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LIPSCHITZ MODULUS OF THE OPTIMAL SET MAPPING IN CONVEX OPTIMIZATION VIA MINIMAL SUBPROBLEMS

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Dedicated to Michel Théra on his 60th Birthday

Abstract: In this paper we relate the Lipschitz modulus of the optimal set mapping of a canonically perturbed convex optimization problem in $\mathbb{R}^n$ with the Lipschitz moduli of suitable subproblems with exactly n constraints. This approach allows us to simplify other expressions for the modulus given in [8] in terms of d.c. functions involving the problem's data.

Key words: *Lipschitz modulus, convex programming, metric regularity*

Mathematics Subject Classification: *49J53, 90C25, 90C31, 52A41*

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1 Introduction

This paper is devoted to express the *Lipschitz modulus* of the *optimal set mapping* (also called *argmin* mapping) of a canonically perturbed convex optimization problem (see (1.1)) as the maximum of the Lipschitz moduli associated with appropriate subproblems with exactly n constraints. These subproblems are minimal in a sense which is made precise below. Some antecedents for the particular case of linear optimization problems can be found in [5], [18] and [19] (see also [1], [17] and [21] for the feasible set mapping). In the context of linear problems with finitely many constraints the Lipschitz modulus of the optimal set mapping becomes itself a Lispchitz constant (referred to as *sharp Lipschitz constant* in [18]). More specifically, for such problems, [18] provides the sharp Lipschitz constant for the feasible set mapping and a Lispchitz constant (which can be sharp or not, depending on the problem) for the optimal set mapping. Paper [5], making use of some tools of [19], provides the sharp Lipschitz constant for the optimal set mapping at any linear problem (with only - finitely many- inequality constraints). On the other hand, [8] provides an upper bound for the modulus in the semi-infinite (infinitely many constraints) convex setting, and the exact modulus in the finite case. Our approach here allows us to simplify the referred latter expression (see (2.1) in §2).

We are concerned with the parametrized convex programming problem, in $\mathbb{R}^n$,

$$P(c,b): \quad \begin{array}{l} \operatorname{Inf} f(x) + \langle c, x\rangle \\ \text{s.t. } g_i(x) \leq b_i, \quad i = 1,\ldots,m, \end{array} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $c \in \mathbb{R}^n$, $\langle .,. \rangle$ represents the usual inner product in $\mathbb{R}^n$, $m \geq n$, $f, g_1, ..., g_m : \mathbb{R}^n \to \mathbb{R}$ are given convex functions, and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$. The pair $(c, b) \in \mathbb{R}^n \times \mathbb{R}^m$ is regarded as the parameter to be perturbed. In other words, our model is subject to canonical perturbations, i.e., linear perturbations of the objective function and right hand side perturbations of the constraints. In the parameter space $\mathbb{R}^n \times \mathbb{R}^m$ we consider the norm

$$\|(c, b)\| := \max \left\{ \|c\|, \|b\|_\infty \right\}, \tag{1.2}$$

where $\mathbb{R}^n$ is equipped with any given norm, $\|\cdot\|$, and $\|b\|_\infty := \max_{1 \leq i \leq m} |b_i|$. The corresponding dual norm in $\mathbb{R}^n$ is given by $\|u\|_* := \max \left\{ \langle u, x \rangle \mid \|x\| \leq 1 \right\}$ and d_* denotes the associated distance.

Given a subset $D \subset \{1, \ldots, m\}$ with $|D| = n$ ($|D|$ denotes the cardinality of D), we consider the associated subproblem

$$\begin{array}{ll} P_D(c, \beta): & \operatorname{Inf} f(x) + \langle c, x \rangle \\ & \text{s.t. } g_i(x) \leq \beta_i, \quad i \in D, \end{array} \tag{1.3}$$

where $\beta = (\beta_i)_{i \in D} \in \mathbb{R}^D$ $(\equiv \mathbb{R}^n)$. The parameter space $\mathbb{R}^n \times \mathbb{R}^D$ is endowed with a norm analogous to (1.2).

Associated with the parametrized problem $P(c, b)$, we consider the *optimal set mapping*, $\mathcal{F}^* : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, which assigns to each parameter $(c, b) \in \mathbb{R}^n \times \mathbb{R}^m$ the *optimal set* of $P(c, b)$; i.e.,

$$\mathcal{F}^*(c, b) := \arg\min \left\{ f(x) + \langle c, x \rangle \mid g_i(x) \leq b_i, \ i = 1, \ldots, m \right\}.$$

For (1.3), the mapping $\mathcal{F}_D^* : \mathbb{R}^n \times \mathbb{R}^D \rightrightarrows \mathbb{R}^n$ is defined in a completely analogous way.

Paper [9] introduces a sufficient condition for the *strong Lipschitz stability* of $\mathcal{F}^*$ at a given $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph} \mathcal{F}^*$ (the graph of $\mathcal{F}^*$); in other words, for the single-valuedness and Lipschitz continuity of $\mathcal{F}^*$ in a neighborhood of $\left(\overline{c}, \overline{b}\right)$ (equivalently, the strict continuity of $\mathcal{F}^*$ at $\left(\overline{c}, \overline{b}\right)$, according to [22, Def. 9.1(b)]). Although the standard definition requires local single-valuedness (see, e.g., [16]), we can say single-valuedness because $\mathcal{F}^*$ is convex-valued. This sufficient condition (given in §2) is called *Extended Nürnberger Condition*, ENC for short, in [7]. That paper shows that, under ENC at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$, the Lipschitz modulus of $\mathcal{F}^*$ at $\left(\overline{c}, \overline{b}\right)$, denoted by $\operatorname{lip} \mathcal{F}^*\left(\overline{c}, \overline{b}\right)$ (recall that $\mathcal{F}^*$ is single-valued around $\left(\overline{c}, \overline{b}\right)$) turns out to be equal to $\operatorname{lip} \widetilde{\mathcal{F}}\left(\overline{b}\right)$, where $\widetilde{\mathcal{F}}(b) := \mathcal{F}^*(\overline{c}, b)$ for $b \in \mathbb{R}^m$. In other words, under ENC, small perturbations of the objective function are negligible in our analysis and

