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GENERALIZED AFFINE MAPS AND GENERALIZED CONVEX FUNCTIONS

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Abstract: We study some classes of generalized convex functions, using a generalized derivative approach. We establish some links between these classes and we devise some optimality conditions for constrained optimization problems. In particular, we get Lagrange-Kuhn-Tucker multipliers for mathematical programming problems.

Key words: *colinvex, colinfine, generalized derivative, mathematical programming, optimality conditions, pseudoconvex function, pseudolinear function, quasiconvex function*

Mathematics Subject Classification: *26B25, 46G05, 49K27, 90C26, 90C32*

1 Introduction

Prompted by the needs of mathematical economics, some mathematicians have scrutinized several notions of generalized convexity or concavity. While some of them, such as quasiconvexity and pseudoconvexity are rather general, some others are rather peculiar. This is the case for functions which are both pseudoconvex and pseudoconcave. Initially, these functions have been called pseudomonotone (Martos [45]); nowadays these functions are usually called pseudolinear, but we prefer to call them pseudoaffine since a function which is both convex and concave is affine; moreover, this class includes affine functions which are not linear. Albeit especial, this class of functions contains interesting, non trivial examples and enjoys striking properties, as shown in [1], [2], [5], [7], [8], [13], [16], [18], [27], [31], [33], [36], [51], [54], [59], [61] among others. In all the references dealing with this subject, pseudoaffine functions are defined through the use of a directional derivative function or a Dini derivative. We also follow this line, taking a general bifunction h as a substitute for such a derivative. The use of subdifferentials for dealing with this class will be made elsewhere.

It is our purpose here to study related classes of functions, to give characterizations and to detect some relationships. In particular, we introduce and study properties which are related to some quasiconvexity, pseudoconvexity and invexity properties. We also present extensions to vector-valued maps and we study some composition properties (section 4).

Some relations between solutions of variational inequalities and optimization problems were discussed in the papers quoted above. We also tackle this question in section 5; in particular, we study necessary and sufficient optimality conditions in the form of a Karush-Kuhn-Tucker condition when the data belong to the class of functions under study. We provide examples allowing a comparison with previous results in [38], [41], [43] and [44].

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$\boxed{2}$ Pseudoconvex, Quasiconvex and Protoconvex Functions

In the sequel, X, Y are normed vector spaces (n.v.s.), C is a nonempty subset of X and $f : C \to \mathbb{R}$; it is convenient to extend f by $+\infty$ outside C. We set $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{P} :=]0, +\infty[$. For $w, x \in X$, we set $[w, x] := \{(1-t)w + tx : t \in [0, 1]\}$, $[w, x[:= [w, x] \backslash \{x\}$ and $]w, x[:= [w, x[\backslash \{w\}$. We define the *visibility cone* $V(C, x)$ *of* C at $x \in C$ as the cone generated by $C - x$:

$$V(C, x) := \mathbb{P}(C - x) := \{r(c - x) : r \in \mathbb{P}, c \in C\}.$$

The *visibility bundle of* C is the set

$$VC := \{(x, u) \in C \times X : \exists r \in \mathbb{P}, w \in C, u = r(w - x)\} = \bigcup_{x \in C} \{x\} \times V(C, x).$$

It contains the *radial tangent bundle of* C which is the set

$$T^r C := \{(x, u) \in C \times X : \exists\, (r_n) \to 0_+, x + r_n u \in C\ \forall n\} = \bigcup_{x \in C} \{x\} \times T^r(C, x).$$

We also use the *tangent bundle of* C which is the set

$$TC := \{(x, u) \in C \times X : \exists (r_n) \to 0_+, (u_n) \to u,\ x + r_n u_n \in C\ \forall n\} = \bigcup_{x \in C} \{x\} \times T(C, x).$$

Here $T^r(C, x) := \{u \in X : \exists\, (r_n) \to 0_+, x + r_n u \in C\ \forall n\}$ is the *radial tangent cone* to C at $x \in C$ and $T(C, x) := \{u \in X : \exists\, (r_n) \to 0_+, (u_n) \to u,\ x + r_n u_n \in C\ \forall n\}$ is the *tangent cone* to C at x. One has the obvious inclusions $T^r(C, x) \subset T(C, x)$, $T^r(C, x) \subset V(C, x)$. When C is *starshaped at* $x \in C$, in the sense that $[x, w] \subset C$ for every $w \in C$, one has $V(C, x) = T^r(C, x)$. When C is a convex subset of X, as it will be frequently the case in the sequel, $T(C, x)$ is the closure of $V(C, x) = T^r(C, x)$.

A bifunction $h : VC \to \overline{\mathbb{R}}$ will stand for a generalized derivative of f. All the usual generalized directional derivatives have the common feature that they are positively homogeneous functions of the direction u. Thus, we assume that for all $x \in C$, $h(x, \cdot)$ is positively homogeneous and $h(x, 0) = 0$. There are several possible choices for h. Among them are the Dini derivatives of f. The upper and the lower radial derivatives (or upper and lower Dini derivatives) of f at $x \in C$, in the direction $u \in T^r(C, x)$, are defined by

$$\begin{aligned} D^+ f(x, u) &= \limsup_{t \to 0+,\ x + tu \in C} \frac{1}{t}[f(x + tu) - f(x)]; \\ D_+ f(x, u) &= \liminf_{t \to 0+,\ x + tu \in C} \frac{1}{t}[f(x + tu) - f(x)]. \end{aligned}$$

If for some $(x, u) \in T^r C$ the relation $D^+ f(x, u) = D_+ f(x, u)$ holds, then the *radial derivative* of f at x in the direction u is the common value of the preceding limits:

$$Df(x, u) = D^+ f(x, u) = D_+ f(x, u).$$

This derivative is often called the *directional derivative of* f at x in the direction u, but we prefer to keep this term for the limit

$$f'(x, u) = \lim_{\substack{(t,v) \to (0_+, u) \\ x + tv \in C}} \frac{1}{t}[f(x + tv) - f(x)] \qquad (x, u) \in TC,$$

when it exists. Of course, when this directional derivative (also called the Hadamard derivative) exists for some $(x, u) \in T^r C$, the radial derivative exists and has the same value. Note that the limit in the definition of $f'(x, u)$ is $+\infty$ if u does not belong to the tangent cone $T(C, x)$ to C at x which is the set of limits of sequences of the form $(t_n^{-1}(x_n - x))$ where $(t_n) \to 0_+$ and $x_n \in C$ for every $n \in \mathbb{N}$. Similarly, $D_+ f(x, u)$ is $+\infty$ if u does not belong to the $T^r(C, x)$.

