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THE ATTOUCH-WETS TOPOLOGY IN METRIC AND NORMED SPACES

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Abstract: We survey the basic facts about Attouch-Wets convergence, also called bounded Hausdorff convergence, and the associated Attouch-Wets topology for the nonempty closed subsets of a metric space as well as for the proper lower semicontinuous convex functions defined on a normed linear space X as identified with their convex epigraphs in $X \times \mathbb{R}$. Some new results and simpler proofs of old results are included. With our emphasis on topological results, this article complements [58].

Key words: *Attouch-Wets convergence, bounded Hausdorff convergence, bornological convergence, convex function, Fenchel conjugate*

Mathematics Subject Classification: *54B20; 46A55, 26A51*

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1 Introduction

The most well-known topology on the nonempty subsets $\mathscr{P}_0(X)$ of a metric space $\langle X, d\rangle$ is the *topology of Hausdorff distance* [14, 47, 49]. To describe Hausdorff distance in $\mathscr{P}_0(X)$, for each nonempty subset A and $\varepsilon > 0$, put

$$A^\varepsilon := \{x \in X : d(x, A) < \varepsilon\}.$$

A^ε is called the *ε-enlargement* or *ε-neighborhood* of A. Intuitively the Hausdorff distance H_d between A and B is the minimal amount we can enlarge both sets to contain the other:

$$H_d(A, B) := \inf \{\varepsilon > 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}.$$

Of course, if A were unbounded and B were bounded, the set of such ε over which the infimum is taken is empty and we logically obtain $H_d(A, B) = \infty$. But this can happen more generally, e.g., when A and B are two nonparallel lines in $\mathbb{R}^2$ with the Euclidean metric. Hausdorff distance so defined gives an extended real valued pseudometric on $\mathscr{P}_0(X)$ which becomes an extended real valued metric on the nonempty closed subsets $\mathscr{C}_0(X)$. As $H_d(A, B) = H_d(\text{cl}(A), \text{cl}(B))$, arguably there is no loss of generality in restricted our attention to closed subsets. The resulting metrizable topology mirrors characteristics of the underlying space in important ways : if $\langle X, d\rangle$ is complete or totally bounded or compact, then $\langle \mathscr{C}_0(X), H_d\rangle$ inherits the same property.

While Hausdorff distance works well for bounded sets, it is much too strong for applications to unbounded sets. In any reasonable hyperspace topology, one would like the sequence of lines $\langle L_n \rangle$ in $\mathbb{R}^2$ where each L_n has slope $\frac{1}{n}$ and passes through the origin to converge to the horizontal axis. With respect to this criterion, the H_d topology is just not reasonable.

The *Attouch-Wets topology*, also know as the *bounded Hausdorff topology*, is a weakening of the Hausdorff metric topology that seems to best retain its desirable properties. While this notion appears first in the Walkup-Wets isometry theorem [69] and was subsequently noticed by Mosco [55] in a landmark paper in which his eponymous convergence was introduced, it was first comprehensively studied by Attouch and Wets [5, 6] in the context of epigraphical analysis. Its inclusion in texts devoted to nonlinear and set-valued analysis is now standard [14, 54, 62]. Convergence of a net $\langle A_\lambda \rangle$ of nonempty sets to $A \in \mathscr{P}_0(X)$ requires that for each $\varepsilon > 0$, each truncation of A (resp. A_λ) by a prescribed bounded set eventually be contained in the ε-enlargement of A_λ (resp. A). Evidently for nets of uniformly bounded sets, this reduces to convergence in Hausdorff distance.

After setting forth some notation and basic terminology in Section 2, Section 3 lays out the basic facts about the Attouch-Wets topology for the nonempty closed subsets of a metric space. In Section 4, we look at the the so-called *epi-distance topology* on the proper lower semicontinuous convex functions $\Gamma(X)$ defined on a normed linear space X equipped with the Attouch-Wets topology it inherits from $\mathscr{C}_0(X \times \mathbb{R})$ where we associate each such function f with $\{(x, \alpha) : x \in X,\ \alpha \in \mathbb{R} \text{ and } \alpha \geq f(x)\}$. Most notably, this topology is stable with respect to duality : the Fenchel transform $f \mapsto f^*$ is bicontinuous.

2 Preliminaries

In the sequel, all metric spaces are assumed to contain at least two points. If $\langle X, d \rangle$ is a metric space and $A \subseteq X$, we denote its closure, interior and set of limit points by $\mathrm{cl}(A)$, $\mathrm{int}(A)$, and A', respectively. If $\langle Y, \rho \rangle$ is a second metric space, the box metric $d \times \rho$ will be understood on $X \times Y$. If $f : X \to Y$, its graph will be denoted by $\mathrm{Gr} f$. If f is an extended real-valued function on X, its *epigraph* is the following subset of $X \times \mathbb{R}$:

$$\mathrm{epi}\ f := \{(x, \alpha) : x \in X, \alpha \in \mathbb{R},\ \text{and}\ \alpha \geq f(x)\}.$$

Such a function f is called *lower semicontinuous* if its epigraph is a closed subset of the product. Equivalently, this means that $\forall \alpha \in \mathbb{R}$, each sublevel set at height α $\{x : f(x) \leq \alpha\}$ is a closed subset of X. As the epigraph of a supremum of a family of functions is the intersection of their epigraphs, an arbitrary supremum of lower semicontinuous functions is again lower semicontinuous.