$$\operatorname{lip} \mathcal{F}^*\left(\overline{c}, \overline{b}\right) = \operatorname{lip} \widetilde{\mathcal{F}}\left(\overline{b}\right) := \limsup_{b, b' \to \overline{b}, \ b \neq b'} \frac{\left\| \mathcal{F}^*(\overline{c}, b) - \mathcal{F}^*(\overline{c}, b') \right\|}{\|b - b'\|_\infty}, \tag{1.4}$$

where we have identified $\mathcal{F}^*(\overline{c}, b)$ with its only element, for b near $\overline{b}$, in order to avoid additional notation. Nevertheless, we have preferred to give the statements of Theorems 1 and 2 in terms of $\left(\overline{c}, \overline{b}\right)$ instead of $\overline{b}$ only.

The strong Lipschitz stability of $\mathcal{F}^*$ at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ is known to be equivalent to the *metric regularity* of the inverse mapping $\mathcal{G}^* := \left(\mathcal{F}^*\right)^{-1}$ at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right)$ (see, e.g., [15, Cor. 4.7]). Recall that $\left(\mathcal{F}^*\right)^{-1}$ is defined by $(c, b) \in \left(\mathcal{F}^*\right)^{-1}(x) \Leftrightarrow x \in \mathcal{F}^*(c, b)$. The metric regularity of $\mathcal{G}^*$ at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \operatorname{gph} \mathcal{G}^*$ is defined as the existence of certain neighborhoods, U of $\overline{x}$ and W of $(\overline{c}, \overline{b})$, and a constant $\kappa \geq 0$ such that

$$d\left(x, \mathcal{F}^*(c, b)\right) \leq \kappa d\left((c, b), \mathcal{G}^*(x)\right) \text{ for all } x \in U \text{ and all } (c, b) \in V. \tag{1.5}$$

The infimum of κ for which associated U and W verifying (1.5) exist is the so-called *modulus of metric regularity*, denoted by $\operatorname{reg}\mathcal{G}^*\left(\overline{x} \mid (\overline{c},\overline{b})\right)$, which coincides with $\operatorname{lip}\mathcal{F}^*\left(\overline{c},\overline{b}\right)$ under the ENC assumption. Papers [7] and [9] follow the terminology of metric regularity, whereas [8] is in terms of strong Lipschitz stability and Lipschitz modulus.

Concerning the structure of the paper, §2 gathers the additional notation and preliminary results needed along the paper, and §3 is oriented to establish Theorem 3.6, which is the main contribution of the paper. Final conclusions are presented in §4. At this moment we point out the fact that the approach of the present paper does not extend to the semi-infinite case (with infinitely many constraints), on spite of being strongly connected with [8]. In fact, the question of whether or not Theorem 3.6 extends to the semi-infinite case remains as an open problem even in the particular case of linear problems (see [5, Sect. 5]).

2 Preliminaries

This section provides the necessary notation and preliminary results needed later on. Given $\varnothing \neq X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by $\operatorname{co}(X)$ and $\operatorname{cone}(X)$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that $\operatorname{cone}(X)$ always contains the zero-vector, 0_p, and so $\operatorname{cone}(\varnothing) = \{0_p\}$. From the topological side, $\operatorname{int}(X)$ denotes the *interior* of X.

For a given $b = (b_1, ..., b_m) \in \mathbb{R}^m$, we denote the constraint system of (1.1) by $\sigma(b)$, i.e., $\sigma(b) := \{g_i(x) \leq b_i,\ i = 1, \ldots, m\}$, and $\mathcal{F}(b)$ represents the associated feasible set, i.e., $\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid g_i(x) \leq b_i,\ i = 1, \ldots, m\}$. Given $b \in \mathbb{R}^m$ and $x \in \mathcal{F}(b)$, we denote by $T_b(x)$ the associated *active index set* given by

$$T_b(x) := \{i \in \{1, ..., m\} \mid g_i(x) = b_i\}.$$

System $\sigma(b)$ is said to satisfy the *Slater constraint qualification* (SCQ) if $T_b(x^0)$ is empty for some feasible point $x^0 \in \mathcal{F}(b)$.

Next we recall the well-known Karush-Kuhn-Tucker (KKT) optimality conditions. Here ∂ represents the classical subdifferential in convex analysis.

Lemma 2.1. (see, for instance, [12, Ch. VII]) *Let $(c, b) \in \mathbb{R}^n \times \mathbb{R}^m$ and $x \in \mathcal{F}(b)$. If*

$$(c + \partial f(x)) \cap \operatorname{cone}\left(\cup_{i\in T_b(x)}(-\partial g_i(x))\right) \neq \varnothing$$

then $x \in \mathcal{F}^*(c, b)$. The converse holds when $\sigma(b)$ satisfies SCQ.

Definition 2.2. The *extended Nürnberger condition (ENC*, for short*)* is said to be satisfied at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$ if

$$\begin{gathered}\sigma\left(\overline{b}\right) \text{ satisfies } SCQ \text{ and there is no } D \subset T_{\overline{b}}(\overline{x}) \\ \text{with } |D| < n \text{ such that } (\partial f(\overline{x}) + \overline{c}) \cap \operatorname{cone}\left(\bigcup_{i\in D}(-\partial g_i(x))\right) \neq \varnothing.\end{gathered}$$

Observe that, because of Carathéodory's Theorem and KKT conditions, the existence of such a D with $|D| \leq n$ is guaranteed. So, ENC entails $|D| = n$.