Other generalized derivatives can be used. For instance, one can use the lower directional derivative (or contingent derivative or lower Hadamard derivative) *df* given by

$$df(x, u) := \liminf_{\substack{(t,v) \to (0_+, u) \\ x + tv \in C}} \frac{1}{t}[f(x + tv) - f(x)]$$

and its upper version, or some epiderivative. One can also use the Clarke-Rockafellar derivative $f^{\uparrow}$ of f ([10], [55]) or the moderate derivative $f^{\diamondsuit}$ of Michel-Penot ([46]).

Slightly modifying a notion introduced in [57], we say that a mapping $s : X \to \overline{\mathbb{R}}$ is *subodd on* $C \subset X$ if the following condition is satisfied

$$(S) \qquad\qquad w, x \in C \Longrightarrow s(w - x) \geq -s(x - w).$$

Clearly, if s is finitely valued and positively homogenous, s satisfies (S) if s is sublinear. We say that $s : C - C \to \overline{\mathbb{R}}$ *reverses signs on* $C \subset X$ if the following condition is satisfied

$$(R) \qquad\qquad w, x, y \in C,\ x \in]w, y[\Longrightarrow s(w - x)s(y - x) \leq 0.$$

Conditions (R) and (S) are independent, as the following simple examples show.

Example 1. Let $s_1 : \mathbb{R} \to \mathbb{R}$ be given by $s_1(u) = |u|$; then s_1 is subodd, but $s_1(u)s_1(-u) > 0$ for all $u \in \mathbb{R}\backslash\{0\}$.

Example 2. Let $s_2 : \mathbb{R} \to \mathbb{R}$ be given by $s_2(u) = 0$ if $u \geq 0$ and $s_2(u) = u$ if $u < 0$; then $s_2(u)s_2(-u) \leq 0$ for all $u \in \mathbb{R}$ but $s_2(1) \ngeq -s_2(-1)$.

Both conditions (R) and (S) imply the following condition:

$$(T) \qquad\qquad w, x, y \in C,\ x \in]w, y[,\ s(w - x) < 0 \Longrightarrow s(y - x) \geq 0.$$

Note that s satisfies condition (R) if, and only if s and $-s$ satisfy condition (T). On the other hand, s and $-s$ satisfy condition (S) if, and only if, s is odd on $C - C$.

A function $f : C \to \mathbb{R}$ is said to be is *quasiconvex at* $x \in C$ if for all $w \in C$ and $x_t := tx + (1 - t)w \in C$ for $t \in [0, 1]$ then $f(x_t) \leq \max\{f(x), f(w)\}$.

Let us recall the following definitions (see [24], [30], [32], [36] and [57]).

Definition 2.1. Given a bifunction $h : VC \to \overline{\mathbb{R}}$, a function $f : C \to \mathbb{R}$ is said to be

(a) h-pseudoconvex at $x \in C$, if

$$w \in C, \quad h(x, w - x) \geq 0 \Rightarrow f(w) \geq f(x).$$

(b) h-quasiconvex at $x \in C$, if

$$w \in C, \quad h(x, w - x) > 0 \Rightarrow f(w) \geq f(x).$$

(c) h-protoconvex at $x \in C$, if

$$w \in C, \quad h(x, w - x) > 0 \Rightarrow f(w) > f(x).$$

(d) h-pseudoconcave at $x \in C$, if $-f$ is $-h$-pseudoconvex at $x \in C$.

(e) h-quasiconcave at $x \in C$, if $-f$ is $-h$-quasiconvex at $x \in C$.

(f) h-protoconcave at $x \in C$, if $-f$ is $-h$-protoconvex at $x \in C$.

We say that f is h-pseudoconvex on C or, in short, h-pseudoconvex (resp. h-quasiconvex, h-protoconvex, h-pseudoconcave, h-quasiconcave, h-protoconcave) if f is h-pseudoconvex (resp. h-quasiconvex, h-protoconvex, h-pseudoconcave, h-quasiconcave, h-protoconcave) at every $x \in C$.

Clearly, f is h-pseudoconcave (resp. h-quasiconcave, h-protoconcave) if, and only if, for any $w, x \in C$,

$$\begin{array}{rrcl} & f(x) & < & f(w) \Rightarrow h(x, w-x) > 0 \\ \text{(resp.} & f(x) & < & f(w) \Rightarrow h(x, w-x) \geq 0, \\ \text{(resp.} & f(x) & \leq & f(w) \Rightarrow h(x, w-x) \geq 0.) \end{array}$$

Also,

$$\begin{array}{rcl} f \text{ is } h\text{-pseudoconvex} & \Rightarrow & f \text{ is } h\text{-quasiconvex}, \\ f \text{ is } h\text{-protoconvex} & \Rightarrow & f \text{ is } h\text{-quasiconvex}. \end{array}$$

Let us observe that h-pseudoconvexity at x and h-protoconvexity at x are independent properties. Both properties are consequences of the following condition:

$$w \in C \backslash \{x\}, \quad f(w) \leq f(x) \Rightarrow h(x, w-x) < 0.$$

This rather stringent condition will not be considered here.

