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DUALITY BETWEEN COHERENT RISK MEASURES AND STOCHASTIC DOMINANCE CONSTRAINTS IN RISK-AVERSE OPTIMIZATION*

Darinka Dentcheva and Andrzej Ruszczyński

Abstract: We consider optimization problems with nonlinear second order stochastic dominance constraints formulated as relations of Lorenz curves. We demonstrate that mean-risk models with law invariant coherent risk measures appear as dual optimization problems to the problems with stochastic dominance constraints.

Key words: *stochastic dominance, risk measures, dual utility, rank dependent utility, Kusuoka representation, duality, stochastic program*

Mathematics Subject Classification: *90C15, 90C34, 90C48; 46N10, 60E15, 91B06*

1 Introduction

The theory of optimal decision under uncertainty and risk has gained much attention due to its practical relevance and theoretical challenges. Several approaches to risk-aversion in optimization are established. One classical approach is based on expected utility theory. Von Neumann and Morgenstern ([30]) have established from a set of axioms that every rational decision maker has a utility function $u(\cdot)$ such that she prefers random outcome X over outcome Y if and only if $\mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$. In practice, however, it is almost impossible to elicit the utility function of a decision maker explicitly. Additional difficulties arise when there is a group of decision makers with different utility functions who have to come to a consensus.

Recently, the *dual utility theory* (or *rank dependent expected utility theory*) has attracted much attention in economics ([22], [33]). It is related to Choquet integral theory (see [7, 29]). From a different system of axioms, than those of von Neumann and Morgenstern, one derives that every decision maker has a certain *rank dependent utility function* $w : [0,1] \to \mathbb{R}$. The decision maker prefers a nonnegative outcome X over a nonnegative outcome Y, if and only if

$$\int_0^1 F_{(-1)}(X;p)\,dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\,dw(p). \tag{1.1}$$

Here, $F_{(-1)}(X;\cdot)$ denotes the inverse distribution function of X. For a comprehensive treatment, see [23]. The dual utility theory plays an important role in actuarial mathematics, see [31, 32].

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Another approach to risk-averse optimization is represented by mean-risk models. In these models the objective is a combination of a certain mean outcome (calculated with respect to some fixed probability measure $\bar{\mu}$), and some dispersion or deviation statistics, representing the uncertainty of the outcome. This direction was pioneered by Markowitz [16], who introduced the mean-variance model. Recently, there is a lot of attention devoted to the development of proper mathematical models of risk (see, e.g., [4, 28]).

An approach to risk-averse optimization based on stochastic dominance constraints has been proposed and developed in ([8, 9, 10, 11]). Our analysis demonstrated that both utility theories provide dual models to the problem with stochastic dominance constraints. Von Neumann–Morgenstern utility functions play the role of Lagrange multipliers associated with the dominance constraint formulated in the direct form, and rank-dependent utility functions appear as Lagrange multipliers associated with the inverse dominance constraint.

The main goal of this paper is to demonstrate similar connection to the theory of risk measures. We demonstrate that stochastic dominance constraints lead to a certain implied risk measure. In this way an optimization model with a coherent measure of risk can be interpreted as a dual problem to a problem with stochastic dominance constraints.

We use $(\Omega, \mathcal{F}, P)$ to denote an abstract probability space, and $\mathcal{L}_p(\Omega, \mathcal{F}, P)$, where $p \geq 1$, denotes the space of measurable real functions $X : \Omega \to \mathbb{R}$ such that $|X|^p$ is integrable. The symbol $\mathcal{P}(I)$ denotes the set of probability measures on the interval $I \subset \mathbb{R}$.

2 Measures of Risk

A *measure of risk* ρ assigns to an uncertain outcome $X \in \mathcal{L}_p(\Omega, \mathcal{F}, P)$ a real value $\rho(X)$ and can take values in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. For $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$, the notation $Y \succeq X$ means that $Y(\omega) \geq X(\omega)$ for all $\omega \in \Omega$. A *coherent measure of risk* is a functional $\rho : \mathcal{L}_1(\Omega, \mathcal{F}, P) \to \overline{\mathbb{R}}$ satisfying the following axioms:

Convexity : $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$ for all $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and $\alpha \in [0,1]$.

Monotonicity : If $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and $Y \succeq X$, then $\rho(Y) \leq \rho(X)$.

Translation Equivariance : If $a \in \mathbb{R}$ and $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$, then $\rho(X + a) = \rho(X) - a$.

Positive homogeneity : If $t > 0$ and $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$, then $\rho(tX) = t\rho(X)$.