In [9, Thm. 10] ENC is shown to be sufficient for the strong Lipschitz stability of $\mathcal{F}^*$ at $\left(\overline{c},\overline{b}\right)$ in a semi-infinite setting more general than (1.1), namely, when the index set is assumed to be a compact metric space and the associated g_i's depend continuously on i. In this setting, but restricted to linear problems, ENC is in fact equivalent to the strong Lipschitz stability of $\mathcal{F}^*$ at $\left(\overline{c},\overline{b}\right)$ [9, Thm. 16]. For linear problems, ENC constitutes a reformulation of the condition given in [20] for characterizing those problems in the interior set of problems having a strongly unique optimal solution. The following lemma is a direct consequence of [9, Prop. 9(i)].

Lemma 2.3. *Assume that* ENC *holds at* $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)\in\operatorname{gph}\left(\mathcal{F}^*\right).$ *Then there exists a neighborhood* W *of* $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ *such that* ENC *is satisfied at any* $\left(\left(c,b\right),x\right)\in W\cap\operatorname{gph}\left(\mathcal{F}^*\right).$

Taking [13] and [14] as a starting point, papers [6], [7], and [8] provide different steps in the way of computing (through the formula (2.1)) the Lipschitz modulus. The first step, provided in [6], is highly connected with the problem of finding error bounds for a certain lower semicontinuous function related to functions f_b defined in (2.2) (see [2] for characterizations of error bounds for generic lower semicontinuous functions). Indeed, the same idea of studying the Lipschitz behavior of $\mathcal{F}^*$ by means of the distance function $(x,b)\mapsto f_b(x)$ has been exploited by different authors in more general settings, as pointed out in [3] (see also references therein). Under ENC, and in the referred semi-infinite setting, [8, Thm. 4] provides an upper bound on $\operatorname{lip}\mathcal{F}^*\left(\overline{c},\overline{b}\right)$ in terms of the nominal problem's data, i.e., functions f and g_i's. In the finite case, which is our current setting, this upper bound equals the exact modulus and admits a certain simplification.

Specifically, for our problem (1.1) and assuming that ENC holds at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)\in\operatorname{gph}\left(\mathcal{F}^*\right),$ we have (see [8, Thm. 4]):

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b})=\limsup_{\substack{(z,b)\to(\overline{x},\overline{b})\\ f_b(z)>0}}\ \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_b(z)=f_b^D(z)}}\left(d_*\left(0_n,\widehat{\partial}f_b^D(z)\right)\right)^{-1},\tag{2.1}$$

where

- f_b is defined by
$$f_b(x)=d_\infty\left(b,\widetilde{\mathcal{G}}(x)\right),\tag{2.2}$$
- d_∞ refers to $\left\|\cdot\right\|_\infty,$
- $\widetilde{\mathcal{G}}:=\widetilde{\mathcal{F}}^{-1}$ is the inverse multifunction of $\widetilde{\mathcal{F}}$ (given by $b\in\widetilde{\mathcal{G}}(x)\Leftrightarrow x\in\widetilde{\mathcal{F}}(b):=\mathcal{F}^*(\overline{c},b)$),
- $$\mathcal{T}_{\overline{b}}(\overline{x}):=\left\{\begin{array}{c}D\subset T_{\overline{b}}(\overline{x})\mid |D|=n,\text{ and}\\ (\partial f(\overline{x})+\overline{c})\cap\operatorname{cone}\left(\cup_{i\in D}\left(-\partial g_i(\overline{x})\right)\right)\neq\varnothing\end{array}\right\}\tag{2.3}$$
  is the set of *minimal subsets of indices* involved in KKT conditions, according to ENC,
- for each $D\in\mathcal{T}_{\overline{b}}(\overline{x}),$ f_b^D is given by
$$f_b^D(x):=\max\{\left|g_i(x)-b_i\right|,\ i\in D;\ g_i(x)-b_i,\ i\in\{1,...,m\}\setminus D\},\tag{2.4}$$
  and
- $\widehat{\partial}$ stands for the *regular* (*Fréchet*) *subdifferential.* We recall that, for $\varphi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ and $z\in\mathbb{R}^n$ with $\varphi(z)<+\infty$, one has
$$\widehat{\partial}\varphi(z):=\left\{u\in\mathbb{R}^n\mid\liminf_{\tau\searrow0,\ w'\to w}\frac{\varphi(z+\tau w')-\varphi(z)}{\tau}\geq\langle u,w\rangle\text{ for all }w\in\mathbb{R}^n\right\}.$$
  (For convex functions it coincides with the ordinary subdifferential in convex analysis.)

The following lemma is a straightforward consequence of [8, Thm. 3] and provides an operative expression for $f_b(x)$ with (x,b) near $\left(\overline{x},\overline{b}\right).$

Lemma 2.4. *Assume that* ENC *is satisfied at* $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}^{*}\right)$. *Then there exist some neighborhoods* U *and* V *of* $\overline{x}$ *and* $\overline{b}$, *respectively, verifying*

$$f_{b}\left(x\right)=\min_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} f_{b}^{D}(x),\ \textit{for all } x\in U \textit{ and all } b\in V. \tag{2.5}$$

For further details about these ingredients, the reader is addressed to [7] and [8]. At the moment we emphasize that each f_{b}^{D} is a *d.c. function* (difference of convex functions), and a d.c. decomposition may be obtained in terms of f and g_i's. In such a way, $\widehat{\partial}f_{b}^{D}$ may be expressed in terms of the ordinary subdifferentials of the convex functions appearing in the d.c. decomposition, which are also given in terms of ∂g_i's. An explicit expression for $\widehat{\partial}f_{b}^{D}$ can be found in [8, Lem. 2].