The extended real valued function f is called *proper* if (i) $\forall x,\ f(x) > -\infty$, and (ii) $\exists x$ with $f(x) \in \mathbb{R}$. In geometric terms, such a function is proper if its epigraph is nonempty and contains no vertical line. For a proper function f, we call $\mathrm{dom}(f) := \{x \in X : f(x) < \infty\}$ its *effective domain.*

In the case that X is a normed linear space, $f : X \to [-\infty, \infty]$ is called *convex* if its epigraph is a convex subset of the product. We denote the proper lower semicontinuous convex functions on a normed linear space by $\Gamma(X)$. Two elements of $\Gamma(X)$ associated with a nonempty closed convex subset A of a normed linear space are its *distance function* $d(\cdot, A)$ which is actually Lipschitz continuous and its *indicator function* $\iota(\cdot, A)$ defined by

$$\iota(x, A) = \begin{cases} 0 & \text{if } x \in A \\ \infty & \text{otherwise} \end{cases}.$$

By a *bornology* [43] $\mathscr{B}$ on $\langle X, d\rangle$ we mean a cover of X by nonempty subsets that is closed under taking finite unions and that is hereditary : $B_1 \in \mathscr{B}$ and $B_2 \subseteq B_1$ with $B_2 \neq \emptyset$ imply $B_2 \in \mathscr{B}$. Evidently $\mathscr{P}_0(X)$ is the largest bornology and the nonempty finite subsets $\mathscr{F}(X)$ is the smallest. Intermediate in strength are the nonempty d-bounded subsets $\mathscr{B}_d(X)$ (the so-called *metric bornology* determined by d) and the nonempty subsets with compact closure. By a *base* for a bornology $\mathscr{B}$, we mean a subfamily that is cofinal with respect to inclusion. For example, a countable base for $\mathscr{B}_d(X)$ consists of all open balls with fixed center $x_0 \in X$ and integral radius. A celebrated result of S.-T. Hu [46] describes when a bornology $\mathscr{B}$ in a metrizable space X is $\mathscr{B}_d(X)$ for some metric d compatible with the topology: $\mathscr{B}$ has a countable base and $\forall B_1 \in \mathscr{B}\ \exists B_2 \in \mathscr{B}$ with $\mathrm{cl}(B_1) \subseteq \mathrm{int}(B_2)$. Note that the bornology of sets with compact closure forms a metric bornology if and only if X is locally compact and separable as first noted by Vaughan [65]. When the metrizable space is compact there is obviously just one metric bornology: $\mathscr{P}_0(X)$. Otherwise, there are uncountably many distinct ones [17].

If A and B are nonempty subsets of $\langle X, d\rangle$, we call

$$e_d(B, A) := \sup_{b\in B}\ d(b, A) = \inf\{\varepsilon > 0 : B \subseteq A^{\varepsilon}\}$$

the *excess* of B over A. Consistent with this formula, we agree that $e_d(\emptyset, A) = 0$. Dually $D_d(B, A) := \inf_{b\in B} d(b, A)$ is called the *gap* between B and A. Note that gap is a symmetric functional whereas excess is not, and that the Hausdorff distance between A and B is $H_d(A, B) = \max\ \{e_d(A, B), e_d(B, A)\}$. Note also that excess and gap are invariant under replacing sets by their closures and that both $e_d(\cdot, A)$ and $D_d(\cdot, A)$ extend distance $d(\cdot, A)$ from X to $\mathscr{P}_0(X)$ in different ways: for $x \in X$ and A nonempty, we have

$$e_d(\{x\}, A) = D_d(\{x\}, A) = d(x, A).$$

We record two important formulas, the first better known than the second (see, e.g., [14, pp. 29-31]).

Proposition 2.1. *Suppose A and B are nonempty subsets of $\langle X, d\rangle$. Then*

(1) $e_d(B, A) = sup_{x\in X}\ d(x, A) - d(x, B)$;

(2) $D_d(B, A) = inf_{x\in X}\ d(x, B) + d(x, A)$.

The first formula implies that the Hausdorff distance between A and B is the uniform distance between their distance functionals $d(\cdot, A)$ and $d(\cdot, B)$. A natural way to topologize the nonempty subsets of X is to identify each with its distance functional and then to equip such functionals with a topology of uniform convergence on some bornology $\mathscr{B}$ within X, a program initiated by B. Cornet [35], later taken up more generally in [28, 53]. There is no loss of generality in assuming for this purpose that the bornology has closed base, and it was discovered independently in [28, 53] that two such bornologies determine the same topology on $\{d(\cdot, A) : A \in \mathscr{C}_0(X)\}$ if and only if the families of closed sets within each have the same closures with respect to Hausdorff distance. The Hausdorff metric topology, where we effectively take for our bornology $\mathscr{P}_0(X)$, is the strongest such. The weakest such topology results when we use the bornology $\mathscr{F}(X)$, in which case we get the extremely well-known *Wijsman topology*, so named in recognition of its application by R. Wijsman in his seminal study of convex duality in finite dimensions [70]. It was first seriously studied in the setting of a general metric space by Levi and Lechicki [51].

3 The Attouch-Wets Topology for a Metric Space

We begin with a formal definition of Attouch-Wets convergence for nets of nonempty sets in a metric space $\langle X, d\rangle$ as described in the Introduction.

Definition 3.1. A net $\langle A_\lambda\rangle$ in $\mathscr{P}_0(X)$ is called *Attouch-Wets convergent* to a nonempty subset A of X if $\forall B \in \mathscr{B}_d(X)$ and $\forall \varepsilon > 0$, we have eventually both

$$A \cap B \subseteq A_\lambda^\varepsilon \text{ and } A_\lambda \cap B \subseteq A^\varepsilon.$$

When this occurs we will write $A = \mathrm{AW}_d - \lim A_\lambda$.

Evidently, Attouch-Wets convergence of $\langle A_\lambda\rangle$ to A can be expressed in terms of excess: for each $B \in \mathscr{B}_d(X)$, both $\lim_\lambda e_d(A \cap B, A_\lambda) = 0$ and $\lim_\lambda e_d(A_\lambda \cap B, A) = 0$. From this perspective and paralleling convergence in Hausdorff distance, it is easy to see that $A = \mathrm{AW}_d - \lim A_\lambda$ if and only if $\mathrm{cl}(A) = \mathrm{AW}_d - \lim \mathrm{cl}(A_\lambda)$ because for nonempty sets E and F, we have for all $B \in \mathscr{B}_d(X)$ and $\varepsilon > 0$,

$$e_d(E \cap B, F) \leq e_d(\mathrm{cl}(E) \cap B, \mathrm{cl}(F)) \leq e_d(E \cap B^\varepsilon, F).$$

One can also see easily see that if $\langle x_\lambda\rangle$ is a net in X then $\lim\ d(x, x_\lambda) = 0$ if and only if $\{x\} = \mathrm{AW}_d - \lim \{x_\lambda\}$. This means that Attouch-Wets convergence agrees with d-metric convergence for nets of points. In Definition 3.1, one could have replaced the bornology $\mathscr{B}_d(X)$ by an arbitrary bornology of subsets (see, e.g., [22, 23, 28, 31, 40, 52]). This more encompassing notion of convergence, now called *bornological convergence*, will not be considered further in the current survey.