The concept of coherent measures of risk was introduced by Artzner, Delbaen, Eber and Heath in [4] for functionals ρ defined on $\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$. For further developments, see [27, 28] and the references therein.

In the theory of measures of risk a special role is played by the functional called the Average Value at Risk and denoted AVaR$(\cdot)$ (see [2, 26]). To introduce this concept we define the left-continuous inverse of the cumulative distribution function $F_1(X;\eta) = P\{X \leq \eta\}$ as follows:

$$F_{(-1)}(X;p) = \inf\{\eta : F_1(X;\eta) \geq p\} \quad \text{for} \quad 0 < p < 1.$$

It is clear that $F_{(-1)}(X;p)$ is the left p-quantile of X. The *Value at Risk* of X at level p is defined as $\text{VaR}_p(X) = -F_{(-1)}(X;p)$. Next, we define the *absolute Lorenz function* $F_{(-2)}(X;\cdot) : [0,1] \to \overline{\mathbb{R}}$, introduced in [15], as the cumulative quantile function:

$$F_{(-2)}(X;p) = \int_0^p F_{(-1)}(X;t)\,dt \quad \text{for} \quad 0 < p \leq 1, \tag{2.1}$$

and $F_{(-2)}(X;0) = 0$. The *Average Value at Risk* of X at level p is defined as

$$\mathrm{AVaR}_p(X) = -\frac{1}{p}F_{(-2)}(X;p) = \frac{1}{p}\int_0^p \mathrm{VaR}_t(X)\,dt.$$

A measure of risk ρ is called *spectral* (see [1]) if there exists a probability measure $\mu \in \mathcal{P}((0,1])$ such that for all X

$$\rho(X) = \int_0^1 \mathrm{AVaR}_p(X)\,\mu(dp).$$

We say that a measure of risk ρ is a *Kusuoka measure*, if there exists a convex set $\mathcal{M} \subset \mathcal{P}((0,1])$ such that for all X we have

$$\rho(X) = \sup_{\mu\in\mathcal{M}} \int_0^1 \mathrm{AVaR}_p(X)\,\mu(dp).$$

A fundamental result in the theory of coherent measures of risk is the Kusuoka theorem [14]: *For a nonatomic space Ω, every law invariant, finite-valued coherent measure of risk on $\mathcal{L}_\infty(\Omega,\mathcal{F},P)$ is a Kusuoka measure.*

3 Stochastic Dominance

For a random variable $X \in \mathcal{L}^m(\Omega,\mathcal{F},P)$ we construct the integrated distribution functions

$$F_k(X;\eta) = \int_{-\infty}^{\eta} F_{k-1}(X;\alpha)\;d\alpha \quad \text{for} \quad \eta\in\mathbb{R}, \quad k = 2,\dots,m+1. \tag{3.1}$$

The stochastic dominance relation of order k is defined as follows:

$$X \succeq_{(k)} Y \quad \Leftrightarrow \quad F_k(X;\eta) \le F_k(Y;\eta) \quad \text{for all} \quad \eta\in\mathbb{R}. \tag{3.2}$$

The relations express that X is stochastically larger than Y. Of special importance is the second order stochastic dominance relation, as a fundamental model of risk-averse preferences.

The stochastic dominance relation has an equivalent characterization by utility functions. Relation $X \succeq_{(1)} Y$ means that $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for every nondecreasing utility function $u(\cdot)$ for which these expected values exist. The second order stochastic dominance relation $X \succeq_{(2)} Y$ means that $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for every nondecreasing and concave utility function $u(\cdot)$ (see, e.g., [12, 18]).

Stochastic dominance relations can also be characterized by quantile functions. From the definition of the first order dominance we see that

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_{(-1)}(X;p) \ge F_{(-1)}(Y;p) \quad \text{for all} \quad 0 < p < 1. \tag{3.3}$$

The second order dominance can be characterized by the use of the Lorenz function (2.1):

$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_{(-2)}(X;p) \ge F_{(-2)}(Y;p) \quad \text{for all} \quad 0 \le p \le 1. \tag{3.4}$$

This fact is widely used in economics and statistics (see [3, 13, 17, 21]).

In [10], we provide an equivalent characterization by rank dependent utility functions, which is analogous to the characterization by expected utility functions. Consider the set $\mathcal{W}_0$ of all continuous nondecreasing functions $w : [0,1] \to \mathbb{R}$, and the set $\mathcal{W}_1 \subset \mathcal{W}_0$ of functions which are concave and subdifferentiable at 0.