These definitions are closely related. Let us note that here the terminology slightly differs from the one in [30], [37] and [57]. Such a change will allow a better concordance with the classical terminology for the case a subdifferential is used (see [2], [23], [39], [52]). Here, we do not need any assumption on the bifunction h, but in some cases, we will require some comparisons with lower and upper Dini derivatives.

Proposition 2.2. *If f has no local minimizer on $f^{-1}(f(x))$ and if $h(x, \cdot)$ is lower semicontinuous (l.s.c.) then h-quasiconvexity at x coincides with h-protoconvexity at x.*

Proof. Let $w \in C$ be such that $f(w) = f(x)$. Since f has no local minimizer on $f^{-1}(f(x))$, there exists $(w_n) \to w$ such that $f(w_n) < f(w) = f(x)$. Since f is h-quasiconvex at x, one has $h(x, w_n - x) \leq 0$ hence, $h(x, w - x) \leq 0$ by lower semicontinuity of $h(x, \cdot)$. Thus, f is h-protoconvex at x. □

Links with usual quasiconvexity are given in the next proposition. It shows that, under mild assumptions, quasiconvexity, h-quasiconvexity and h-protoconvexity coincide. We will use the following complement to the Three Points Lemma of [33]; as there, we say that f satisfies the *nonconstancy property*, in short, the NC-property, if there is no line segment $[a, b]$ in C on which f is constant.

Lemma 2.3. *(Three Points Lemma) Let C be convex. Let $f : C \to \mathbb{R}$ be either radially continuous or radially lower semicontinuous and satisfy the NC-property. Let $w, z \in C$ with $w \neq z$ and let $y = z + t(w - z)$, with $t \in]0, 1[$ be such that $f(y) > f(w) \geq f(z)$. Then there exist some $x, x' \in [w, z]$ such that $f(x') < f(x)$ and $D_+ f(x, x' - x) > 0$.*

Proof. If f is radially l.s.c. and satisfies the NC-property, the result is given in [33, Lemma 10.1]. If f is radially continuous, by the intermediate value theorem, there exists some $p \in]0, t[$ such that $f(w) < f(u) < f(y)$ for $u := z + p(w - z)$. Taking p close enough to t, we may assume that $f(w) < f(v)$ for all $v \in [u, y]$.

Let $s := \sup\{r \in [p,t] : f(z + r(w - z)) \leq f(u)\}$ and let $v := z + s(w - z)$. Since f is radially l.s.c., one has $f(v) \leq f(u) < f(y)$. By the mean value theorem [15], there exists some $x \in [v, y[$ such that $D_+f(x, y - v) \geq f(y) - f(v) > 0$. Then $D_+f(x, w - x) > 0$ and $f(w) < f(x)$ since $f(z + t'(w - z)) > f(w)$ for all $t' \in [p, t]$. Setting $x' := w$, one has the conclusion. □

Proposition 2.4. *Let C be convex.*

(a) *If f is quasiconvex and if $h \leq D^+f$, then f is h-protoconvex, hence h-quasiconvex.*

(b) *Conversely, if f is h-quasiconvex, if $h \geq D_+f$, and if f is either radially continuous or radially lower semicontinuous on C and satisfies the NC-property, then f is quasiconvex.*

(c) *If f is radially differentiable and if $h := Df$, then f is quasiconvex, if, and only if, it is h-quasiconvex, if, and only if, it is h-protoconvex..*

Proof. (a) Suppose f is quasiconvex and $f(w) \leq f(x)$ for some $w, x \in C$. Then for $t \in]0, 1[$ we have $f(x + t(w - x)) \leq f(x)$, hence $D^+f(x, w - x) \leq 0$. Thus $h(x, w - x) \leq 0$ if $h(x, \cdot) \leq D^+f(x, \cdot)$.

(b) Suppose that f is not quasiconvex. Then, there exist w, y, z such that $f(y) > f(w) \leq f(z)$, where $y = z + t(w - z)$, $t \in]0, 1[$. By the preceding lemma, there exist $x, x' \in [w, z]$ such that $h(x, x' - x) \geq D_+f(x, x' - x) > 0$ and $f(x') < f(x)$: a contradition with the h-quasiconvexity of f.

(c) In view of the fact that a radially differentiable function is radially continuous, the assertion follows from (a) and (b). □

A comparison between h-pseudoconvexity and quasiconvexity is given in the next proposition. Note that its assertion (b) does not involve Dini derivatives; it has been proved in [57, Theorem 5.1] when for all $x \in C$, if $h(x, \cdot)$ is subodd. Its last assertion is well known.

Proposition 2.5. *Suppose f is h-pseudoconvex on a convex set C. If one of the following two conditions is satisfied, then f is quasiconvex:*

(a) *f is radially lower semicontinuous on C and $h \geq D_+f$.*

(b) *for all $x \in C, h(x, \cdot)$ satisfies (T).*

In particular, if f is radially differentiable and f is h-pseudoconvex with $h := Df$ (in short, f is pseudoconvex), then f is quasiconvex.

Proof. (a) It is proved in [33, Theorem 10.5].

(b) Suppose $h(x, \cdot)$ satisfies (T) for all $x \in C$ and f is h-pseudoconvex. To prove that f is quasiconvex, it suffices to show that if there exist $w, x, y \in C$ with $x \in]w, y[$ such that $f(x) > f(w)$ and $f(x) > f(y)$, one is led to a contradiction. By h-pseudoconvexity of f, one gets $h(x, w - x) < 0$ and $h(x, y - x) < 0$. Since $h(x, \cdot)$ satisfies (T), one obtains the required contradiction. □

When f is both h-pseudoconvex and h-protoconvex, we get a special form of invexity. Here, in the relation

$$\forall w, x \in C \qquad f(w) \geq f(x) + h(x, v(w, x))$$

which characterizes *h-invexity* of f on C, the vector $v(w, x)$ is of the form $\lambda(w, x)(w - x)$, with $\lambda(w, x) \in \mathbb{P}$, i.e. $v(w, x)$ is positively colinear to the vector $w - x$. This fact explains the terminology chosen in the next statement. This terminology may also have an interest in the case one needs to avoid a possible confusion with the notion of semiconvexity used in [60], where semiconvexity is equivalent to quasiconvexity plus invexity or a confusion with the notion used for the study of Hamilton-Jacobi equations as in [9].