Attouch-Wets convergence on $\mathscr{P}_0(X)$ so defined is compatible with a pseudometrizable topology on $\mathscr{P}_0(X)$, as it is obviously compatible with a uniformity $\sum_d$ having as a countable base all sets of the form

$$\textstyle\sum_d(n) := \{(A_1, A_2) : A_1 \cap B_n \subseteq A_2^{\frac{1}{n}} \text{ and } A_2 \cap B_n \subseteq A_1^{\frac{1}{n}}\}$$

where $n \in \mathbb{N}$ and $\{B_1, B_2, B_3, \ldots\}$ is an increasing cofinal family of bounded sets with respect to set inclusion (the uniformity is independent of the particular choice of the B_n). If we look at the trace of this uniformity on $\mathscr{C}_0(X)$, it is separated so that the pseudometrizable topology becomes Hausdorff and thus metrizable. We will confine our attention to the metrizable Attouch-Wets topology τ_{AW_d} on $\mathscr{C}_0(X)$ in this section, and state convergence results for sequences of nonempty closed sets as they determine the topology.

The standard metric for this metrizable hyperspace arises from a different uniformity - an often stronger uniformity - that is indicated by the following result, discovered in normed spaces by Azé and Penot [7] and separately by the author [11]. For a proof in metric spaces, the reader may consult [14, Prop. 3.1.6]

Theorem 3.2. *Let $A, A_1, A_2, A_3, \ldots$ be a sequence of nonempty closed sets in a metric space $\langle X, d\rangle$. Then $A = AW_d - lim\ A_n$ if and only if $\langle d(\cdot, A_n)\rangle$ converges to $d(\cdot, A)$ uniformly on bounded subsets of X. Thus, the mapping $A \mapsto d(\cdot, A)$ is an embedding of $\langle \mathscr{C}_0(X), \tau_{AW_d}\rangle$ into the metrizable locally convex space of continuous real functions that are bounded on bounded sets of X, equipped with the topology of uniform convergence on bounded sets.*

Thus, the Attouch-Wets topology fits within the program of Cornet even though he did not consider it explicitly. With this in mind, another compatible uniformity $\diamondsuit_d$ on $\mathscr{C}_0(X)$ (or on $\mathscr{P}_0(X)$) has as a countable base all sets of the form

$$\diamondsuit_d(n) := \{(A_1, A_2) : \sup_{x \in B_n} |d(x, A_1) - d(x, A_2)| < \frac{1}{n}\}$$

where again $\{B_1, B_2, B_3, \ldots\}$ is an increasing cofinal family of bounded sets with respect to set inclusion. The natural metric corresponding to this function space uniformity on $\mathscr{C}_0(X)$ is given by

$$\rho_{AW_d}(A_1, A_2) := \sum_{n=1}^{\infty} 2^{-n} \min\{1,\ \sup\ \{|d(x, A_1) - d(x, A_2)| : x \in B_n\}\}.$$

The next fact about this metric was identified by Attouch, Lucchetti and Wets [3].

Theorem 3.3. *Let $\langle X, d\rangle$ be a complete metric space. Then the metric space $\langle \mathscr{C}_0(X), \rho_{AW_d}\rangle$ is a complete metric space.*

Showing that the limit of a ρ_{AW_d}-Cauchy sequence of distance functionals exists as a bounded continuous function is standard; what takes a little work is showing that that the limit is itself a distance function for a nonempty closed set. We do not know whether or not there is a weaker topology on the continuous real functions that are bounded on bounded sets in which the distance functionals are closed.

It is easy to verify that for each n we have $\diamondsuit_d(n) \subseteq \sum_d(n)$ which makes the second uniformity stronger than the first. That it can be properly stronger was shown in [19]. We also do not know of a useful metric whose uniformity is $\sum_d$.

Consistent with our notation for the Attouch-Wets topology on $\mathscr{C}_0(X)$, we now write τ_{H_d} and τ_{W_d} for the Hausdorff metric and Wijsman topologies determined by d. Viewed from the perspective of the Cornet program we have $\tau_{W_d} \subseteq \tau_{AW_d} \subseteq \tau_{H_d}$. Coincidence of τ_{AW_d} with the other two is provided by the next two results, the first of which is totally obvious.

Theorem 3.4. *Let $\langle X, d\rangle$ be a metric space. Then $\tau_{H_d} = \tau_{AW_d}$ if and only if X is bounded.*

Proof. Sufficiency comes the fact that $X \in \mathscr{B}_d(X)$. For necessity, if $x_0 \in X$ and $\langle x_n\rangle$ is a sequence with $\lim_{n\to\infty} d(x_n, x_0) = \infty$, then $\{x_0\} = \mathrm{AW}_d - \lim\ \{x_0, x_n\}$. □

A proof of the next result can be found in [14].

Theorem 3.5. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:*

(1) $\tau_{W_d} = \tau_{AW_d}$ *on* $\mathscr{C}_0(X)$*;*

(2) τ_{AW_d} *on* $\mathscr{C}_0(X)$ *is second countable;*

(3) *Each bounded subset of $\langle X, d\rangle$ is totally bounded.*

Recall that a metrizable space is called *Polish* if it is second countable and has a compatible complete metric.