3 Lipschitz Modulus via Minimal Subproblems

In this section we approach the calculus of $\operatorname{lip}\mathcal{F}^{*}\left(\overline{c},\overline{b}\right)$ (see (1.4)) by means of subproblems (1.3), with $D\in\mathcal{T}_{\overline{b}}(\overline{x})$. It is obvious from the definition that, if ENC holds at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}^{*}\right)$, then this property also holds for subproblem $P_D\left(\overline{c},\overline{b}_D\right)$, with a fixed $D\in\mathcal{T}_{\overline{b}}(\overline{x})$, at $\left(\left(\overline{c},\overline{b}_D\right),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}_D^{*}\right)$, where $\overline{b}_D:=\left(\overline{b}_i\right)_{i\in D}$. Note that we are just considering the subproblem of the nominal one consisting only of those constraints whose indices are in D. Moreover, for subproblem $P_D\left(\overline{c},\overline{b}_D\right)$, the role played by $\mathcal{T}_{\overline{b}}(\overline{x})$ in the whole problem is now played by $\{D\}$, i.e., D itself is the only possible subset of D associated with KKT conditions at $\left(\left(\overline{c},\overline{b}_D\right),\overline{x}\right)$.

The following theorem is a straightforward consequence of (2.1) and Lemma 2.4. In it we use the notation

$$h_{\beta}^{D}(z):=\max\left\{\left|g_i\left(z\right)-\beta_i\right|:i\in D\right\},\ \text{for } z\in\mathbb{R}^n,\ \beta\in\mathbb{R}^D. \tag{3.1}$$

Theorem 3.1. *Assume that* ENC *is satisfied at* $\left((\overline{c},\overline{b}),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}^{*}\right)$ *and let* $D\in\mathcal{T}_{\overline{b}}(\overline{x})$. *Then,* $\mathcal{F}_D^{*}$ *is strongly Lipschitz stable at* $\left((\overline{c},\overline{b}_D),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}_D^{*}\right)$ *and its Lipschitz modulus is given by*

$$\operatorname{lip}\mathcal{F}_D^{*}(\overline{c},\overline{b}_D)=\limsup_{\substack{(z,\beta)\to(\overline{x},\overline{b}_D)\\ h_{\beta}^{D}(z)>0}}\left(d_{*}\left(0_n,\widehat{\partial}h_{\beta}^{D}(z)\right)\right)^{-1}.$$

The following corollary is used in [5] for approaching the case of linear programs (see Corollary 3.7 below). Recall that in this case ENC is equivalent to the strong Lipschitz stability of the optimal set mapping. Although the result comes straightforwardly from Lemma 2.3 and [10, Exa. 1.1], here we provide a self-contained proof as an illustration of Theorem 3.1.

Corollary 3.2. *Let* $f\equiv 0$ *and let* $g_i\equiv\langle a_i,\cdot\rangle$, $i=1,\ldots,m$, *be given linear functions, and assume that* ENC *holds at* $\left((\overline{c},\overline{b}),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}^{*}\right)$. *Then, for every* $D\in\mathcal{T}_{\overline{b}}(\overline{x})$, $\mathcal{F}_D^{*}$ *is strongly Lipschitz stable at* $\left((\overline{c},\overline{b}_D),\overline{x}\right)$ *and*

$$\operatorname{lip}\mathcal{F}_D^{*}(\overline{c},\overline{b}_D)=\left\|A_D^{-1}\right\|,$$

where A_D *is the matrix whose rows are* a_i, $i\in D$.

Remark 3.3. *Here, under ENC, Lemma 2.3 entails the following equivalence:* $((\overline{c},\beta),x)\in \operatorname{gph}\left(\mathcal{F}_D^{*}\right)$ *for* β *close enough to* $\overline{b}_D$ *if and only if* $A_Dx=\beta$. *Then [10, Exa. 1.1] can be applied.*

Proof. Fix $D \in \mathcal{T}_{\overline{b}}(\overline{x})$. We shall use the expression

$$\left\| A_D^{-1} \right\| = \left(\min_{\|\lambda\|_1 = 1} \|A_D^{\intercal}\lambda\|_* \right)^{-1}, \tag{3.2}$$

which can be found in [5, Sect. 4] and follows from [4, Cor. 3.2] ($A_D^{\intercal}$ is the transpose of A_D). Keeping the notation of the previous theorem, we have

$$h_\beta^D(z) := \max\left\{ |\langle a_i, z\rangle - \beta_i| : i \in D \right\}, \text{ for } z \in \mathbb{R}^n \text{and } \beta \in \mathbb{R}^D,$$

and, so, h_β^D is itself a convex function. By applying Valadier's formula (see for instance [12, Cor. VI.4.3.2]) we obtain

$$\widehat{\partial} h_\beta^D(z) = \partial h_\beta^D(z) = \mathrm{co}\{a_i,\ i \in I_\beta(z); -a_i,\ i \in J_\beta(z)\},$$

where

$$I_\beta(z) := \{i \in D \mid h_\beta^D(z) = \langle a_i, z\rangle - \beta_i\},\ J_\beta(z) := \{i \in D \mid h_\beta^D(z) = \beta_i - \langle a_i, z\rangle\}.$$

Thus,

$$\widehat{\partial} h_\beta^D(z) \subset \mathrm{co}\{\pm a_i,\ i \in D\} = \left\{ A_D^{\intercal}\lambda : \|\lambda\|_1 = 1 \right\},$$

hence

$$d_*\left(0_n, \widehat{\partial} h_\beta^D(z)\right) \geq \min_{\|\lambda\|_1 = 1} \|A_D^{\intercal}\lambda\|_*,$$

and then we conclude $\mathrm{lip}\, \mathcal{F}_D^*(\overline{c}, \overline{b}_D) \leq \left\| A_D^{-1} \right\|$ by applying Theorem 3.1 and (3.2).