Proposition 2.6. *For any function $f : C \to \mathbb{R}$ and any bifunction $h : VC \to \overline{\mathbb{R}}$, the following assertions are equivalent:*

(a) *f is h-semiconvex (i.e. f is h-pseudoconvex and h-protoconvex).*

(b) *f is h-colinvex: there exists $\lambda : C \times C \to \mathbb{P}$ such that for all $w, x \in C$,*

$$f(w) \geq f(x) + \lambda(w, x)h(x, w - x).$$

Proof. (b)⇒(a) is obvious.

(a)⇒(b) Let f be h-pseudoconvex and h-protoconvex. Let $w, x \in C$.

If $h(x, w-x) = 0$ then one has $f(w) \geq f(x)$ by h-pseudoconvexity of f and one can take $\lambda(w, x) = 1$ or any element in $\mathbb{P}$.

If $h(x, w - x) > 0$ then $f(w) > f(x)$ by h-protoconvexity of f. Then one can take

$$\lambda(w, x) = [f(w) - f(x)]/h(x, w - x).$$

If $h(x, w - x) < 0$ then one can take

$$\lambda(w, x) := \max\{[f(w) - f(x)]/h(x, w - x), 0\} + 1,$$

and then $f(w) \geq f(x) + \lambda(w, x)h(x, w - x)$. □

Remark 2.7. When f is convex and if h is the radial derivative of f, then (b) is satisfied with $\lambda(w, x) = 1$. As mentioned above, the specific choice of the function v given by $v(w, x) := \lambda(w, x)(w - x)$ as a vector colinear with $w - x$ pin points this class of functions among invex functions.

3 Pseudoaffine, Protoaffine and Colinfine Functions

In the present section we study classes of functions which are still more restrictive. Their interests lie in their striking properties. In particular, they enjoy nice composition properties with the classes studied in the preceding section. Their behaviors will be presented in Proposition 3.4 and their mutual relationships will be examined in Proposition 3.8 below.

The first concept of the following definition has been introduced by Chew and Chow [13] in the differentiable case and has been used in [1], [7], [8], [31], [36], [59], [61]. The terminology we adopt in this definition is new. For the notions in (a)-(c), it stems from the fact that a function which is both convex and concave is affine. For the notion in (d), as above, it is motivated by the special invexity condition (3.1) in which the vector field is colinear to $w - x$.

Definition 3.1. Given a bifunction $h : VC \to \overline{\mathbb{R}}$, a function $f : C \to \mathbb{R}$ is said to be

(a) h-pseudoaffine at $x \in C$ if f is h-pseudoconvex and h-pseudoconcave at $x \in C$.
(b) h-protoaffine at $x \in C$ if f is h-protoconvex and h-protoconcave at $x \in C$.
(c) h-semiaffine at $x \in C$ if f is h-semiconvex and h-semiconcave at $x \in C$.
(d) h-colinfine at $x \in C$ if the following implications hold for all $w \in C$:

$$\begin{aligned} h(x, w - x) &= 0 \Longrightarrow f(w) = f(x), \\ h(x, w - x) &> 0 \Longrightarrow f(w) > f(x), \\ h(x, w - x) &< 0 \Longrightarrow f(w) < f(x). \end{aligned}$$

These implications can be summarized by saying that $f(w) - f(x)$ must have the same sign as $h(x, w - x)$ (with the convention that the sign of 0 is 0).

We say that f is h-pseudoaffine on C (in short, h-pseudoaffine) (resp. h-protoaffine, h-semiaffine, h-colinfine) if f is h-pseudoaffine (resp. h-protoaffine, h-semiaffine, h-colinfine) at every $x \in C$. In the case h is the radial derivative of f, we simply say that f is *pseudoaffine* (resp. *protoaffine, semiaffine, colinfine*) if f is h-pseudoaffine (resp. h-protoaffine h-semiaffine, h-colinfine) on C. Observe that a radially differentiable function f is pseudoaffine (resp. protoaffine) if, and only if, f and $-f$ are pseudoconvex (resp. protoconvex).

It is easy to show that f is h-colinfine if, and only if, f is h-colinvex and $-f$ is $-h$-colinvex. Also, it is easy to see that f is h-colinfine at x if and only if there exists some function $\lambda : C \times C \to \mathbb{P}$ such that

$$\forall w \in C \qquad f(w) - f(x) = h(x, \lambda(w, x)(w - x)). \tag{3.1}$$

Such an observation justifies the terminology. If the implications of Definition 3.1 (d) hold true then one can introduce λ satisfying Definition 3.1 (d) by setting $\lambda(w, x) = [f(w) - f(x)]/h(x, w - x)$ when $h(x, w - x) \neq 0$ and $\lambda(w, x) \in \mathbb{P}$ arbitrary if $h(x, w - x) = 0$.

The following observation is immediate. We will prove a related property in Proposition 3.8.

Proposition 3.2. *Given $h : VC \to \overline{\mathbb{R}}$, a function $f : C \to \mathbb{R}$ is h-colinfine if, and only if, it is h-semiaffine.*

Proof. Suppose that f is h-colinfine. Then there exists some function $\lambda : C \times C \to \mathbb{P}$ such that

$$\forall w, x \in C \qquad f(w) - f(x) = \lambda(w, x)h(x, w - x).$$

Hence, f is h-colinvex and $-f$ is $-h$-colinvex. By Proposition 2.6, one has that f is h-semiconvex and $-f$ is $-h$-semiconvex. Thus, f is h-semiaffine.

