g^* + \mathrm{epi} h^*) \text{ and } \mathrm{epi}(\sup_{i\in I} g_i)^* = \mathrm{cl\ co}(\bigcup_{i\in I} \mathrm{epi} g_i^*).$$

The closure in the first equation is superfluous if one of g and h is continuous at some $x_0 \in$ dom $g \cap$ dom h (see [13] for details). Let S be a closed convex cone in Y. Denote by S^+ the dual cone of S, defined as

$$S^+ = \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0 \text{ for any } y \in S\}.$$

We say that the map $g : X \to Y$ is $S-$convex if for any $x_1,\ x_2 \in X$ and any $\lambda \in [0,1]$,

$$g(\lambda x_1 + (1-\lambda)x_2) \in \lambda g(x_1) + (1-\lambda) g(x_2) - S.$$

Note that $g^{-1}(-S) := \{x \in X \mid -g(x) \in S\}$.

Let A be a nonempty (closed) convex subset of X and let B be a nonempty convex subset of Y. Let $h : A \times B \to \mathbb{R} \cup \{+\infty\}$ be a function, which is proper lower semicontinuous convex in x and concave in y. We extend h on $X \times B$ by setting for any $y \in B$, $h(x,y) = +\infty$ if $x \notin A$.

Lemma 2.1. *Let $h : A \times B \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function that is convex in x and concave in y. Then*

$$\bigcup_{y\in B} epi(h(\cdot, y))^*$$

is a convex set.

Proof. Let $(u_i, \alpha_i) \in\ \mathcal{E} := \bigcup_{y\in B} \mathrm{epi}(h(\cdot,y))^*,\ i = 1,2$ and $\lambda \in (0,1)$. Then there exist $y_i \in B,\ i = 1,2$ such that, for $i = 1,2$, $(h(\cdot,y_i))^*(u_i)\ \leq\ \alpha_i$. Thus,

$$\sup\{\langle u_i, x\rangle - h(x,y_i) \mid x \in A\}\ \leq\ \alpha_i.$$

Let $\bar{\alpha}\ = \lambda_1\alpha_1 + (1-\lambda)\alpha_2,\ \bar{y} = \lambda y_1 + (1-\lambda) y_2,\ \bar{u} = \lambda u_1 + (1-\lambda) u_2$. Then by convexity of B and concavity of $h(x,\cdot)$, $\bar{y}\ \in\ B$ and for each $x\ \in\ A$,

$$\begin{aligned} h(x,\overline{y}) &\geq \lambda h(x,y_1) + (1-\lambda) h(x,y_2) \\ &\geq \lambda[\langle u_1, x\rangle - \alpha_1] + (1-\lambda)[\langle u_2, x\rangle - \alpha_2] \\ &= \langle \overline{u}, x\rangle - \overline{\alpha}. \end{aligned}$$

Hence $(h(\cdot,\bar{y}))^*(\bar{u})\ \leq\ \bar{\alpha}$, and so, $(\bar{u},\bar{\alpha}) \in \mathcal{E}$. In passing note that only concavity in y is needed for the preceding Lemma. □

3 A Globally Stable Minimax Theorem

In this Section, we establish our globally stable minimax theorem.

Theorem 3.1. (Globally Stable Minimax Theorem) *Let $h : A \times B \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex in x and concave in y. Assume that there exists $\hat{x} \in A$ such that $\sup_{y\in B} h(\hat{x}, y) < +\infty$. Then the following statements are equivalent:*

(i) $\bigcup_{y\in B}$ epi $(h(\cdot, y))^*$ is w^*-closed in $X' \times \mathbb{R}$;

(ii) For each continuous convex function $f : X \to \mathbb{R}$,

$$\inf_{x\in A} \sup_{y\in B} \Big[f(x) + h(x, y)\Big] \;=\; \max_{y\in B} \inf_{x\in A} \Big[f(x) + h(x, y)\Big].$$

Proof. Suppose that (i) holds. Let $f : X \to \mathbb{R}$ be a continuous convex function. Then by Lemma 2.1, and by (i), we have,

$$\begin{aligned}\bigcup_{y\in B} \mathrm{epi}[f(\cdot) + h(\cdot, y)]^* &= \bigcup_{y\in B} [\mathrm{epi} f^* + \mathrm{epi}(h(\cdot, y))^*] \\ &= \mathrm{epi} f^* + \bigcup_{y\in B} \mathrm{epi}(h(\cdot, y))^* \\ &= \mathrm{epi} f^* + clco \bigcup_{y\in B} \mathrm{epi}(h(\cdot, y))^* \\ &= \mathrm{epi} f^* + \mathrm{epi}(\sup_{y\in B} h(\cdot, y))^* \\ &= \mathrm{epi}(f(\cdot) + \sup_{y\in B} h(\cdot, y))^*\end{aligned}$$