Conversely, for any given $D \in \mathcal{T}_{\overline{b}}(\overline{x})$, let $I \subset D$, and $\bar{\lambda}_1, \ldots, \bar{\lambda}_n \geq 0$ be such that $\bar{\lambda}_1 + \cdots + \bar{\lambda}_n = 1$ and

$$\min_{\|\lambda\|_1 = 1} \|A_D^{\intercal}\lambda\|_* = \left\| \sum_{i \in I} \bar{\lambda}_i a_i - \sum_{i \in D \setminus I} \bar{\lambda}_i a_i \right\|_*.$$

Then, considering the vector $\beta^r \in \mathbb{R}^D$ defined for $r = 1, 2, \ldots$ by

$$\beta_i^r = \overline{b}_i - \frac{1}{r} \text{ if } i \in I \text{ and } \beta_i^r = \overline{b}_i + \frac{1}{r} \text{ if } i \in D \setminus I,$$

we obtain, taking Theorem 3.1 into account,

$$\begin{aligned} \mathrm{lip}\, \mathcal{F}_D^*(\overline{c}, \overline{b}_D) &\geq \limsup_{r \to \infty} d_*(0_n, \partial h_{\beta^r}^D(\overline{x}))^{-1} \\ &= d_*(0_n, \mathrm{co}\{a_i,\ i \in I;\ -a_i,\ i \in D \setminus I\})^{-1} \\ &= \left(\min_{\|\lambda\|_1 = 1} \|A_D^{\intercal}\lambda\|_* \right)^{-1} = \left\| A_D^{-1} \right\|. \end{aligned}$$

□

Theorem 3.6 below constitutes the main original contribution of the present paper. In the proof we shall apply the following two lemmas. In the first one we appeal to the notation

$$\mathcal{D}(x) := \{D \subset \{1, ..., m\} : |D| = n,\ (\partial f(x) + \overline{c}) \cap \mathrm{cone}\left(\cup_{i \in D}\left(-\partial g_i(x)\right)\right) \neq \varnothing\}$$

for $x \in \mathbb{R}^n$, and according to (2.3) we define, for $b \in \mathbb{R}^m$ and $x \in \mathcal{F}(b)$

$$\mathcal{T}_b(x) := \{D \in \mathcal{D}(x) : D \subset T_b(x)\}.$$

Roughly speaking, $\mathcal{T}_b(x)$ refers to sets of indices D (if any) associated with KKT conditions at point x for problem $P(\overline{c}, b)$, whereas $\mathcal{D}(x)$ provides the part of KKT conditions which only depend on x, not referring to any b. In this way, if $x \in \mathcal{F}(b)$ and some $D \in \mathcal{D}(x)$ is included in $T_b(x)$ for some b, then $x \in \mathcal{F}^*(\overline{c}, b)$. The following lemma is an immediate consequence of [7, Thm. 4].

Lemma 3.4. *Assume that* ENC *is satisfied at* $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$. *Then, there exist neighborhoods U and V of $\overline{x}$ and $\overline{b}$, respectively, verifying*

$$\varnothing \neq \mathcal{T}_b(x) \subset \mathcal{T}_{\overline{b}}(\overline{x}) \subset \mathcal{D}(x) \quad \text{for all } x \in U \text{ and all } b \in V. \tag{3.3}$$

In the following paragraphs let V be an open convex subset of $\mathbb{R}^m$. Recall that a function $\varphi : V \to \mathbb{R}^n$ is said to be Lipschitz continuous on V with rank $\lambda \geq 0$, denoted by $\varphi \in Lip(\lambda, V)$, if

$$\|\varphi(b) - \varphi(b')\| \leq \lambda \|b - b'\|_\infty \text{ for all } b, b' \in V.$$

We say that φ is calm on V with rank $\lambda \geq 0$, denoted by $\varphi \in Clm(\lambda, V)$, if for any $b \in V$ there exists a neighborhood of b, $W \subset V$, such that

$$\|\varphi(b) - \varphi(b')\| \leq \lambda \|b - b'\|_\infty \text{ for all } b' \in W.$$

Obviously, $\varphi \in Clm(\lambda, V)$ implies the continuity of φ on V. The following lemma is a particular case of [19, Thm. 2.1] (see also [5, Lem. 3]).

Lemma 3.5. *With the previous notation, if $\varphi \in Clm(\lambda, V)$ then $\varphi \in Lip(\lambda, V)$.*

Now we establish the aimed relationship between the Lipschitz modulus of the whole problem $P\left(\overline{c}, \overline{b}\right)$ -see (1.1)- and the moduli of subproblems $P_D\left(\overline{c}, \overline{b}_D\right)$ with $D \in \mathcal{T}_{\overline{b}}(\overline{x})$. Recall that the latter moduli are given by Theorem 3.1.

Theorem 3.6. *Assume that* ENC *holds at* $\left((\overline{c}, \overline{b}), \overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$. *Then, we have*

$$\operatorname{lip} \mathcal{F}^*(\overline{c}, \overline{b}) = \max_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \operatorname{lip} \mathcal{F}_D^*(\overline{c}, \overline{b}_D). \tag{3.4}$$

Proof. According to the negligibility of the perturbations of $\overline{c}$ pointed out in (1.4), we will refer to mappings $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{G}}$ $(:= \widetilde{\mathcal{F}}^{-1})$, $\widetilde{\mathcal{F}}_D := \mathcal{F}_D^*(\overline{c}, \cdot)$, and $\widetilde{\mathcal{G}}_D := \widetilde{\mathcal{F}}_D^{-1}$, with $D \in \mathcal{T}_{\overline{b}}(\overline{x})$. Recall that $\operatorname{lip} \mathcal{F}^*(\overline{c}, \overline{b}) = \operatorname{lip} \widetilde{\mathcal{F}}(\overline{b})$ and $\operatorname{lip} \mathcal{F}_D^*(\overline{c}, \overline{b}_D) = \operatorname{lip} \widetilde{\mathcal{F}}_D(\overline{b}_D)$ (see (1.4)).