Corollary 3.6. *Let $\langle X, d\rangle$ be metric space. Then $\langle \mathscr{C}_0(X), \tau_{AW_d}\rangle$ is Polish if and only if X is Polish and each bounded subset of $\langle X, d\rangle$ is totally bounded.*

Proof. As a result of the important theorem of Costantini regarding Polishness of the Wijsman topology for a Polish space [36], if $\langle X, d\rangle$ is completely metrizable and each bounded subset of $\langle X, d\rangle$ is totally bounded, then $\langle \mathscr{C}_0(X), \tau_{AW_d}\rangle$ is a Polish space because X is second countable. Conversely, if $\langle \mathscr{C}_0(X), \tau_{AW_d}\rangle$ is completely metrizable, then $\langle X, d\rangle$ being homeomorphic to the closed subhyperspace $\{\{x\} : x \in X\}$ is completely metrizable. Further, since the hyperspace is second countable, each bounded subset of $\langle X, d\rangle$ is totally bounded. □

An intricate result in the domain of infinite dimensional topology that bares on the last corollary has been recently obtained by R. Voytsitskyy [67], who has given necessary and sufficient conditions for $\langle \mathscr{C}_0(X), \tau_{AW_d}\rangle$ to be homeomorphic to the Polish space ℓ_2 (see also [9, 63, 68]).

Theorem 3.7. *Let $\langle X, d\rangle$ be a metric space. Then $\langle \mathscr{C}_0(X), \tau_{AW_d}\rangle$ is homeomorphic to ℓ_2 if and only if*

(1) *X is a Polish space;*

(2) *no bounded subset of X has locally compact complement;*

(3) *each closed and bounded subset of the completion of $\langle X, d\rangle$ is compact;*

(4) *the completion of $\langle X, d\rangle$ has no bounded component but is locally connected.*

Let X be a metrizable space and let $D(X)$ be the family of metrics on X that are compatible with its topology. If $d \in D(X)$ and $\rho \in D(X)$, then it is well known that $\tau_{H_\rho} \subseteq \tau_{H_d}$ if and only if the identity map id : $\langle X, d\rangle \to \langle X, \rho\rangle$ is uniformly continuous. This of course means that the metric uniformity on X induced by ρ is contained in the one determined by d. Sufficiency is trivial, while necessity is most easily established using the Efremovic Lemma [14, p. 92]. The analogous and more difficult question for Wijsman topolgies was resolved in [37]. A natural question to ask is this: given a metrizable topological space, what are necessary and sufficient conditions on metrics d and ρ compatible with the topology such that $\tau_{AW_\rho} \subseteq \tau_{AW_d}$? This question was answered by Beer and Di Concilio [19].

To state our result we first need a definition of independent interest.

Definition 3.8. Let B be a nonempty subset of a metric space $\langle X, d\rangle$ and let $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ be a function. We say f is *strongly uniformly continuous on* B if $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $d(x, w) < \delta$ and $\{x, w\} \cap B \neq \emptyset$, then $\rho(f(x), f(w)) < \varepsilon$.

Obviously, strong uniform convergence of f on B implies the restriction of f to B is uniformly continuous. Further, f is strongly uniformly continuous on a singleton set $\{x\}$ if and only if f is continuous at x in the usual sense. For f arbitrary, it is easy to show that $\mathscr{B}_f := \{B \subseteq X : f$ is strongly uniformly continuous on $B\}$ is hereditary and closed under finite unions. Thus $\mathscr{B}_f$ is a bornology if and only f is globally continuous. For f globally continuous, it is clear that $\mathscr{B}_f$ contains the bornology of nonempty subsets with compact closure. A number of intriguing characterisations of strong uniform continuity are presented in [24]. Here is one such characterization: f is strongly uniformly continuous on B if and only if its associated direct image map $\hat{f} : \langle \mathscr{P}_0(X), H_d\rangle \to \langle \mathscr{P}_0(Y), H_\rho\rangle$ is continuous at each nonempty subset of B.

We are now ready to answer our question.

Theorem 3.9. *Let X be a metrizable topological space with compatible metrics d and ρ. The following conditions are equivalent:*

(1) *$\tau_{AW_\rho} \subseteq \tau_{AW_d}$ on $\mathscr{C}_0(X)$;*

(2) *$\sum_\rho \subseteq \sum_d$ on $\mathscr{C}_0(X)$;*

(3) *$\mathscr{B}_\rho(X) \subseteq \mathscr{B}_d(X)$ and the identity map id : $\langle X, d\rangle \to \langle X, \rho\rangle$ is strongly uniformly continuous on each element of $\mathscr{B}_\rho(X)$;*

(4) *$\mathscr{B}_\rho(X) \subseteq \mathscr{B}_d(X)$ and if $f : X \to Y$ is ρ-strongly uniformly continuous on each element of $\mathscr{B}_d(X)$, then f is d-strongly uniformly continuous on each element of $\mathscr{B}_\rho(X)$.*

Given a metrizable space X and any $d \in D(X)$ it is clear from Theorem 3.9 that τ_{AW_d} is contained in the Attouch-Wets topology determined by the bounded uniformly equivalent metric $d' = \min\{d, 1\}$ which agrees with the Hausdorff metric topology. Thus the supremum of the Attouch-Wets topologies $\{\tau_{AW_d} : d \in D(X)\}$ coincides with $\sup\{\tau_{H_d} : d \in D(X)$ and d is bounded$\}$ which is no smaller than $\sup\{\tau_{H_d} : d \in D(X)\}$, because a Hausdorff metric topology τ_{H_d} for an unbounded metric d is unchanged by replacing d by $d' = \min\{d, 1\}$.

Theorem 3.10. *Let X be a metrizable topological space with compatible metrics $D(X)$. The following are equivalent:*

(a) *$\{\tau_{AW_d} : d \in D(X)\}$ has a largest member;*

(b) *$\{\tau_{H_d} : d \in D(X)\}$ has a largest member;*

(c) *the set of limit points X' of X is compact.*

Proof. The equivalence of (a) and (b) has already been argued. Now it is well-known [61] that (c) is equivalent to the existence of $d_0 \in D(X)$ such that whenever $\langle Y, \rho\rangle$ is a metric space and $f : \langle X, d_0\rangle \to \langle Y, \rho\rangle$ is continuous, then f is uniformly continuous. Thus if X' is compact, then $\forall d \in D(X)$, $\mathrm{id} : \langle X, d_0\rangle \to \langle X, d\rangle$ is uniformly continuous, so there is a largest metric uniformity and hence a largest Hausdorff metric topology. Conversely, if X' is not compact, let $d \in D(X)$ be arbitrary. Then there is a continuous function f defined on $\langle X, d\rangle$ with values in some metric space $\langle Y, \rho\rangle$ that is not uniformly continuous. Evidently, $d_1 \in D(X)$ defined by

$$d_1(x, w) = d(x, w) + \rho(f(x), f(w))$$

determines a properly larger uniformity than d, so $\tau_{H_d} \subset \tau_{H_{d_1}}$. □