Since $\mathrm{epi}(f(\cdot) + \sup_{y\in B} h(\cdot, y))^*$ is w^*-closed, it follows that $\bigcup_{y\in B} \mathrm{epi}(f(\cdot) + h(\cdot, y))^*$ is w^*-closed. Moreover, $\bigcup_{y\in B} \mathrm{epi}(f(\cdot) + h(\cdot, y))^*$ is convex. Let $\alpha = \inf_{x\in A} \sup_{y\in B}[f(x) + h(x, y)]$. The existence of $\hat{x} \in A$ such that $\sup_{y\in B} h(\hat{x}, y) < +\infty$, ensures that $f(\hat{x}) + \sup_{y\in B} h(\hat{x}, y) < +\infty$, and so, $\alpha < +\infty$. If $\alpha = -\infty$, the result holds trivially. Assume that α is finite. Then we have

$$\begin{aligned}-\alpha &= \sup_{x\in A} \inf_{y\in B} [-f(x) - h(x, y)] = \sup_{x\in X} \inf_{y\in B} [-f(x) - h(x, y)] \\ &= \sup_{x\in X} (-\sup_{y\in B}[f(x) + h(x, y)]) = (\sup_{y\in B}[f(\cdot) + h(\cdot, y)])^*(0).\end{aligned}$$

Thus $(0, -\alpha) \in$ epi $(\sup_{y\in B}[f(\cdot) + h(\cdot, y)])^* =$ clco $\bigcup_{y\in B}$ epi $[f(\cdot) + h(\cdot, y)]^*$ as $\sup_{y\in B}[f(\cdot) + h(\cdot, y)]$ is proper, lower semicontinuous and convex. Since $\bigcup_{y\in B}$ epi $[f(\cdot) + h(\cdot, y)]^*$ is w^*-closed and convex,

$$(0, -\alpha) \in \bigcup_{y\in B} \mathrm{epi}\ [f(\cdot) + h(\cdot, y)]^*.$$

So, there exists $y \in B$ such that $[f(\cdot) + h(\cdot, y)]^*(0) \leq -\alpha$, that is, $\inf_{x\in A}[f(x) + h(x, y)] \geq \alpha$. This gives us that ,

$$\max_{y\in B} \inf_{x\in A}[f(x) + h(x, y)] \geq \inf_{x\in A} \sup_{y\in B}[f(x) + h(x, y)].$$

Hence (ii) holds.

Conversely, suppose that (ii) holds. Let $\mathcal{E} := \bigcup_{y\in B} \text{epi}(h(\cdot,y))^*$ and let $(u,r) \in \text{ cl}\mathcal{E}$. Then there exist nets $\{y_t\} \subset B$ and $\{(u_t,r_t)\} \subset X' \times \mathbb{R}$ such that $(u_t,r_t) \in\text{epi}(h(\cdot,y_t))^*$ and $\lim(u_t,r_t) = (u,r)$ in the w^*-topology in $X' \times \mathbb{R}$. This gives, for each $x \in X$,

$$\langle u_t, x\rangle - h(x,y_t) \le r_t.$$

So, for each $x \in A$,

$$0 \le r_t - \langle u_t, x\rangle + \sup_{y\in B} h(x,y).$$

Hence in the limit, $0 \le r - \langle u,x\rangle + \sup_{y\in B} h(x,y)$ for each $x \in A$.

Let $f(x) = r - \langle u,x\rangle$. Then, clearly, f is continuous and convex and so,

$$\inf_{x\in A}\sup_{y\in B}\Big[f(x) + h(x,y)\Big] \ge 0.$$

By (ii),

$$\max_{y\in B}\inf_{x\in A}\Big[f(x) + h(x,y)\Big] \ge 0.$$

This means that there exists $y \in B$ such that for each $x \in A$, $f(x) + h(x,y) \ge 0$. By the construction of f, for each $x \in A$, $\langle u,x\rangle - h(x,y) \le r$. As $h(x,y) = +\infty$ for $x \notin A$, we get that $(u,r) \in \text{epi}(h(\cdot,y))^*$. Thus $(u,r) \in \mathcal{E}$ and hence $\mathcal{E}$ is w^*-closed. □

The following minimax theorem which is closely related to the well known convex minimax theorem (see e.g, Theorem 2.12 [11]) follows easily from Theorem 3.1.

Theorem 3.2. *Let $h : A \times B \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex in x and concave in y. Assume that there exists $\hat{x} \in A$ such that $\sup_{y\in B} h(\hat{x},y) < +\infty$. If $\bigcup_{y\in B}$ epi $(h(\cdot,y))^*$ is w^*-closed, then*

$$\inf_{x\in A}\sup_{y\in B} h(x,y) \;=\; \max_{y\in B}\inf_{x\in A} h(x,y).$$

Proof. The conclusion follows from (ii) of Theorem 3.1 in the case where $f = 0$. □

Now we give examples illustrating Theorem 3.1. First we give an example where the usual minmax theorem holds, but the globally stable minimax theorem does not hold.