First we establish $\geq$ in (3.4). Fix any $\lambda > \operatorname{lip} \widetilde{\mathcal{F}}(\overline{b})$ and let us see that $\operatorname{lip} \widetilde{\mathcal{F}}_D(\overline{b}_D) \leq \lambda$ for all $D \in \mathcal{T}_{\overline{b}}(\overline{x})$. Appealing to the metric regularity terminology, together with (2.2), consider neighborhoods U and V of $\overline{x}$ and $\overline{b}$, respectively, such that

$$d\left(x, \widetilde{\mathcal{F}}(b)\right) \leq \lambda d_\infty\left(b, \widetilde{\mathcal{G}}(x)\right) = f_b(x), \text{ for all } x \in U \text{ and all } b \in V. \tag{3.5}$$

It is not restrictive to assume that U and V are small enough to ensure that (2.5) and (3.3) hold, $\widetilde{\mathcal{F}}$ is single-valued and Lipschitz continuous on V, and ENC is satisfied at any $((\overline{c}, b), x)$ with $(b, x) \in (V \times U) \cap \operatorname{gph}(\widetilde{\mathcal{F}})$ (the last assumption is due to Lemma 2.3).

Recall that the fulfillment of ENC at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ ensures that this property also holds at $\left(\left(\overline{c}, \overline{b}_D\right), \overline{x}\right)$, associated with subproblem $P_D\left(\overline{c}, \overline{b}_D\right)$, for all $D \in \mathcal{T}_{\overline{b}}(\overline{x})$ -see our comments at the beginning of this section-. In particular, for all $D \in \mathcal{T}_{\overline{b}}(\overline{x})$, $\widetilde{\mathcal{F}}_D$ is single-valued in some neighborhood of $\overline{b}_D$ and $\widetilde{\mathcal{F}}_D\left(\overline{b}_D\right) = \{\overline{x}\}$.

Now, if we fix arbitrarily $D_0 \in \mathcal{T}_{\overline{b}}(\overline{x})$ we can see that $\operatorname{lip} \widetilde{\mathcal{F}}_{D_0}(\overline{b}_{D_0}) \leq \lambda$. Specifically, we are going to find certain neighborhoods of $\overline{x}$ and $\overline{b}_{D_0}$, say $U_0 \subset \mathbb{R}^n$ and $V_0 \subset \mathbb{R}^{D_0}(\equiv \mathbb{R}^n)$ such that

$$d\left(z, \widetilde{\mathcal{F}}_{D_0}(\beta)\right) \leq \lambda d_\infty\left(\beta, \widetilde{\mathcal{G}}_{D_0}(z)\right) \text{ for all } z \in U_0 \text{ and all } \beta \in V_0.$$

Take $\varepsilon > 0$ such that $\left\{b \in \mathbb{R}^m : \left\|b - \overline{b}\right\|_\infty < \varepsilon\right\} \subset V$. The continuity of the g_i's and the ENC at $\left(\left(\overline{c}, \overline{b}_{D_0}\right), \overline{x}\right)$ (together with its consequences commented above) ensure the existence of neighborhoods $U_0 \subset \mathbb{R}^n$ and $V_0 \subset \mathbb{R}^{D_0}$ of $\overline{x}$ and $\overline{b}_{D_0}$, respectively, such that

$$U_0 \subset U \tag{3.6}$$

$$\max_{i=1,\cdots,m} |g_i(x) - g_i(\bar{x})| < \varepsilon \text{ for all } x \in U_0, \tag{3.7}$$

$$\widetilde{\mathcal{F}}_{D_0} \text{ is single-valued on } V_0 \tag{3.8}$$

$$\widetilde{\mathcal{F}}_{D_0}(V_0) \subset U_0, \tag{3.9}$$

$$d_\infty(\beta, \widetilde{\mathcal{G}}_{D_0}(z)) = h_\beta^{D_0}(z) \text{ for all } z \in U_0, \text{ and all } \beta \in V_{D_0}, \tag{3.10}$$

where $h_\beta^{D_0}(z)$ is defined as in (3.1). In relation to (3.10), observe that, as a consequence of the ENC assumption, $\widetilde{\mathcal{G}}_{D_0}(z)) = \{(g_i(z),\ i \in D_0)\}$ for z close enough to $\overline{x}$. Now let us see that the aimed Lipschitzian inequality holds for these neighborhoods U_0 and V_0.

Pick any $z \in U_0$ and $\beta \in V_0$. Let y be the only point of $\widetilde{\mathcal{F}}_{D_0}(\beta)$, which entails $g_i(y) = \beta_i$ for all $i \in D_0$, and define $b \in \mathbb{R}^m$ as follows:

$$b_i = \begin{cases} \beta_i\ (= g_i(y)), & \text{if } i \in D_0, \\ \max\{\overline{b}_i, g_i(z), g_i(y)\}, & \text{if } i \notin D_0. \end{cases} \tag{3.11}$$

Let us see that $b \in V$. For $i \in D_0$ one has $\left|b_i - \overline{b}_i\right| = |g_i(y) - g_i(\overline{x})| < \varepsilon$, according to (3.7) and (3.9). For $i \notin D_0$ one has, according to (3.11), and denoting by $[\alpha]_+ = \max\{\alpha, 0\}$ the positive part of $\alpha \in \mathbb{R}$,

$$\left|b_i - \overline{b}_i\right| = \max\left\{\left[g_i(z) - \overline{b}_i\right]_+, \left[g_i(y) - \overline{b}_i\right]_+\right\}.$$

Recalling that $z \in U_0$ we have

$$\left[g_i(z) - \overline{b}_i\right]_+ \leq [g_i(z) - g_i(\overline{x})]_+ + \left[g_i(\overline{x}) - \overline{b}_i\right]_+ \leq |g_i(z) - g_i(\overline{x})| + 0 < \varepsilon,$$

and a completely analogous inequality holds for y instead of z, because $y \in U_0$ from (3.9). Thus, $\left\|b - \overline{b}\right\|_\infty < \varepsilon$ and, so, $b \in V$.