Even if there is no largest hyperspace, it is possible to describe the supremum in general. It is the celebrated *locally finite topology* τ_{LF} as idenitified in [21]. To describe a base for the topology, given a family of nonempty subsets $\mathscr{V}$ of X, put

$$\mathscr{V}^- := \{A \in \mathscr{C}_0(X) : \ \forall V \in \mathscr{V},\ V \cap A \neq \emptyset\}.$$

Then a base for the locally finite topology on $\mathscr{C}_0(X)$ consists of all sets of the form $\mathscr{V}^- \cap \{A : A \subseteq W\}$ where $\mathscr{V}$ is a locally finite family of nonempty open subsets of X and W is a nonempty open subset of X.

A different but related question is this: given a metric bornology $\mathscr{B}$ for a metrizable space X - that is, a bornology satisfying the conditions of Hu's Theorem [46] listed in the introduction - what is the supremum of $\{\tau_{AW_d} : d \in D(X)$ and $\mathscr{B} = \mathscr{B}_d(X)\ \}$? The desired supremum topology, not surprisingly a variant of the locally finite topology, was identified in [27]. As to when there is a largest Attouch-Wets topology corresponding to a prescribed metric bornology, the reader may consult [20, 44]

While the infimum of $\{\tau_{H_d} : d \in D(X)\}$ has been identified by Costantini and Vitolo [38], the infimum of $\{\tau_{AW_d} : d \in D(X)\}$ does not seem to be known at this writing.

The topology τ of each Tychonoff space $\langle X, \tau\rangle$ can always be expressed as a weak topology induced by some family of real valued functions defined on X. For example the continuous real valued functions on X with values in $[0,1]$ induce the topology. One of the remarkable facts about Tychonoff hyperspace topologies is that they are often weak topologies determined by geometric real-valued set functionals, and the Attouch-Wets topology is no exception. As shown by Beer and Lucchetti [27], τ_{AW_d} is the weakest topology on $\mathscr{C}_0(X)$ such that for each $B \in \mathscr{B}_d(X)$, both

$$A \mapsto D_d(B,A)$$
$$A \mapsto e_d(B,A)$$

are continuous functions on $\mathscr{C}_0(X)$. Thus, the Attouch-Wets topology is the weak topology determined by all gap and excess functionals whose left argument runs over the nonempty d-bounded subsets of X.

We take this opportunity to provide a much simpler proof that does not involve splitting hyperspace topologies into their upper and lower halves, taking advantage of Theorem 3.2. We are obliged to argue using nets, as we don't know a priori that the weak topology is first countable. We call upon a general folk-theorem about real functions: if $\langle f_\lambda \rangle$ is a net of real valued functions defined on a set S that converges uniformly to a real-valued function f, then $\inf \{f(s) : s \in S\} = \lim_\lambda \inf \{f_\lambda(s) : s \in S\}$ and $\sup \{f(s) : s \in S\} = \lim_\lambda \sup \{f_\lambda(s) : s \in S\}$, whether or not the infimum/supremum is finite and without any continuity assumptions whatever.

Theorem 3.11. *Let $\langle X, d\rangle$ be a metric space. Then the Attouch-Wets topology τ_{AW_d} on $\mathscr{C}_0(X)$ is the weak topology determined by the family of functionals*

$$\{e_d(B,\cdot) : B \in \mathscr{B}_d(X)\} \ \cup \ \{D_d(B,\cdot) : B \in \mathscr{B}_d(X)\}.$$

Proof. Suppose $\langle A_\lambda \rangle$ in $\mathscr{C}_0(X)$ is Attouch-Wets convergent to a nonempty closed subset A. Fix $B \in \mathscr{B}_d(X)$. By Theorem 3.2, $\langle d(\cdot, A_\lambda)\rangle$ converges uniformly to $d(\cdot, A)$ on B. As a result, both $\lim_\lambda \sup_{b\in B} d(b, A_\lambda) = \sup_{b\in B} d(b,A)$ and $\lim_\lambda \inf_{b\in B} d(b, A_\lambda) = \inf_{b \in B} d(b, A)$. In view of the definitions of excess and gap, this proves that τ_{AW_d} is finer than the weak topology.

To show the reverse inclusion, fix B_0 nonempty and bounded and $\varepsilon > 0$. Assuming that for each $B \in \mathscr{B}_d(X)$, $e_d(B,A) = \lim_\lambda e_d(B, A_\lambda)$ and $D_d(B,A) = \lim_\lambda D_d(B, A_\lambda)$, we intend to show that

(i) eventually $A \cap B_0 \subseteq A_\lambda^\varepsilon$,

(ii) eventually $A_\lambda \cap B_0 \subseteq A^\varepsilon$.

For (i), if $A \cap B_0 = \emptyset$, there is nothing to show. Otherwise, $A \cap B_0 \neq \emptyset$ and $e_d(A\cap B_0, A) = 0$. Since $A \cap B_0$ is nonempty and bounded, we have

$$\lim_\lambda e_d(A \cap B_0, A_\lambda) = e_d(A\cap B_0, A) = 0.$$

Thus, given $\varepsilon > 0$, eventually $e_d(A \cap B_0, A_\lambda) < \varepsilon$. In particular, this means that eventually $A \cap B_0 \subseteq A_\lambda^\varepsilon$. For (ii), suppose to the contrary that frequently $A_\lambda \cap B_0 \nsubseteq A^\varepsilon$. There exists a cofinal set of indices Λ_0 in the underlying directed set for the net such that

$$\forall \lambda \in \Lambda_0 \ \exists x_\lambda \in B_0 \cap A_\lambda \text{ with } d(x_\lambda, A) \geq \varepsilon.$$

Put $B_1 := \{x_\lambda : \lambda \in \Lambda_0\}$, a nonempty bounded set. Frequently, $D_d(B_1, A_\lambda) = 0$ whereas $D_d(B_1, A) \geq \varepsilon$. This contradicts $D_d(B_1, A) = \lim_\lambda D_d(B_1, A_\lambda)$. $\square$

There is an enormous literature on graph convergence of continuous functions between metric spaces $\langle X, d\rangle$ and $\langle Y, \rho\rangle$, including many results involving the Attouch-Wets convergence of graphs with respect to the box metric $d\times\rho$ on $X\times Y$ (see, e.g., [19, 20, 32, 39, 44, 60]). It is completely obvious that uniform convergence on bounded subsets of X yields Attouch-Wets convergence of graphs. With respect to the converse, we can state the following result [19].