Lemma 3.1.
(i) *For any two random variables* $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ *the relation* $X \succeq_{(1)} Y$ *holds true if and only if for all* $w \in \mathcal{W}_0$

$$\int_0^1 F_{(-1)}(X;p)\,dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\,dw(p). \tag{3.5}$$

(ii) *For any two random variables* $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ *the relation* $X \succeq_{(2)} Y$ *holds true if and only if* (3.5) *is satisfied for all* $w \in \mathcal{W}_1$.

The functions $w(\cdot)$ appearing in the characterization are rank dependent (dual) utility functions.

First relations between stochastic orders and mean-risk models are developed in [19], where mean-risk models providing stochastically non-dominated solutions are called consistent with the stochastic dominance relation. Further consistency results for specific mean-risk models are presented in [20, 21, 24]. Conditions for consistency of general risk function with the stochastic dominance relation of second order are established in ([28]).

4 Dominance Constrained Optimization

In our earlier paper [8] we have introduced the following optimization problem:

$$\begin{aligned} &\max f(X) \\ &\text{subject to } \; X \succeq_{(k)} Y \\ &X \in C \end{aligned} \tag{4.1}$$

Here C is a closed convex subset of $\mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$ and $f : \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a concave continuous functional. The random variable Y is a benchmark outcome in $\mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$, $k \geq 1$. From now on we concentrate on the case of $k = 2$.

In [9], we developed optimality and duality theory for the following relaxation of model (4.1):

$$\max f(X) \tag{4.2}$$

$$\text{subject to } F_2(X;\eta) \leq F_2(Y;\eta) \quad \forall\, \eta \in [a,b], \tag{4.3}$$

$$X \in C. \tag{4.4}$$

In constraint (4.3), we restrict the range of η to a compact interval $[a,b]$, rather than the whole real line, in order to formulate constraint qualification conditions. When $a = -\infty$ no reasonable constraint qualification condition can be satisfied. We associated with problem (4.2)–(4.4) the Lagrangian

$$L(X,u) = f(X) + \mathbb{E}\big[u(X) - u(Y)\big], \tag{4.5}$$

where the "Lagrange multiplier" $u(\cdot)$ is a certain von Neumann–Morgenstern utility function. If $f(X) = \mathbb{E}[X]$, then in (4.5) the function $g(x) = x + \hat{u}(x)$ becomes the *implied utility function* resulting from the benchmark Y. Thus, the expected utility approach may be regarded as dual to the model with stochastic dominance constraints with respect to a random benchmark.

Now, we consider the following optimization problem:

$$\max\ f(X) \tag{4.6}$$

$$\text{subject to } F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \forall\, p \in [\alpha,\beta], \tag{4.7}$$

$$X \in C. \tag{4.8}$$

Here, $[\alpha,\beta] \subset (0,1)$. We call (4.7) the *inverse dominance constraint.* The restriction of the inequality $F_{(-2)}(X;p) \geq F_{(-2)}(Y;p)$ to an interval $[\alpha,\beta] \subset (0,1)$ is needed for a constraint qualification condition. We use the following constraint qualification condition for problem (4.6)–(4.8):

Definition 4.1. Problem (4.6)–(4.8) satisfies the *uniform inverse dominance condition* if there exists a point $\tilde{X} \in C$ such that

$$\inf_{p\in[\alpha,\beta]} \Big\{ F_{(-2)}(\tilde{X};p) - F_{(-2)}(Y;p) \Big\} > 0.$$

In [10], we developed optimality and duality theory for this optimization problem. Let us introduce the set $\mathcal{W}_1([\alpha,\beta])$ of concave and nondecreasing functions $w : [0,1] \to \mathbb{R}$ satisfying the following conditions:

$$w(p) = 0 \text{ for all } p \in [\beta,1];$$

$$w(p) = w(\alpha) + c(p-\alpha), \text{ with some } c > 0, \text{ for all } p \in [0,\alpha].$$

We define the Lagrangian of problem (4.6)–(4.8), $\Phi : C \times \mathcal{W}_1([\alpha,\beta]) \to \mathbb{R}$, as follows

$$\Phi(X,w) = f(X) + \int_0^1 F_{(-1)}(X;p)\, dw(p) - \int_0^1 F_{(-1)}(Y;p)\, dw(p). \tag{4.9}$$