Conversely, suppose that f is h-semiaffine. Using the definition of a h-semiaffine function, we see that the above implications are satisfied, so that f is h-colinfine by the construction of λ we have pointed out. □

Now, let us give an answer to the question: when is a colinfine function a convex function?

Proposition 3.3. *Let f be h-colinfine on a convex set C and let $\lambda : C \times C \to \mathbb{P}$ be such that*

$$f(w) - f(x) = \lambda(w, x)h(x, w - x).$$

Assume that for all $w, x \in C$, $t \in [0, 1]$ one has $\lambda(x_t, x) \leq \lambda(w, x)$ whenever $f(x) \leq f(w)$, where $x_t := x + t(w - x)$. Then f is convex on C.

Proof. Let $w, x \in C$, $x_t := x + t(w - x)$ for $t \in [0, 1]$. Without loss of generality, we may suppose $f(x) \leq f(w)$. Then $h(x, w - x) \geq 0$. Since f is h-colinfine, we have

$$\begin{aligned} f(x_t) - f(x) &= \lambda(x_t, x)h(x, x_t - x) = t\lambda(x_t, x)h(x, w - x), \\ t[f(w) - f(x)] &= t\lambda(w, x)h(x, w - x). \end{aligned}$$

By assumption, we have $\lambda(x_t, x) \leq \lambda(w, x)$, hence $f(x_t) - f(x) \leq t(f(w) - f(x))$. Thus, f is convex. □

Even for a smooth bifunction h, an h-colinfine function maybe nonsmooth, and not even continuous.

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$ for $x > 0$, $f(x) = -x^2$ for $x \leq 0$, and let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $h(x, u) = u$ for all $x, u \in \mathbb{R}$. Then f is h-colinfine. Note that h is neither the radial derivative nor one of the Dini derivatives of f. The function f is not continuous at 0 and f is neither convex nor concave.
Example 4. If h is as in the preceding example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$ is not pseudoaffine but f is h-colinfine.

The preceding example shows that a function may be h_1-colinfine for some bifunction $h_1 : (x, u) \to u$ but not h_2-colinfine (and not even h_2-pseudoconvex) for some other bifunction $h_2 := Df$, even when f, h_1 and h_2 are smooth.

The next statement shows that h-colinfine functions have a very special behavior. Here, we do not need any continuity of f.

Proposition 3.4. *Let $f : C \to \mathbb{R}$. There exists $h : VC \to \overline{\mathbb{R}}$ such that f is h-colinfine on C if and only if for all $w, x \in C$, the function $t \mapsto f(x + t(w - x))$ is either increasing or decreasing or constant on the interval on which it is defined.*

Proof. Let f be h-colinfine on C. Let $w, x \in C$. If $f(w) = f(x)$ then $f(x + t(w - x)) = f(x)$ for all $t \in \mathbb{R}$ such that $x_t := x + t(w - x) \in C$ since we have $h(x, x_t - x) = th(x, w - x) = 0$.

If $f(w) > f(x)$ then $h(x, w - x) > 0$. For all $t > 0$ such that $x_t := x + t(w - x) \in C$, one has $h(x, x_t - x) > 0$ by homogeneity. Thus $f(x + t(w - x)) > f(x)$. Also, given $u := x + r(w - x) \in C$ and $v := x + s(w - x) \in C$ with $s > r$, one has $f(v) > f(u)$. Otherwise, one would have either $f(v) = f(u)$ and then $f(w) = f(x)$ by what precedes or $f(v) < f(u)$ and then $h(v, u - v) > 0$, $h(v, x - v) < 0$ (as $f(x) < f(v)$), a contradiction since $x - v = q(u - v)$ for some $q > 0$. A similar proof shows that for every $t > 0$ we have $f(w + t(x - w)) < f(w)$ and for $s > r$, $f(w + s(x - w)) < f(w + r(x - w))$ when the involved points are in C.

If $f(x) > f(w)$, then we also have the conclusion by interchanging the role of w and x in what precedes.

Conversely, a possible choice for h is as follow. Given $(x, u) \in VC$ with $\|u\| = 1$, we can find some $t \in \mathbb{P}$ such that $w := x + tu \in C$. If $f(w) = f(x)$, for $r \in \mathbb{P}$ we take $h(x, ru) = 0$ and $\lambda(x + ru, x) := 1$. If $f(w) \neq f(x)$, then for $r \in \mathbb{P}$, we set

$$h(x, ru) := rt^{-1}(f(w) - f(x)), \qquad \lambda(x + ru, x) := \frac{t(f(x + ru) - f(x))}{r(f(x + tu) - f(x))}.$$

Then h is clearly positively homogeneous in its second variable, $\lambda(x + ru, x) \in \mathbb{P}$ by our assumption, and if $(x + ru, x) \in C \times C$ we have $f(x + ru) - f(x) = \lambda(x + ru, x)h(x, ru)$, as required. □

From Proposition 3.4, we see that the set of all minimizers and the set of all maximizers of a non constant h-colinfine function are contained in the boundary of its domain.

Corollary 3.5. *Let f be h-colinfine and let $w, x \in C$. Then for any $t \in \mathbb{R}\backslash\{0, 1\}$ such that $x_t := x + t(w - x) \in C$, the following statements are equivalent:*

(a) $f(x) = f(w)$.

(b) $h(x, w - x) = 0$.

(c) $h(w, x - w) = 0$.

(d) $f(x_t) = f(x)$.

(e) $h(x_t, w - x) = 0$.

(f) $f(x_t) = f(w)$.