Theorem 3.12. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be a metric spaces. Suppose $f, f_1, f_2, f_3, \ldots$ is a sequence of continuous functions from X to Y such that f is uniformly continuous on bounded sets and $\forall B \in \mathscr{B}_d(X)$ $\exists k \in \mathbb{N}$ such that $\cup_{n\geq k} f_n(B)$ is bounded in Y. Then $\langle f_n\rangle$ converges uniformly to f on bounded subsets of X if and only if $Gr\ f = AW_{d\times\rho} - lim\ Gr\ f_n$.*

It is clear that the conditions of the Theorem 3.12 imply that the limit function in either sense is bounded on bounded subsets of X. It is not difficult to show that if $\mathscr{B}_d(X)$ has a base of connected sets, and if f is assumed both bounded and uniformly continuous when restricted to bounded sets, then we get the eventual uniform boundedness of $\{f_n : n \in \mathbb{N}\}$ on bounded sets for free. In particular, this is true in a normed linear space. Since convergence for continuous linear transformations in the operator norm is uniform convergence on bounded subsets, we get as a special case the following result of Penot and Zalinescu [58] (for earlier results on linear functionals, see also [18]).

Theorem 3.13. *Let X and Y be normed linear spaces and let $T, T_1, T_2, T_3, \ldots$ be a sequence of continuous linear transformations from X to Y. Then $\langle T_n\rangle$ converges to T in the operator norm topology if and only if $Gr\ T = AW_{d\times\rho} - lim\ Gr\ T_n$.*

As noted earlier, one can identify a lower semicontinuous extended real valued function with its closed epigraph in $X\times\mathbb{R}$, and so one can embed such functions in $\mathscr{C}_0(X\times\mathbb{R})$ equipped with the Attouch-Wets topology. This topology, often called the *epi-distance topology*, has become the topology of choice for "one-sided analysts". In the next section we provide the reader with some qualitative results for the space of proper convex lower semicontinuous functions equipped with the epi-distance topology. The cognoscenti and those more interested in applications and precise estimation are invited to consult the 2005 survey of Penot and Zalinescu [58] and the references therein. But additional applications continue to appear: most recently, Bauschke, Lucet and Trienis [10] have considered homotopies between convex functions with respect to the topology.

4 Attouch-Wets Convergence for Convex Functions

Let $\langle X, ||\cdot||_X\rangle$ be a real normed linear space with closed unit ball U_X and origin θ. In the dual space X^* of continuous linear functionals on X, we denote the closed unit ball by U_{X^*} and the origin by θ^*.

In this section we will restrict our attention to the epi-distance topology on $\Gamma(X)$. We will not subscript AW in the sequel with the understanding that the metric is the one induced by the natural norm for the space, whether the space be X, X^*, or a product such as $X\times\mathbb{R}$, $X^*\times\mathbb{R}$, or $X\times\mathbb{R}\times X^*$. The results we present here are in no way meant to be exhaustive; rather they are a selection of nontechnical yet representative results that at the same time expose the uninitiated to some of the basic constructs of convex analysis. For further information on these constructs, the reader may consult [14, 41, 45, 59, 71].

Our first result describes how convergence in the epi-distance topology can be explained in terms of the convergence of sublevel sets in certain cases [25].

Theorem 4.1. *Suppose X is a normed linear space and $f, f_1, f_2, \ldots$ is a sequence in $\Gamma(X)$.*

(1) *If epi $f = AW - lim$ epi f_n, then $\forall \alpha > inf\ \{f(x) : x \in X\}$, we have $f^{-1}(-\infty, \alpha] = AW - lim\ f_n^{-1}(-\infty, \alpha]$.*

(2) *If $inf\ \{f(x) : x \in X\} = lim_{n\to\infty} \inf\ \{f_n(x) : x \in X\}$, then the converse of (1) holds.*

It is easy show that without convexity, we always have $\inf\ \{f(x) : x \in X\} \geq \limsup_{n\to\infty} \inf\ \{f_n(x) : x \in X\}$, and we note that (2) holds without the convexity assumption also.

It is obvious that for a sequence of closed convex sets $A, A_1, A_2, \ldots$, we have $A = \mathrm{AW} - \lim A_n$ if and only if epi $\iota(\cdot, A) = \mathrm{AW}-\lim$ epi $\iota(\cdot, A_n)$. One way to obtain the equivalence of these conditions with epi $d(\cdot, A) = \mathrm{AW} - \lim$ epi $d(\cdot, A_n)$ is to apply the following attractive result of Borwein and Vanderwerff [31] linking uniform convergence on bounded sets to a real-valued limit with the weaker Attouch-Wets convergence of epigraphs.

Theorem 4.2. *Let $f \in \Gamma(X)$ be real valued. The following conditions are equivalent:*

(1) *f is bounded on bounded sets;*

(2) *whenever $\langle f_n \rangle$ is a sequence in $\Gamma(X)$ convergent to f in the epi-distance topology, then $\langle f_n \rangle$ converges uniformly to f on bounded sets.*

One can try to impose standard function operations on $\Gamma(X)$, or alternatively one can perform geometric set operations on epigraphs. For example, we can add two elements of $\Gamma(X)$ using ordinary functional addition to get another, provided the effective domains of the functions overlap. With respect to epigrahical convergence, the best result available is the following [57, 58]:

Theorem 4.3. *Suppose $f, f_1, f_2, f_3, \ldots$ and $g, g_1, g_2, g_3, \ldots$ are sequences in $\Gamma(X)$ with epi $f = AW - lim$ epi f_n and epi $g = AW - lim$ epi g_n. Suppose $X = \mathbb{R}_+(dom(f) - dom(g))$. Then epi $f + g = AW - lim$ epi $f_n + g_n$*

Notice that the result is valid if f is continuous at some point of $\mathrm{dom}(g)$, as observed in [26].