Example 3.3. Let $A = \mathbb{R}$, $B = [0,\infty)$, $f(x) = -x$ and let $h(x,y) = y\Big[\max\{0,x\}\Big]^2$. Then, $\inf_{x\in A}\sup_{y\in B} h(x,y) = 0$. For any $y \in B$, $\inf_{x\in A} h(x,y) = 0$ and hence $\max_{y\in B}\inf_{x\in A} h(x,y) = 0$. Thus

$$\inf_{x\in A}\sup_{y\in B} h(x,y) \;=\; \max_{y\in B}\inf_{x\in A} h(x,y).$$

Morover, we have,

$$\sup_{y\in B}\Big[f(x) + h(x,y)\Big] = \begin{cases} -x & \text{if } \; x \le 0 \\ +\infty & \text{if } \; x > 0 \end{cases}$$

and

$$\inf_{x\in A}\Big[f(x)+h(x,y)\Big]=\begin{cases}-\infty & \text{if} \quad y=0\\ -\dfrac{1}{4y} & \text{if} \quad y>0.\end{cases}$$

Thus, $\inf\limits_{x\in A}\sup\limits_{y\in B}\Big[f(x)+h(x,y)\Big]=0$ and $\sup\limits_{y\in B}\inf\limits_{x\in A}\Big[f(x)+h(x,y)\Big]=0$. However there does not exist $y\in B$ such that $\inf\limits_{x\in A}\Big[f(x)+h(x,y)\Big]=\max\limits_{y\in B}\inf\limits_{x\in A}\Big[f(x)+h(x,y)\Big]$. One can check that

$$\bigcup_{y\in B}\text{epi}(h(\cdot,y))^*=\{0\}\times\mathbb{R}_+\bigcup\{(x,y)\in\mathbb{R}^2 \mid x>0,\ y>0\},$$

where $\mathbb{R}_+=\{x\in\mathbb{R}\mid x\geq 0\}$ and $\bigcup\limits_{y\in B}\text{epi }(h(\cdot,y))^*$ is not closed in $\mathbb{R}^2$.

Example 3.4. Let $A=\mathbb{R}^2$, $B=[0,\infty)$, $f(x_1,x_2)=x_1$ and let $h(x_1,x_2,y)=y(\sqrt{{x_1}^2+{x_2}^2}-x_2)$. Then for each $(x_1,x_2)\in A$ $\sup\limits_{y\in B}\Big[f(x_1,x_2)+h(x_1,x_2,y)\Big]=\begin{cases}0 & \text{if} \quad \sqrt{{x_1}^2+{x_2}^2}-x_2\leq 0\\ +\infty & \text{otherwise.}\end{cases}$

Note that for any fixed x_1 with $x_1<0$, $\lim\limits_{x_2\to+\infty}(\sqrt{{x_1}^2+{x_2}^2}-x_2)=0$. So we have, for each $y\in B$,

$$\inf_{(x_1,x_2)\in A}\Big[f(x_1,x_2)+h(x_1,x_2,y)\Big]=-\infty.$$

Thus, $\inf\limits_{(x_1,x_2)\in A}\sup\limits_{y\in B}\Big[f(x_1,x_2)+h(x_1,x_2,y)\Big]=0$, but $\sup\limits_{y\in B}\inf\limits_{(x_1,x_2)\in A}\Big[f(x_1,x_2)+h(x_1,x_2,y)\Big]=-\infty$. Moreover, one can check that

$$\bigcup_{y\in B}\text{epi}(h(\cdot,y))^*=\{(0,0,z)\in\mathbb{R}^3 \mid z\geq 0\}\bigcup\{(x,y,z)\in\mathbb{R}^3 \mid x\in\mathbb{R},\ y>0,\ z\geq 0\},$$

Thus. $\bigcup\limits_{y\in B}\text{epi }(h(\cdot,y))^*$ is not closed in $\mathbb{R}^3$.

Example 3.5. Let $A=[0,\infty)$, $B=[0,\infty)$ and $h(x,y)=x(-y^2)$. Then, one can easily check that

$$\bigcup_{y\in B}\text{epi}(h(\cdot,y))^*=\{(x,y)\in\mathbb{R}^2 \mid x\leq 0,y\geq 0\}.$$

which is closed in $\mathbb{R}^2$. Moreover, we can easily check that for each convex function $f:\mathbb{R}\to\mathbb{R}$,

$$\begin{aligned}\inf_{x\in A}\sup_{y\in B}[f(x)+h(x,y)] &= \inf_{x\in A}f(x)\\ &= \max_{y\in B}\inf_{x\in A}[f(x)+h(x,y)].\end{aligned}$$

So, Theorem 3.1 holds.