It is well defined, because $w(\cdot)$ has bounded subgradients and for every $X \in \mathcal{L}_1(\Omega,\mathcal{F},P)$ we have the following identity:

$$\int_0^1 F_{(-1)}(X;p)\, dp = \mathbb{E}[X]. \tag{4.10}$$

Theorem 4.2 ([10], Thm.2). *Assume that the uniform inverse dominance condition is satisfied. If $\hat{X}$ is an optimal solution of* (4.6)–(4.8) *then there exists a function $\hat{w} \in \mathcal{W}_1([\alpha,\beta])$ such that*

$$\Phi(\hat{X},\hat{w}) = \max_{X\in C} \Phi(X,\hat{w}); \quad \textit{and} \tag{4.11}$$

$$\int_0^1 F_{(-1)}(\hat{X};p)\, d\hat{w}(p) = \int_0^1 F_{(-1)}(Y;p)\, d\hat{w}(p). \tag{4.12}$$

Conversely, if for some function $\hat{w} \in \mathcal{W}_1([\alpha,\beta])$ an optimal solution $\hat{X}$ of (4.11) *satisfies* (4.7) *and* (4.12)*, then $\hat{X}$ is an optimal solution of* (4.6)–(4.8).

Similarly to the results in ([8]), we observe that the "Lagrange multipliers" w to the stochastic dominance constraints are directly related to rank dependent expected utility theory. A point $\hat{X}$ is a solution to (4.6)–(4.8) if there exists a dual utility function $\hat{w}(\cdot)$ such that $\hat{X}$ maximizes over C the objective functional $f(X)$ augmented with this dual utility. In particular, when $f(X) = \mathbb{E}[X]$, we can use the identity (4.10) to transform the Lagrangian (4.9) in the following way:

$$\begin{aligned}\Phi(X, w) &= \int_0^1 F_{(-1)}(X;p)\,dp + \int_0^1 F_{(-1)}(X;p)\,dw(p) - \int_0^1 F_{(-1)}(Y;p)\,dw(p)\\ &= \int_0^1 F_{(-1)}(X;p)\,dv(p) - \int_0^1 F_{(-1)}(Y;p)\,dw(p),\end{aligned}$$

where $v(p) = p + w(p)$. At the optimal solution the function $\hat{v}(p) = p + \hat{w}(p)$ is the *implied rank-dependent utility function* associated with the benchmark Y. Since $\int_0^1 F_{(-1)}(Y;p)\,dw(p)$ is fixed, the problem at the right hand side of (4.11) becomes a problem of maximizing the implied rank dependent expected utility in C.

A duality theory can be based on Lagrangian (4.9). For every function $w \in \mathcal{W}_1([\alpha,\beta])$ the problem

$$\max_{X\in C} \Big\{ f(X) + \int_0^1 F_{(-1)}(X;p)\,dw(p) - \int_0^1 F_{(-1)}(Y;p)\,dw(p) \Big\} \tag{4.13}$$

is a Lagrangian relaxation of problem (4.6)–(4.8). Its optimal value

$$\Psi(w) = \sup_{X\in C} \Phi(X, w)$$

is always greater than or equal to the optimal value of (4.6)–(4.8). Indeed, any feasible solution X of (4.6)–(4.8) is feasible for (4.13), and the dominance relation (4.7) implies that

$$\int_0^1 F_{(-1)}(X;p)\,dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\,dw(p),$$

by virtue of Lemma 3.1. We define the dual problem as

$$\min_{w\in\mathcal{W}_1([\alpha,\beta])} \Psi(w). \tag{4.14}$$

The set $\mathcal{W}_1([\alpha,\beta])$ is a closed convex cone in $\mathcal{C}([0,1])$ and $\Psi(\cdot)$ is a convex functional, so problem (4.14) is a convex optimization problem.

Theorem 4.3 ([10], Thm.3). *Assume that the uniform inverse dominance condition is satisfied and problem* (4.6)–(4.8) *has an optimal solution. Then problem* (4.14) *has an optimal solution and the optimal values of both problems coincide. Furthermore, the set of optimal solutions of* (4.14) *is the set of functions* $\hat{w} \in \mathcal{W}_1([\alpha,\beta])$ *satisfying* (4.11)–(4.12) *for an optimal solution* $\hat{X}$ *of* (4.6)–(4.8).

5 The Implied Measure of Risk

The objective of this section is to demonstrate that optimization of a measure of risk can be interpreted as a Lagrangian dual to a problem with dominance constraints in the inverse form. First, we establish a correspondence between rank dependent utility functions and spectral measures of risk.