One can also add two functions in $\Gamma(X)$ by taking the "vertical closure" of the vector sum of their epigraphs. This operation $\Box$, called the *infimal convolution* or the *epi-sum* in the literature, is defined formally by

$$(f \Box g)(x) := \inf\ \{f(w) + g(x - w) : w \in X\} \quad (x \in X).$$

As important examples, note that

$$\iota(\cdot, A) \Box \iota(\cdot, B) = \iota(\cdot, A + B),$$

and

$$\iota(\cdot, A) \Box ||\cdot||_X = d(\cdot, A).$$

But here too there can be pathology: if y is a nonzero continuous linear functional on X, then epi $y \Box - y = X \times \mathbb{R}$. Even if both f and g are lower bounded, $f \Box g$ need not be lower semicontinuous: let $y \in X^*$ be a norm one functional that is not norm achieving on U_X, and take the epi-sum of the indicator function of $\{x : < y, x >= 1\}$ with $\iota(\cdot, U_X)$.

A standard tool of convex analysis is to form *regularizations* of elements of $\Gamma(X)$ by taking epi-sums with so-called smoothing kernels. For example one obtains Lipschitz regularizations using $\alpha|| \cdot ||$ for large enough values of α. Regularizations by kernels of the form $\alpha|| \cdot ||^2$ are called *quadratic regularizations* or *Moreau-Yosida regularizations* [1, 5] in the literature.

One of the most important facts about elements of $\Gamma(X)$ is that each is the supremum of the continuous affine functionals that it majorizes (see, e.g. [41, p. 114]). In terms of epigraphs, if $f \in \Gamma(X)$ then

$$\text{epi } f = \bigcap \{\text{epi } g : g \text{ is continuous and affine and epi } f \subseteq \text{epi } g\}.$$

Now each continuous affine functional g on X depends on two parameters: an element y of X^* and a scalar β where we may write

$$g(x) = g_{(y,\beta)}(x) = < y, x > - \beta \quad (x \in X).$$

The set of ordered pairs $\{(y, \beta) : g_{(y,\beta)} \leq f\}$ forms the epigraph of a proper convex function defined on X^* called the *Fenchel conjugate* f^* of f. Formally

$$f^*(y) = \sup_{x \in X} \ < y, x > - f(x) \quad (y \in X^*).$$

As an example, if A is a nonempty closed convex subset of X, then

$$\iota^*(\cdot, A)(y) = \sup_{x \in A} < y, x >$$

. The conjugate of $\iota(\cdot, A)$ is called the *support functional* for the set A.

Not only is f^* lower semicontinuous - it is weak* lower semicontinuous, i.e., epi f^* is closed in $X^* \times \mathbb{R}$ where X^* is equipped with the weak* topology. Another important point is that for $f \in \Gamma(X)$ we have $(f^*)^* = f$, where we are restricting the second conjugate from X^{**} to the natural embedding of X within [45, p. 46]. Two easily verified formulae that we shall use in the sequel are

$$(1) \qquad (f \,\Box\, g)^* = f^* + g^* \qquad f, g \text{ in } \Gamma(X),$$

and

$$(2) \qquad (\mu|| \cdot ||_X)^* = \iota(\cdot, \mu U_{X^*}) \qquad (\alpha > 0).$$

The operations $\Box$ and $+$ cannot be interchanged in formula (1) without additional assumptions.

The celebrated result of Wijsman [70] in finite dimensions that arguably launched the modern theory of topologies on convex functions goes as follows: if $f, f_1, f_2, \ldots$ is a sequence in $\Gamma(\mathbb{R}^n)$ then $\langle \text{epi} \ \ f_n \rangle$ is Wijsman convergent to epi f if and only if $\langle \text{epi} \ \ f_n^* \rangle$ is Wijsman convergent to epi f^*. Now by Theorem 3.5, Attouch-Wets convergence in finite dimensions reduces to Wijsman convergence, and so the next result extends Wijsman's theorem to infinite dimensions in a totally satisfactory way.

Theorem 4.4. *Let X be a normed linear space and let $f, f_1, f_2, f_3, \ldots$ be a sequence in $\Gamma(X)$. The following conditions are equivalent:*

(a) *epi $f = AW - lim$ epi f_n;*

(b) *epi $f^* = AW - lim$ epi f_n^*.*

The initial proof given in [12] is not as transparent as possible, and subsequently there have been additional proofs including those that provide useful quantitative estimates (see, e.g., [8, 34, 56]).

As one consequence of bicontinuity of the Fenchel transform, we see that Attouch-Wets convergence of a sequence of closed convex sets is further equivalent to convergence in the epi-distance topology of the associated sequence of support functionals. Further, if $A = \text{AW} - \lim A_n$, applying Theorem 4.4 and then Theorem 4.1 to support functionals with $\alpha = 1$, we have

$$\{y \in X^* : \forall a \in A < y, a >\leq 1\} = \text{AW} - \lim\{y \in X^* : \forall a \in A_n < y, a >\leq 1\}.$$

This means that we have continuity of polarity with respect to the Attouch-Wets topology, and for closed convex sets that contain θ, bicontinuity of polarity. That this is true for convex cones, as discovered by Walkup and Wets [69] over forty years ago, was the point of departure for the study of the Attouch-Wets topology in the first place.

We now give a proof of a result of the author [16] linking convergence in the epi-distance topology to uniform convergence of Lipschitz regularizations on bounded sets that combines much of the above machinery. We note that for $f \in \Gamma(X)$ and $\mu > 0$, $f \Box \mu || \cdot ||_X$ is proper provided for some $\beta > 0$, epi$f$ does not hit the inverted cone $\{(x, \alpha) : \alpha \leq -\mu ||x||_X - \beta\}$, which in turn occurs if μ exceeds the distance from dom(f^*) to the origin of X^*. When the the epi-sum is proper, it is the largest μ-Lipschitz continuous function that f majorizes [14, pp. 251-252].