Now we show that y is the only point of $\widetilde{\mathcal{F}}(b)$ (recall that $\widetilde{\mathcal{F}}$ is single-valued in V). In fact, (3.11) entails $y \in \mathcal{F}(b)$ and $D_0 \subset T_b(y)$, and we have $D_0 \in \mathcal{D}(y)$ because of (3.3). Hence, $D_0 \in \mathcal{T}_b(y)$, providing KKT conditions, and so $\widetilde{\mathcal{F}}(b) = \{y\}$. Then, we have (see comments after the formula):

$$\begin{array}{rcl} d(z,\widetilde{\mathcal{F}}_{D_0}(\beta)) & = & d(z,\widetilde{\mathcal{F}}\,(b)) \leq \lambda d_\infty\left(b,\widetilde{\mathcal{G}}\,(z)\right) = \lambda f_b\,(z) \\ & = & \lambda \min\limits_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} f_b^D\,(z) \leq \lambda f_b^{D_0}\,(z) \\ & = & \lambda \max\{|g_i\,(z)-b_i|\,,\; i\in D_0;\; g_i\,(z)-b_i,\; i\notin D_0\} \\ & = & \lambda \max\{|g_i\,(z)-g_i\,(y)|\,,\; i\in D_0\} \\ & = & \lambda \max\{|g_i\,(z)-\beta_i|\,,\; i\in D_0\} \\ & = & \lambda h_\beta^{D_0}\,(z) = \lambda d_\infty(\beta,\widetilde{\mathcal{G}}_{D_0}(z)). \end{array}$$

In the first row we use the definitions of b and y, the facts that $b\in V$, $y\in U_0\subset U$, and (3.5); the second row comes from the fact that U and V have been chosen to verify (2.5); in the third we use (2.4); in the fourth, we appeal to the facts that $g_i\,(z)-b_i\leq 0$ for $i\notin D_0$ and $b_i=g_i\,(y)$ for $i\in D_0$, both coming from (3.11); in the fifth we use $g_i(y)=\beta_i$ for $i\in D_0$; and in the last row we appeal to (3.10). So, we conclude $\operatorname{lip}\widetilde{\mathcal{F}}_{D_0}(\overline{b}_{D_0})\leq\lambda$, and the proof of $\geq$ in (3.4) is complete.

Next we prove the converse inequality, i.e., $\operatorname{lip}\widetilde{\mathcal{F}}(\overline{b}) \leq \max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})}\operatorname{lip}\widetilde{\mathcal{F}}_D(\overline{b}_D)$. Take any $\mu>\max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})}\operatorname{lip}\widetilde{\mathcal{F}}_D(\overline{b}_D)$ and let us see that μ is a Lipschitz constant for $\widetilde{\mathcal{F}}$ in a certain open and convex neighborhood of $\overline{b}$, say V'. Specifically, we define $V':=\left\{b\in\mathbb{R}^m:\left\|b-\overline{b}\right\|_\infty<\eta\right\}$ such that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_D$ are single-valued in V' and V'_D, respectively, for all $D\in\mathcal{T}_{\overline{b}}(\overline{x})$, where $V'_D:=\left\{\beta\in\mathbb{R}^D:\left\|\beta-\overline{b}_D\right\|_\infty<\eta\right\}$, and in addition

$$\left\|\widetilde{\mathcal{F}}_D\,(\beta)-\widetilde{\mathcal{F}}_D\,(\beta')\right\|\leq\mu\left\|\beta-\beta'\right\|_\infty \text{ for all } \beta,\beta'\in V'_D \text{ and all } D\in\mathcal{T}_{\overline{b}}(\overline{x}).$$

Here, in order to avoid additional notation, we have identified $\widetilde{\mathcal{F}}_D\,(\beta)$ and $\widetilde{\mathcal{F}}_D\,(\beta')$ with their unique elements. The same will be done with $\widetilde{\mathcal{F}}$ in the following paragraphs. Let us see that $\widetilde{\mathcal{F}}\in Clm\,(\mu,V')$, and then Lemma 3.5 will ensure $\widetilde{\mathcal{F}}\in Lip\,(\mu,V')$, which itself entails $\operatorname{lip}\widetilde{\mathcal{F}}(\overline{b})\leq\mu$. This will conclude the proof.

For any given $b\in V'$ let $W\subset V'$ be a neighborhood of b such that, for all $b'\in W$ one has

$$\mathcal{T}_{b'}\left(\widetilde{\mathcal{F}}(b')\right)\subset\mathcal{T}_b\left(\widetilde{\mathcal{F}}(b)\right) \text{ for all } b'\in W.$$

(Such a neighborhood exists as an immediate consequence of Lemma 3.4.)

Then, for any $b'\in W$ and any $D\in\mathcal{T}_{b'}\left(\widetilde{\mathcal{F}}(b')\right)$ one has

$$\left\|\widetilde{\mathcal{F}}\,(b)-\widetilde{\mathcal{F}}\,(b')\right\|=\left\|\widetilde{\mathcal{F}}_D\,(b_D)-\widetilde{\mathcal{F}}_D\,(b'_D)\right\|\leq\mu\left\|b_D-b'_D\right\|_\infty\leq\mu\left\|b-b'\right\|_\infty.$$

Thus $\widetilde{\mathcal{F}}\in Clm\,(\mu,V')$, as we aimed to prove. □

In the particular case of linear problems of the form

$$\begin{array}{l} \operatorname{Inf}\langle c,x\rangle \\ \text{s.t. } \langle a_i,x\rangle\leq b_i,\;\; i=1,\ldots,m, \end{array} \tag{3.12}$$

we obtain the following corollary, in which we appeal to the notation of Corollary 3.2.