Lemma 5.1. *For every function* $w \in \mathcal{W}_1$ *there exists a spectral risk measure* ρ *and a constant* $\kappa \geq 0$ *such that for all* $X \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$

$$\int_0^1 F_{(-1)}(X;p)\,dw(p) = -\kappa\hat{\rho}(X). \tag{5.1}$$

Conversely, for every spectral measure of risk ρ there exists a function $w \in \mathcal{W}_1$ such that equation (5.1) holds true with $\kappa = 1$.

Proof. Consider the functional

$$\sigma(X) = \int_0^1 F_{(-1)}(X;p)\,dw(p). \tag{5.2}$$

For any $w \in \mathcal{W}_1$, the left derivative of w,

$$w'_-(p) = \lim_{\tau\uparrow p}[w(p) - w(\tau)]/(p-\tau), \quad p \in (0,1],$$

is well-defined, nondecreasing and continuous from the left. By [6, Theorem 12.4], after an obvious adaptation, there exists a unique regular nonnegative measure ν on $[0,1]$ satisfying

$$\nu([p,1]) = \begin{cases} w'_-(p) & \text{for } p \in (0,1], \\ w'_+(0) & \text{for } p = 0. \end{cases}$$

By the concavity of $w(\cdot)$, the measure ν is nonnegative. Substitution into (5.2) and integration by parts yield

$$\begin{aligned}\sigma(X) &= \int_0^1 F_{(-1)}(X;p)\,\nu([p,1])\,dp = \int_0^1 \int_p^1 d\nu(t)\,F_{(-1)}(X;p)\,dp \\ &= \int_0^1 \int_0^t F_{(-1)}(X;p)\,dp\,\nu(dt) = \int_0^1 F_{(-2)}(X;t)\,\nu(dt) = \int_0^1 \frac{1}{t}F_{(-2)}(X;t)\,t\nu(dt).\end{aligned} \tag{5.3}$$

Define $\kappa = \int_0^1 t\,\nu(dt)$. Observe that $\kappa = 0$ means that $w(p) \equiv 0$ for all $p \in [0,1]$. If $\kappa > 0$ we define the measure $\mu(dt) = t\nu(dt)/\kappa$, and we obtain that

$$\sigma(X) = -\kappa \int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt). \tag{5.4}$$

Clearly, μ is a probability measure on $(0,1]$. Hence, $\rho(\cdot) = -\sigma(\cdot)/\kappa$ is a spectral measure of risk.

To prove the converse, consider a spectral measure of risk

$$\rho(X) = \int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt),$$

with some probability measure μ on $(0,1]$. We define the measure ν on $(0,1]$ by the relation $\nu(dt) = (1/t)\mu(dt)$. Now we define the function $\xi(\tau) = \nu([\tau,1])$ as the ν-measure of the set $[\tau,1]$, and the function $w : [0,1] \to \mathbb{R}$ as follows:

$$w(p) = -\int_p^1 \xi(\tau)\,d\tau.$$

We observe that $\xi(\cdot)$ is a nonnegative, non-increasing function on the interval $[0,1]$. Therefore, $w(\cdot)$ is concave. Furthermore, $w(\cdot)$ is nondecreasing by construction. Using the function w, we proceed as in (5.3) (in the reverse order) to derive the representation (5.1) of the risk measure $\rho(\cdot)$. □

Now, we formulate and prove necessary conditions of optimality for problems with dominance constraints where the "Lagrangian" involves a spectral risk measure.

Theorem 5.2. *Assume that the uniform inverse dominance condition is satisfied. If $\hat{X}$ is an optimal solution of* (4.6)–(4.8) *then there exists a spectral risk measure $\hat{\rho}$ and a constant $\kappa \geq 0$ such that $\hat{X}$ is also an optimal solution of the problem*

$$\max_{X \in C} \big\{ f(X) - \kappa \hat{\rho}(X) \big\}; \quad \textit{and,} \tag{5.5}$$

$$\kappa \hat{\rho}(\hat{X}) = \kappa \hat{\rho}(Y). \tag{5.6}$$

Proof. The optimal solution of problem (4.6)–(4.8) is also an optimal solution of (4.13). The last term in this expression does not depend on X and can be skipped. By Lemma 5.1, there exist a constant $\kappa \geq 0$ and a spectral measure of risk $\hat{\rho}(\cdot)$ such that the second term of (4.13) can be written as follows:

$$\sigma(X) = \int_0^1 F_{(-1)}(X;p)\, dw(p) = -\kappa \hat{\rho}(X).$$

An identical equation holds true for Y. Therefore the optimal solution of problem (4.6)–(4.8) is also a solution of the problem (5.5). By the complementarity condition (4.12) of Theorem 4.2, we get $\sigma(X) = \sigma(Y)$, which is the same as (5.6). □

Observe that if the dominance constraint (4.6) is active (i.e., its removal would change the optimal value), we have $\kappa > 0$, and the complementarity condition (5.6) takes on the form

$$\hat{\rho}(\hat{X}) = \hat{\rho}(Y).$$

It is also clear that the support of the measure μ in the spectral representation of $\hat{\rho}(\cdot)$ is included in $[\alpha, \beta]$.