4 Applications

In this section, we apply our minimax Theorem of Section 3 to derive a globally stable Lagrangian duality theorem for convex optimization problems and a stable Farkas' lemma for cone-convex systems.

Theorem 4.1 (Globally Stable Lagrangian Duality). *Let* $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function. Let* $S \subset Y$ *be a closed convex cone and let* $g : X \longrightarrow Y$ *be a continuous* $S-$*convex mapping with* $\mathrm{dom}\varphi \cap \{x \in X \mid g(x) \in -S\} \neq \emptyset$. *Then the following statements are equivalent.*

(i) $\mathrm{epi}\varphi^* + \bigcup_{y^* \in S^+} \mathrm{epi}(y^* \circ g)^*$ is w^*-closed;

(ii) For each continuous convex function $f : X \to \mathbb{R}$,

$$\inf_{x \in g^{-1}(-S)} \{\varphi(x) + f(x)\} = \max_{y^* \in S^+} \inf_{x \in X} \{\varphi(x) + (y^* \circ g)(x) + f(x)\}.$$

Proof. Let $A = X,\ B = S^+ \subset Y'$ and $h(x, y^*) = \varphi(x) + (y^* \circ g)(x)$ for $x \in A$ and $y^* \in B$. Since $\mathrm{dom}\varphi \cap \{x \in X \mid g(x) \in -S\} \neq \emptyset,\ \sup_{y^* \in B} h(x, y^*) = \varphi(x)$ for each $x \in \mathrm{dom}\varphi \cap g^{-1}(-S)$ and hence $\sup_{y^* \in B} h(\cdot, y^*)$ is a proper, lower semicontinuous convex function. As $y^* \circ g$ is continuous on $\mathrm{dom}\varphi \cap \{x \in X \mid g(x) \in -S\}$,

$$\mathrm{epi}(\varphi + (y^* \circ g))^* = \mathrm{epi}\varphi^* + \mathrm{epi}(y^* \circ g)^*.$$

Hence the conclusion follows from Theorem 3.1. □

As a special case of Theorem 4.1, we derive a constraint qualification which completely characterizes strong Lagrangian duality of convex optimization problems.

Corollary 4.2. *Let* $S \subset Y$ *be a closed convex cone and let* $g : X \to Y$ *be a continuous* S*-convex mapping. Suppose that* $\{x \in X \mid g(x) \in -S\} \neq \emptyset$. *Then the following statements are equivalent.*

(i) $\bigcup_{y^* \in S^+} \mathrm{epi}(y^* \circ g)^*$ *is* w^**-closed;*

(ii) *For each continuous convex function* $f : X \to \mathbb{R}$,

$$\inf_{x \in g^{-1}(-S)} f(x) = \max_{y^* \in S^+} \inf_{x \in X} \{f(x) + (y^* \circ g)(x)\}.$$

Proof. Let $\varphi = 0$. Then $\mathrm{epi}\varphi^* + \bigcup_{y^* \in S^+} \mathrm{epi}(y^* \circ g)^* = \bigcup_{y^* \in S^+} \mathrm{epi}(y^* \circ g)^*$. Hence the conclusion follows from Theorem 4.1. □

Theorem 4.3 (Globally Stable Farkas' Lemma). *Let* $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function. Let* $S \subset Y$ *be a closed convex cone and let* $g : X \longrightarrow Y$ *be a continuous* $S-$*convex mapping with* $\mathrm{dom}\varphi \cap \{x \in X \mid g(x) \in -S\} \neq \emptyset$. *Then the following statements are equivalent.*

(i) $\mathrm{epi}\varphi^* + \bigcup_{y^* \in S^+} \mathrm{epi}(y^* \circ g)^*$ is w^*-closed;

(ii) For each continuous convex function $f : X \to \mathbb{R}$,

$$\big[-g(x) \in S \Rightarrow \varphi(x) + f(x) \geq 0 \big] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in X) \quad \varphi(x) + (\lambda \circ g)(x) + f(x) \geq 0.$$

Proof. Let $A = X,\ B = S^+$ and $h(x, \lambda) = \varphi(x) + (\lambda \circ g)(x)$ for $x \in A$ and $\lambda \in B$. Then, one can check that (ii) is equivalent to the condition that, for each continuous convex function $f : X \to \mathbb{R}$,

$$\inf_{x \in X} \sup_{\lambda \in S^+} \big[f(x) + h(x, \lambda)\big] \geq 0 \ \Rightarrow \max_{\lambda \in S^+} \inf_{x \in X} \big[f(x) + h(x, \lambda)\big] \geq 0.$$

Hence we can check that the conclusion follows from Theorem 3.1. □

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