Proof. One has (a)⇔(b)⇔(c) by the definition of a h-colinfine function and (a)⇔(d)⇔(f) by the preceding proposition. If $t > 0$ then $x_t - x = t(w - x)$, hence (d)⇔(e); otherwise if $t < 0$ then $w - x_t = (1 - t)(w - x)$, hence (f)⇔(e). □

Although the preceding definitions are quite restrictive, they are satisfied in some cases of significant interest.

Example 5. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be positively homogeneous in its second variable and such that $h(x, 1) > 0$, $h(x, -1) < 0$. Then f is h-colinfine. Note that we do not require h is the directional derivative nor some Dini derivative of f.

Example 6. Let X be a n.v.s., let $a, b \in X^*, \alpha, \beta \in \mathbb{R}$. For $C := \{x \in X : bx + \beta > 0\}$, let $f : C \to \mathbb{R}$ be given by

$$\forall x \in C \qquad f(x) = (ax + \alpha)/(bx + \beta).$$

Since

$$f'(x, w - x) = (awbx - axbw + (\beta a - \alpha b)(w - x))/(bx + \beta)^2,$$

one has

$$f(w) - f(x) = \frac{awbx - axbw + (\beta a - \alpha b)(w - x)}{(bw + \beta)(bx + \beta)} = \frac{bx + \beta}{bw + \beta} f'(x, w - x)$$

and f is colinfine on C with $\lambda(w, x) := (bx + \beta)(bw + \beta)^{-1}$. But when a and b are linearly independent and $\alpha = 0$, $\beta = 0$, f is not convex since one can find x such that $ax = 0$, $bx = 1$ and then $f''(x)(u, u) = -2aubu$ whose sign may be negative for some choices of u.

Example 7. Let $g : X \to \mathbb{R}$ be colinfine. For $C := \{x \in X : g(x) > 0\}$, let $f(x) = \sqrt{g(x)}$. Since g is colinfine, we have $g(w) - g(x) = \lambda_g(w, x)g'(x, w - x)$ with $\lambda_g(w, x) > 0$. Since

$$f'(x, w - x) = \frac{1}{2\sqrt{g(x)}} g'(x, w - x),$$

one has

$$f(w) - f(x) = (g(w) - g(x))/(\sqrt{g(w)} + \sqrt{g(x)}) = \lambda_g(w, x)g'(x, w - x)/(\sqrt{g(w)} + \sqrt{g(x)}).$$

Thus f is colinfine with $\lambda_f(w, x) := 2\lambda_g(w, x)\sqrt{g(x)}/(\sqrt{g(w)} + \sqrt{g(x)})$.

Example 8. More generally, let X be a n.v.s., $g : X \to \mathbb{R}$ be colinfine on $C := \{x \in X : g(x) > 0\}$. Then, for any $p \in \mathbb{N}$, $f(\cdot) := g^p(\cdot) := (g(\cdot))^p$ is colinfine on C. In fact, since g is colinfine on C, for all $w, x \in C$ we have $g(w) - g(x) = \lambda_g(w, x)g'(x, w - x)$ with $\lambda_g(w, x) > 0$. Since $f'(x, w - x) = p(g(x))^{p-1}g'(x, w - x)$, we get

$$\begin{aligned} f(w) - f(x) &= g^p(w) - g^p(x) = (g(w) - g(x))(\sum_{k=0}^{p-1} g^k(w)g^{p-k-1}(x)) \\ &= \lambda_g(w, x)g'(x, w - x)(\sum_{k=0}^{p-1} g^k(w)g^{p-k-1}(x)). \end{aligned}$$

Thus f is colinfine with $\lambda_f(w, x) := \lambda_g(w, x)p^{-1}g^{1-p}(x)(\sum_{k=0}^{p-1} g^k(w)g^{p-k-1}(x))$. □

Composition properties which will be established in the next section will enlarge the field of colinfine functions. For the moment, we delineate some properties and characterizations. For the rest of this section, we assume that C is convex. In the sequel, given a bifunction h viewed as a generalized derivative of f, we say that f satisfies (H$^+$), (H$^-$), (H$_0^+$), (H$_0^-$) if the following assertions are satisfied:

(H$^+$) If $h(x, w-x) > 0$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) > f(x)$.

(H$^-$) If $h(x, w-x) < 0$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) < f(x)$.

(H$_0^+$) If $h(x, w-x) > 0$, $f(w) = f(x)$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) > f(x)$.

(H$_0^-$) If $h(x, w-x) < 0$, $f(w) = f(x)$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) < f(x)$.

Remark 3.6. (a) Clearly (H$^+$) implies (H$_0^+$) and (H$^-$) implies (H$_0^-$).

(b) These assumptions are rather mild. Hypothesis (H$^+$) (resp. (H$^-$)) is weaker than h-protoconvexity (resp. h-protoconcavity) of f. They are automatically satisfied if h is the radial derivative of f. More generally, we can make a precise comparison with the Dini derivatives of f.

Lemma 3.7. (a) *If, for all $x \in C$, $h(x, \cdot) \leq D^+ f(x, \cdot)$ (resp. $h(x, \cdot) \geq D_+ f(x, \cdot)$) then assumption (H$^+$) (resp. (H$^-$)) is satisfied. In fact, if for all $w, x \in C$,*

$$h(x, w-x) > 0 \Longrightarrow D^+ f(x, w-x) > 0 \tag{3.2}$$

$$(\textit{resp. } h(x, w-x) < 0 \Longrightarrow D_+ f(x, w-x) < 0), \tag{3.3}$$

then f satisfies (H$^+$) (resp. (H$^-$)).