Consider the special case of finite Ω with elementary events ω_i, $i = 1, \ldots, m$, occurring with probability $1/m$ each. Then we identify random variables on Ω with vectors in $\mathbb{R}^m$ and the set C is a subset of $\mathbb{R}^m$. For a random variable X we denote $x_i = X(\omega_i)$, $i = 1, \ldots, m$. We use the symbol $x_{[i]}$ to denote the ordered coordinates of X, that is,

$$x_{[1]} \leq x_{[2]} \leq \cdots \leq x_{[m]}.$$

We have shown in [10] that, in this case, problem (4.6)–(4.8) with $[\alpha, \beta] = [0, 1]$ takes on the form

$$\max f(X) \tag{5.7}$$

$$\text{subject to } \sum_{k=1}^{i} x_{[k]} \geq \sum_{k=1}^{i} y_{[k]}, \quad i = 1, \ldots, m, \tag{5.8}$$

$$X \in C. \tag{5.9}$$

By [10, Lem.2], constraints (5.8) are convex polyhedral. Slater's constraint qualification condition can be formulated for problem (5.7)–(5.9) as follows: there exists $\tilde{X} \in \operatorname{relint} C$ such that

$$\sum_{k=1}^{i} \tilde{x}_{[k]} > \sum_{k=1}^{i} y_{[k]}, \quad i = 1, \ldots, m.$$

The Lagrangian takes on the form

$$\Psi(x, \nu) = f(x) + \sum_{i=1}^{m} \nu_i \Big(\sum_{k=1}^{i} x_{[k]} - \sum_{k=1}^{i} y_{[k]} \Big). \tag{5.10}$$

Using necessary conditions of optimality, e.g., [25, Thm. 28.2], we obtain the existence of Lagrange multipliers ν such that the optimal solution $\hat{X}$ is also an optimal solution of the problem

$$\max_{X \in C} \Psi(x, \nu).$$

Setting $\kappa = \frac{1}{m}\sum_{i=1}^{m} i\nu_i$ and $\mu_i = i\nu_i/(\kappa m)$, we see that the results of Theorem 4.2 remain valid for $[\alpha, \beta] = [0, 1]$. The implied spectral measure of risk has the form:

$$\hat{\rho}(X) = -\sum_{i=1}^{m} \nu_i \sum_{k=1}^{i} x_{[k]}.$$

Consider now the special case when $f(X) = \mathbb{E}[X]$.

Corollary 5.3. *Assume that the uniform inverse dominance condition is satisfied. If $\hat{X}$ is an optimal solution of* (4.6)–(4.8) *then there exists a spectral risk measure $\bar{\rho}$ such that $\bar{X}$ is also an optimal solution of the problem*

$$\min_{X \in C} \bar{\rho}(X); \quad \textit{and} \tag{5.11}$$

$$\mathbb{E}[X] + \bar{\kappa}\bar{\rho}(\hat{X}) = \mathbb{E}[Y] + \bar{\kappa}\bar{\rho}(Y) \tag{5.12}$$

with some $\bar{\kappa} \geq 1$.

Proof. Using formula (5.4) from the proof of Theorem 4.2 we obtain

$$\mathbb{E}[X] + \sigma(X) = -\text{AVaR}_1(X) - \kappa \int_0^1 \text{AVaR}_t(X)\,\mu(dt).$$

Now we define the measure $\bar{\mu} = (\kappa\mu + \delta_1)/(\kappa + 1)$, where δ_1 is an atom of mass 1 at 1. Again, $\bar{\mu}$ is a probability measure. We conclude that $\mathbb{E}[X] + \sigma(X) = -(1 + \kappa)\bar{\rho}(X)$, with $\bar{\rho}(X) = \int_0^1 \text{AVaR}_t(X)\,\bar{\mu}(dt)$. Therefore (5.5) takes on the form (5.11). From (5.6) we get $\sigma(\hat{X}) = \sigma(Y)$ and thus

$$\mathbb{E}[X] + (1 + \kappa)\bar{\rho}(X) = \mathbb{E}[Y] + (1 + \kappa)\bar{\rho}(Y).$$

□

In fact, the spectral measure $\bar{\rho}$ has an atom $\lambda \in (0, 1)$ at 1, and $\bar{\kappa}$ satisfies $\lambda(1 + \bar{\kappa}) = 1$.