Theorem 4.5. *Let X be a normed linear space and let $f, f_1, f_2, f_3, \ldots$ be a sequence in $\Gamma(X)$. For each $\mu > 0$ let f_μ be the epi-sum of f with $\mu || \cdot ||_X$ and let $f_{n,\mu}$ be the corresponding epi-sum with f_n. Then the following conditions are equivalent:*

(1) *epi $f = AW - lim$ epi f_n;*

(2) $\forall \mu > d(\theta^*, \text{epi } f^*)$, $\langle f_{n,\mu} \rangle$ *converges uniformly on bounded sets to f_μ.*

Proof. Suppose $\langle f_n \rangle$ converges to f in the epi-distance topology. Now fix $\mu > d(\theta^*, \text{epi } f^*)$. We have $\text{dom}(f^*) \cap \text{int dom}(\iota(\cdot, \mu U_{X^*})) \neq \emptyset$. Thus by the continuity of the Fenchel transform and the remark following Theorem 4.3, we get convergence of $\langle f_n^* + \iota(\cdot, \mu U_{X^*}) \rangle$ to $f^* + \iota(\cdot, \mu U_{X^*})$. Applying bicontinuity of the Fenchel transform and the fact that the conjugate of an epi-sum is the sum of the conjugates, we get $\text{epi} f_\mu = \text{AW} - \lim \text{epi } f_{n,\mu}$, which in turn gives uniform convergence on bounded sets by Theorem 4.2, as Lipschitz functions are bounded on bounded sets.

For the converse, fix $\mu > d(\theta^*, \text{epi } f^*)$. The convergence of $\langle f_{n,\mu} \rangle$ to f_μ uniformly on bounded sets forces convergence in the epi-distance topology, and so convergence of $\langle f_n^* + \iota(\cdot, \mu U_{X^*}) \rangle$ to $f^* + \iota(\cdot, \mu U_{X^*})$. Since this is true for all large μ and since adding the indicator function of a set to a member of $\Gamma(X)$ simply restricts its effective domain to that set, this gives epi $f^* = \text{AW} - \lim \text{epi } f_n^*$. Once again applying Theorem 4.4, the implication (2) $\Rightarrow$ (1) follows. □

Notice that when $f = \iota(\cdot, A)$ and $f_n = \iota(\cdot, A_n)$ for nonempty closed convex sets $A, A_1, A_2, \ldots$, Theorem 4.5 reduces to Theorem 3.2. We also mention that a similar result holds for Moreau-Yosida regularizations. In fact, one can work without convexity of the functions for these and related smoothing kernels provided an appropriate minorization requirement is in place, as shown by Attouch and Wets [5, Theorem 3.4].

Michel Théra had a hand in the last result of this section, as well as an anticipatory result in a restricted setting [4]. First a definition.

Definition 4.6. Let $f \in \Gamma(X)$. We call $y \in X^*$ a *subgradient* of f at $x \in X$ if for each $w \in X$ we have $f(w) \geq f(x) + < y, w - x >$.

To say that y is a subgradient of f at x means that the hyperplane in $X \times \mathbb{R}$ that is the graph of $w \mapsto f(x) + < y, w - x >$ supports the epigraph at $(x, f(x))$. Let us write $\partial f(x)$ for the set of subgradients of f at x. While $x \in \text{dom}(f)$ does not guarantee that $\partial f(x)$ is nonempty, it can be shown that in a Banach space X the set $\{x \in \text{dom}(f) : \partial f(x) \neq \emptyset\}$ is dense in dom(f) (see, e.g., [59, p. 51]).

Next, for $f \in \Gamma(X)$ put

$$\triangle(f) := \{(x, f(x), y) : x \in \ \text{dom}(f) \text{ and } y \in \partial f(x)\}.$$

Note that $\triangle(f) \subseteq X \times \mathbb{R} \times X^*$. We now characterize Attouch-Wets convergence in terms of $\triangle$ [29].

Theorem 4.7. *Let X be a Banach space and let $f, f_1, f_2, f_3, \ldots$ be a sequence in $\Gamma(X)$. Then epi $f = AW - lim$ epi f_n if and only if for each bounded subset B of $X \times \mathbb{R} \times X^*$ and each $\varepsilon > 0$, we have eventually*

$$\Delta(f) \cap B \subseteq \Delta(f_n)^\varepsilon.$$

Notice that Theorem 4.7 only asserts that Attouch-Wets convergence of epigraphs guarantees one-half of Attouch-Wets convergence of $\triangle(f_n)$ to $\triangle(f)$. Indeed, convergence in the other half may fail [29, p. 857]. The result is stated in a Banach space because the proof in [29] required the comprehensive Borwein Variational Principle [59, p. 55]. It is possible to state a result in a general normed linear space, but this requires working with approximate subgradients, as executed by the Veronas [66].

We close this section by noting that there is another important topology on convex sets and functions that reduces to the Wijsman topology in finite dimensions. This is the celebrated *Joly topology* [48], now frequently called the *slice topology* [13, 15, 16, 14, 18, 30, 42, 50, 64], which is compatible with Mosco convergence [55] in reflexive spaces but which is stronger in general. While Joly defined his topology in terms of lower semicontinuity of the epigraphical multifunctions $f \rightrightarrows$ epi f and $f \rightrightarrows$ epi f^*, it can be defined much more concretely for closed convex subsets of a normed linear space X, later specializing to epigraphs. For nonempty closed convex sets, it is the weak topology determined by all gap functionals with fixed left argument running over the closed bounded convex subsets. Equivalently, it is the supremum of the Wijsman topologies determined by equivalent renorms of the space X [15]. Theorem 4.5 is valid if when we replace the epi-distance topology by the slice topology, we also replace uniform convergence of regularizations on bounded sets by pointwise convergence of regularizations of both the functions and their conjugates [16]. Theorem 4.7 may of course be viewed as a variant of Attouch's Theorem [2, 33] for slice convergence, originally proved by him for Mosco convergence in reflexive spaces [1]. As to what epigraphical slice convergence of a net in $\Gamma(X)$ means in terms of convergence of the associated net of conjugate functions, the reader may consult [13, 14].

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