Corollary 3.7. [5, Cor. 2] *For the parametrized linear problem (3.12), assume that* ENC *holds at* $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right) \in \operatorname{gph}\mathcal{F}^*$. *Then one has*

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} \left\|A_D^{-1}\right\|$$

where

$$\left\|A_D^{-1}\right\| := \max_{\|y\|_\infty \leq \infty} \left\|A_D^{-1}y\right\| = \left(\min_{\|\lambda\|_1=1} \left\|A_D^\top\lambda\right\|_*\right)^{-1}.$$

4 Conclusions

In this section we point out the main advantages of expression (3.4) in Theorem 3.6 (see also Theorem 3.1) with respect to formula (2.1). Putting all together we have

$$\begin{aligned}\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) &= \limsup_{\substack{(z,b)\to(\overline{x},\overline{b})\\ f_b(z)>0}} \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_b(z)=f_b^D(z)}} \left(d_*\left(0_n,\widehat{\partial}f_b^D(z)\right)\right)^{-1}\\ &= \max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} \limsup_{\substack{(z,\beta)\to(\overline{x},\overline{b}_D)\\ h_\beta^D(z)>0}} \left(d_*\left(0_n,\widehat{\partial}h_\beta^D(z)\right)\right)^{-1}.\end{aligned}$$

In (2.1) (first row), for any pair (z,b) close to $\left(\overline{x},\overline{b}\right)$ we have to find $D\in\mathcal{T}_{\overline{b}}(\overline{x})$ such that $f_b(z) = f_b^D(z)$. This D depends on (z,b). In contrast, (3.4) (second row) each $D\in\mathcal{T}_{\overline{b}}(\overline{x})$ is dealt with separately, and for each D only constraints with indices in D are considered. Moreover, functions h_β^D are easier to handle than f_b^D (the latter considers all indices, not only the ones in D). In such a way, d.c. decompositions of h_β^D is easier than the one of f_b^D, and so the expression of $\widehat{\partial}h_\beta^D(z)$ is also easier than the one of $\widehat{\partial}f_b^D(z)$ (the reader is addressed to [8] for details), and this constitutes a new advantage of our current approach. Of course, the present paper is highly based on [8], but we also should mention that the current approach does not extend to the semi-infinite case (infinitely many indices). In fact, the question of whether or not Corollary 3.7 extends to the semi-infinite case remains as an open problem, as pointed out at the end of [5, Sect. 5].

References

[1] D. Azé and J.-N. Corvellec, On the sensitivity analysis of Hoffman constants for systems of linear inequalities, *SIAM J. Optim.* 12 (2002) 913–927.

[2] D. Azé and J.-N. Corvellec, Characterizations of error bounds for lower semicontinuous functions on metric spaces, *ESAIM Control Optim. Calc. Var.* 10 (2004) 409–425.

[3] M. Bounkhel and L. Thibault, Directionally pseudo-Lipschitz set-valued mappings, *J. Math. Anal. Appl.* 266 (2002) 269–287.

[4] M.J. Cánovas, A.L. Dontchev, M.A. López and J. Parra, Metric regularity of semi-infinite constraint systems, *Math. Program.* 104 (2005) 329–346.

[5] M.J. Cánovas, F.J. Gómez-Senent and J. Parra, On the Lipschitz modulus of the argmin mapping in linear semi-infinite optimization, *Set-Valued Anal.*, published online: September 20, 2007.

[6] M.J. Cánovas, A. Hantoute, M.A. López and J. Parra, Lipschitz behavior of convex semi-infinite optimization problems: A variational approach, *J. Global Optim.* 41 (2008) 1–13.

[7] M.J. Cánovas, A. Hantoute, M.A. López and J. Parra, Stability of indices in KKT conditions an metric regularity in convex semi-infinite optimization, *J. Optim. Theory Appl.*, published online: April 30, 2008

[8] M.J. Cánovas, A. Hantoute, M.A. López and J. Parra, Lipschitz modulus in convex semi-infinite optimization via d.c. functions, *ESAIM Control Optim. Calc. Var.*, published online: July 19, 2008.

[9] M.J. Cánovas, D. Klatte, M.A. López and J. Parra, Metric regularity in convex semi-infinite optimization under canonical perturbations, *SIAM J. Optim.* 18 (2007) 717–732.

[10] A.L. Dontchev, A.S. Lewis and R.T. Rockafellar, The radius of metric regularity, *Trans. Amer. Math. Soc.* 355 (2003) 493–517.

[11] M.A. Goberna and M.A. López, *Linear Semi-Infinite Optimization*, John Wiley & Sons, Chichester (UK), 1998.

[12] J.-B. Hiriart-Urruty and C. Lemarèchal, *Convex analysis and minimization algorithms. I. Fundamentals.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 305. Springer-Verlag, Berlin, 1993.

[13] A.D. Ioffe, Metric regularity and subdifferential calculus, *Uspekhi Mat. Nauk* 55 (333) 103–162; English translation *Math. Surveys* 55 (2000) 501–558.

[14] A.D. Ioffe, On rubustness of the regularity property of maps, *Control Cybernet.* 32 (2003) 543–554.

[15] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications*, Kluwer Academic Publ., Dordrecht, 2002.

[16] D. Klatte and B. Kummer, Strong Lipschitz stability of stationary solutions for nonlinear programs and variational inequalities, *SIAM J. Optim.* 16 (2005) 96–119.

[17] D. Klatte and G. Thiere, A note of Lipschitz constants for solutions of linear inequalities and equations, *Linear Algebra Appl.* 244 (1996) 365–374.

[18] W. Li, The sharp Lipschitz constants for feasible and optimal solutions of a perturbed linear program, *Linear Algebra Appl.* 187 (1993) 15–40.

[19] W. Li, Sharp Lipschitz constants for basic optimal solutions and basic feasible solutions of linear programs, *SIAM J. Control Optim.* 32 (1994) 140–153.

[20] G. Nürnberger, Unicity in semi-infinite optimization, in *Parametric Optimization and Approximation*, B. Brosowski and F. Deutsch (eds.), Birkhäuser, Basel, 1985, pp. 231–246.

[21] S.M. Robinson, Bounds for error in the solution set of a perturbed linear program, *Linear Algebra Appl.* 6 (1973) 69–81.

[22] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

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