(b) *Conversely, if condition (H$^+$) (resp. (H$^-$)) is satisfied, then for all $w, x \in C$ one has*

$$h(x, w-x) > 0 \Longrightarrow D^+ f(x, w-x) \geq 0$$

$$(\textit{resp. } h(x, w-x) < 0 \Longrightarrow D_+ f(x, w-x) \leq 0).$$

Proof. (a) Suppose (3.2) holds. Given $w, x \in C$ such that $h(x, w-x) > 0$, one has $D^+ f(x, w-x) > 0$, so that there exists a sequence $(t_n) \to 0_+$ satisfying $f(x + t_n(w-x)) > f(x)$ for all n. Taking n large enough, we get some $z := x + t_n(w-x) \in (w, x)$ such that $f(z) > f(x)$. Similarly, if (3.3) holds, then f satisfies (H$^-$).

(b) Conversely, when f satisfies (H$^+$) and $w, x \in C$ are such that $h(x, w-x) > 0$, for every $(\varepsilon_n) \to 0_+$ one also has $h(x, \varepsilon_n(w-x)) > 0$, so that there exists $t_n \in]0, \varepsilon_n[$ such that $f(x + t_n(w-x)) > f(x)$. Thus $D^+ f(x, w-x) \geq 0$. The case of (H$^-$) is similar. □

Let us deal with a characterization which has been obtained in [1], [7], [13], [31], [36], [37], [59], [61] under additional assumptions (lower semicontinuity of f or the assumption that h is equal to the radial derivative of f or to one of the Dini derivatives of f). In the following statement, given functions f, h, we say that f satisfies (R) if for every $w, x, y \in C$, $x \in]w, y[$ the function $s := h(x, \cdot)$ satisfies (R).

Proposition 3.8. (a) *A function $f : C \to \mathbb{R}$ is h-pseudoaffine and satisfies (R), (H$_0^+$), (H$_0^-$) if and only if f is h-colinfine.*

(b) *If f is h-pseudoaffine, if $D_+ f \leq h \leq D^+ f$ and if either f satisfies (R) or is radially continuous, then f is h-colinfine.*

Proof. (a) Let f be h-pseudoaffine satisfying (R), (H_0^+), (H_0^-). Given $w, x \in C$, let us find some $\lambda(w,x) \in \mathbb{P}$ such that $f(w) - f(x) = \lambda(w,x)h(x, w-x)$. If $f(w) < f(x)$ then, by h-pseudoconvexity we have $h(x, w-x) < 0$ and we can take

$$\lambda(w,x) = [f(w) - f(x)]/h(x, w-x).$$

If $f(w) > f(x)$ then, by h-pseudoconcavity, we have $h(x, w-x) > 0$ and we can take

$$\lambda(w,x) = [f(w) - f(x)]/h(x, w-x).$$

In the case $f(w) = f(x)$, we will prove $h(x, w-x) = 0$, so that we take $\lambda(w,x) = 1$. We may suppose $x \neq w$. If $h(x, w-x) > 0$, by hypothesis (H_0^+), there exists $z \in]w, x[$ such that $f(z) > f(x) = f(w)$. Thus, $h(z, x-z) < 0$ and $h(z, w-z) < 0$, a contradiction with (R). Similarly, if $h(x, w-x) < 0$, hypothesis (H_0^-) yields a contradiction with (R).

Conversely, suppose f is h-colinfine. Then f is h-semiaffine by Proposition 3.2. Let us first check that f satisfies condition (R), or, equivalently, that, for all $x \in C$, $h(x, \cdot)$ and $-h(x, \cdot)$ satisfy condition (T). Let $w, y \in C$ be such that $x \in]w, y[$ and $h(x, w-x) < 0$; we have to show that $h(x, y-x) \geq 0$. Suppose $h(x, y-x) < 0$. Since f is h-colinfine, we have $f(y) < f(x)$. By Proposition 3.4 we obtain $f(x) < f(w)$, hence $h(x, w-x) > 0$, a contradiction. Thus, (T) is satisfied; since $-f$ is $-h$-colinfine, $-h$ satisfies (T). Now let us prove that (H^+) is satisfied. Let $w, x \in C$ be such that $h(x, w-x) > 0$. Then for all $z \in]x, w[$, one has $h(x, z-x) > 0$ by homogeneity, hence $f(z) > f(x)$ for all $z \in]x, w[$. Thus (H^+) is satisfied. Similarly, (H^-) is satisfied. By Remark 3.6 (a), (H_0^+), (H_0^-) are satisfied too.

(b) When f is h-pseudoaffine and satisfies (R) and when $D_+f \leq h \leq D^+f$, conditions (H_0^+), (H_0^-) are fulfilled by Remark 3.6 and Lemma 3.7; then the conclusion follows from (a). In the case f is h-pseudoaffine, radially continuous and $D_+f \leq h \leq D^+f$, one has that f and $-f$ are quasiconvex by Proposition 2.5(a) and then f and $-f$ are h-protoconvex and $-h$-protoconvex respectively by Proposition 2.4(a). Thus f is h-protoaffine and h-pseudoaffine by assumption, so that f is h-semiaffine. By Proposition 3.2, f is h-colinfine. □

Let us consider the property:

$$(E_0) \qquad \forall x, w \in C, \quad h(x, w-x) = 0 \Rightarrow f(x) = f(w).$$

Remark 3.9. (a) This property is satisfied if f is h-pseudoaffine or if $h(x,u) \neq 0$ for all $x \in C,\ u \in X \backslash \{0\}$.

(b) If f is h-protoaffine the reverse implication $f(x) = f(w) \Rightarrow h(x, w-x) = 0$ holds.

(c) Condition (E_0) implies that if there exist $w, x \in C$ such that $h(x, w-x) = 0$, then for all $t > 0$, $f(x + t(w-x)) = f(x)$.

A related property is presented in the following corollary.

Corollary 3.10. *Let f satisfy properties (R), (H^+), (H^-). If f is h-pseudoaffine on C then the following condition is satisfied:*

$$(E) \qquad \forall x, w \in C, \quad h(x, w-x) = 0 \Leftrightarrow f(x) = f(w).$$

The converse is true if f is radially continuous on C. In fact, if f is radially continuous on C and satisfies conditions (E) and (H^+) (resp. (H^-)) then f is h-pseudoconvex (resp. h-pseudoconcave).