Now, we prove sufficient conditions of optimality for optimization problems with dominance constraints based on the Lagrangian involving spectral risk measures.

Theorem 5.4. *If for some spectral risk measure $\hat{\rho}$ and some $\kappa \geq 0$ an optimal solution $\hat{X}$ of* (5.5) *satisfies* (4.7) *with* $[\alpha, \beta] = [0, 1]$ *and* (5.6)*, then $\hat{X}$ is an optimal solution of* (4.6)–(4.8) *with* $[\alpha, \beta] = [0, 1]$.

Proof. If $\kappa = 0$ the solution of problem $\max_{X \in C} f(X)$ satisfies (4.7) and thus it is optimal for problem (4.6)–(4.8). Assume now that there exist a positive constant κ and a coherent risk measure $\hat{\rho}(\cdot)$ such that (5.5) and (5.6) are satisfied for some feasible random variable $\hat{X}$. Then for every $X \in C$

$$f(X) - \kappa\hat{\rho}(X) \leq f(\hat{X}) - \kappa\hat{\rho}(\hat{X}) = f(\hat{X}) - \kappa\hat{\rho}(Y).$$

Using Lemma 5.1, with the risk measure $\hat{\rho}(\cdot)$ we associate a function $w \in \mathcal{W}_1([0,1])$. We set $\hat{w}(\cdot) = \kappa w(\cdot)$ and substituting the identity (5.1) to obtain

$$f(X) + \int_0^1 F_{(-1)}(X;p)\, d\hat{w}(p) \le f(\hat{X}) + \int_0^1 F_{(-1)}(Y;p)\, d\hat{w}(p).$$

If $X \succeq_{(2)} Y$, Lemma 3.1(b) implies that

$$f(X) \le f(\hat{X}) + \int_0^1 F_{(-1)}(Y;p)\, d\hat{w}(p) - \int_0^1 F_{(-1)}(X;p)\, d\hat{w}(p) \le f(\hat{X}).$$

This means that $\hat{X}$ is an optimal solution of problem (4.6)–(4.8) with $[\alpha,\beta] = [0,1]$. □

6 The Implied Dominance Constraint

One can also ask an inverse question. We are given the problem

$$\max_{X \in C} \big\{ f(X) - \kappa \rho(X) \big\} \tag{6.1}$$

with $\kappa > 0$ and a coherent law invariant measure of risk $\rho(\cdot)$ and we want to construct a problem of form (4.1), such that the solutions of both problems are the same. Denote the solution of (6.1) by $\hat{X}$. We want to determine a benchmark function $F_{(-2)}(Y;p)$ for the right hand side of the dominance constraint (4.7) with $[\alpha,\beta] = [0,1]$:

$$F_{(-2)}(X;p) \ge F_{(-2)}(Y;p) \quad \forall\, p \in [0,1], \tag{6.2}$$

such that $\hat{X}$ is also a solution of problem (4.6)–(4.8).

We assume that ρ is a Kusuoka risk measure with the representation:

$$\rho(X) = \sup_{\mu \in \mathcal{M}} \int_0^1 \mathrm{AVaR}_t(X)\, \mu(dt), \tag{6.3}$$

where $\mathcal{M} \subset \mathcal{P}([0,1])$. We consider $\mathcal{P}([0,1])$ as a subset of the space $\mathrm{rca}([0,1])$ of regular countably additive measures on the interval $[0,1]$, which is the topological dual to the space of continuous functions $\mathcal{C}([0,1])$. We assume that $\mathcal{M}$ is compact in the weak* topology in $\mathrm{rca}([0,1])$.

Theorem 6.1. *Assume that $\hat{X}$ is a solution of problem* (6.1)*, where ρ has a representation* (6.3) *such that the set $\mathcal{M}$ is compact in the weak* topology in* $\mathrm{rca}([0,1])$*. Then there exists a measure $\hat{\mu} \in \mathcal{M}$ such that*

$$\rho(\hat{X}) = \int_0^1 \mathrm{AVaR}_t(\hat{X})\, \hat{\mu}(dt),$$

and for every $Y \in \mathcal{L}_1(\Omega,\mathcal{F},P)$ satisfying the conditions

$$\begin{aligned} F_{(-2)}(Y;t) &\le F_{(-2)}(\hat{X};t), \quad \text{for all} \quad t \in [0,1],\\ F_{(-2)}(Y;t) &= F_{(-2)}(\hat{X};t), \quad \text{for all} \quad t \in \mathrm{supp}(\hat{\mu}), \end{aligned} \tag{6.4}$$

the point $\hat{X}$ is also an optimal solution of problem (4.6)–(4.8) *with* $[\alpha,\beta] = [0,1]$.

Proof. The problem (6.1) corresponds to the following problem:

$$\sup_{X\in C}\big\{f(X)-\kappa\rho(X)\big\}=\sup_{X\in C}\inf_{\mu\in\mathcal{M}}\Big\{f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)\Big\}. \tag{6.5}$$

The supremum on the left hand side is achieved at the point $\hat{X}$ by assumption. Observe that for every $\mu\in\mathcal{M}$ the functional $X\mapsto f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)$ is concave and continuous in the space $\mathcal{L}_1(\Omega,\mathcal{F},P)$. For every X, the function $t\to \mathrm{AVaR}_t(X)$ is continuous on $[0,1]$. Therefore, the functional $\mu\mapsto -\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)$ is linear and continuous in the weak* topology on rca($[0,1]$). The set $\mathcal{M}$ is assumed compact in the weak* topology. Therefore, the assumptions of the asymmetric min-max theorem (see [5, Theorem 6.2.7]) are satisfied and we can exchange the supremum and the infimum operations in (6.5). Moreover, the infimum with respect to $\mu\in\mathcal{M}$ is achieved. We obtain

$$\sup_{X\in C}\inf_{\mu\in\mathcal{M}}\Big\{f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)\Big\}$$
$$=\min_{\mu\in\mathcal{M}}\sup_{X\in C}\Big\{f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)\Big\}. \tag{6.6}$$

Let $\hat{\mu}\in\mathcal{M}$ be the measure at which the minimum on the right hand side is achieved. It follows that the pair $(\hat{X},\hat{\mu})$ is a saddle point of the function $f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)$. Consequently, $\hat{X}$ is a solution of the problem

$$\max_{X\in C}\Big\{f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\hat{\mu}(dt)\Big\}.$$

Defining the spectral measure

$$\hat{\rho}(X)=\int_0^1 \mathrm{AVaR}_t(X)\,\hat{\mu}(dt)$$

we conclude that $\hat{X}$ is also a solution of the problem $\max_{X\in C}\big\{f(X)-\kappa\hat{\rho}(X)\big\}$. We can now apply Theorem 5.4. Conditions (6.4) are equivalent to (4.7) with $[\alpha,\beta]=[0,1]$ and (5.6), and the assertion follows. □

By Lemma 5.1, there is one-to-one correspondence between spectral measures of risk and rank-dependent utility functions. Each measure $\mu\in\mathcal{M}$ gives rise to a rank-dependent utility function $w(\cdot)$ such that

$$\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)=-\int_0^1 F_{(-1)}(X;p)\,dw(p).$$

We denote the collection of rank-dependent utility functions obtained via the measures from $\mathcal{M}$ by $\mathcal{W}_{\mathcal{M}}$, and the function corresponding to the measure $\hat{\mu}$ by $\hat{w}$.

Owing to (6.6), the problem (6.1) corresponds to the following min-max problem:

$$\max_{X\in C}\big\{f(X)-\kappa\rho(X)\big\}=\min_{\mu\in\mathcal{M}}\sup_{X\in C}\Big\{f(X)-\kappa\int_0^1 \mathrm{AVaR}_t(X)\,\mu(dt)\Big\}$$
$$=\min_{w\in\mathcal{W}_{\mathcal{M}}}\sup_{X\in C}\Big\{f(X)+\kappa\int_0^1 F_{(-1)}(X;p)\,dw(p)\Big\}.$$

Therefore, the function $\hat{w}$ is a solution of a certain dual problem, similar to (4.14). There are two differences: the set $\mathcal{W}_1$ is replaced by its subset $\kappa\mathcal{W}_{\mathcal{M}}$, and the linear terms in μ involving the benchmark Y are absent.

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Darinka Dentcheva
Stevens Institute of Technology, Department of Mathematical Sciences
Hoboken, NJ 07030, USA
E-mail address: darinka.dentcheva@stevens.edu

Andrzej Ruszczyński
Rutgers University, Department of Management Science and Information Systems
Piscataway, NJ 08854, USA
E-mail address: rusz@business.rutgers.edu