Proof. The first assertion follows from Proposition 3.8(a) by definition of a h-colinfine function.

To prove the converse assertion, changing f and h into $-f$ and $-h$ respectively, it suffices to show that if $f(w) < f(x)$, then one has $h(x, w - x) < 0$. Since f satisfies condition (E_0), it is enough to prove that assuming $h(x, w - x) > 0$ leads to a contradiction. By hypothesis (H^+), there exists $z \in]w, x[$ such that $f(z) > f(x)$. By continuity of f on $[w, z]$, since $f(x) \in]f(w), f(z)[$, there exists $u \in [w, z]$ such that $f(u) = f(x)$. Then $u \neq x$ and there exists $r \in \mathbb{P}$ such that $w - x = r(u - x)$. By condition (E), $0 = h(x, u - x) = h(x, w - x)$, a contradiction with our assumption $h(x, w - x) > 0$. □

Corollary 3.11. *If $h \leq D^+f$ and if f is radially continuous on C and satisfies condition (E) then f is h-pseudoconvex.*

If $h \geq D_+f$ and if f is radially continuous on C and satisfies condition (E) then f is h-pseudoconcave.

The second assertion of the next corollary slightly extends assertion ii) of Theorem 4.13 in [8]; see also [13], [31], [34].

Corollary 3.12. *If D_+f and D^+f are finite, if $D_+f \leq h \leq D^+f$ and if f satisfies condition (E) then f is h-pseudoaffine. In particular, if f is radially differentiable on C and such that for every $x \in C$ the derivative $Df(x, \cdot)$ is sublinear, then f is pseudoaffine on C if, and only if f satisfies condition (E).*

Let us prepare our main result by the observations contained in the next proposition.

Proposition 3.13. *Let C be an open convex subset of X, let $h : C \times X \to \mathbb{R}$ and let $f : C \to \mathbb{R}$ be a h-colinfine function on C. Then*

(a) *For every $w \in C$ the set $K_w := \{u \in X : h(w, u) = 0\}$ is a linear subspace.*

(b) *If $f(w) = f(x)$ for some $w, x \in C$, then $K_w = K_x$.*

(c) *Moreover, if f is radially continuous and $h : VC \to \mathbb{R}$ is linear and continuous in its second variable, with $\ell(x) := h(x, \cdot)$ for all $x \in C$, then the linear forms $\ell(w)$ and $\ell(x)$ are positively colinear:*

$$w, x \in C : f(w) = f(x) \Rightarrow \exists r \in \mathbb{P}, \quad \ell(w) = r\ell(x). \tag{3.4}$$

Proof. (a) Let $u, v \in K_w$ and let $f : C \to \mathbb{R}$ be a h-colinfine function. Since C is open, $T^r(C, w) = \{w\} \times X$ and for $t \in \mathbb{R}_+$ with t small enough one has $w + tu \in C$ and $f(w + tu) = f(w) + \lambda(w + tu, w)th(w, u) = f(w)$. Thus, by Proposition 3.4, $f(w + ru) = f(w)$ for all $r \in \mathbb{R}$ such that $w + ru \in C$. It follows that $h(w, -tu) = 0$ for $t > 0$ small enough and $-u \in K_w$. Now for $t > 0$ small enough, we also have $w + tv \in C$ and $w + t(u + v) \in C$. Since $f(w + tv) = f(w) = f(w + tu)$, by Proposition 3.4 again, we obtain

$$f(w + t\frac{u + v}{2}) = f(\frac{w + tu}{2} + \frac{w + tv}{2}) = f(w).$$

Thus $u + v \in K_w$. It follows that K_w is a linear subspace.

(b) Now let $w, x \in C$ be such that $f(w) = f(x)$ and let $u \in K_w$. Since f is h-colinfine, we have $x - w \in K_w$, hence $x - w + tu \in K_w$ for all $t \in \mathbb{R}$ since K_w is a linear subspace. Thus, for $|t|$ small enough, we have $x + tu \in C$ and $f(x + tu) = f(w) = f(x)$. Therefore $h(x, u) = 0$ and $u \in K_x$. So, $K_w \subset K_x$. The symmetry of the roles of w and x yields $K_w = K_x$.

(c) Since $K_w = \ker l(w)$, $K_x = \ker l(x)$ and $K_w = K_x$ either $\ell(w)$ and $\ell(x)$ are both 0 and (3.4) is satisfied with $r = 1$ or both of them are non zero and there exits $r \in \mathbb{R}\backslash\{0\}$ such that $\ell(w) = r\ell(x)$. Let us prove that r is positive. Suppose that $r < 0$. Pick $u \in X$ such that $\langle \ell(x), u\rangle = 1$ and set $z_1 = w + tu$ and $z_2 = x + tu$, with $t > 0$ small enough to ensure that $z_1, z_2 \in C$. Then, since f is h-colinfine, one has $f(z_1) < f(w) = f(x) < f(z_2)$ and $\langle \ell(x), w - x\rangle = h(x, w - x) = 0$. Since f is radially continuous, there exists $z \in]z_1, z_2[$ such that $f(z) = f(x)$. Thus, there exists some $s \in]0, 1[$ such that $z := sz_1 + (1-s)z_2$, hence $z - x = sw - sx + tu$ and

$$0 = h(x, z - x) = \ell(x)(z - x) = \ell(x)(sw - sx + tu) = t > 0,$$

a contradiction. Hence, the case $r < 0$ is excluded and (c) is established. □

The following illuminating theorem has been given in [8, Theorems 4.13, 4.14], [31], [34] in the case the function f is differentiable and $h := f'$. Here f is nonsmooth. It also gives a converse of Proposition 3.13. We start with a preparatory lemma dealing with the one-dimensional case.

Lemma 3.14. *Let C be an open interval of $\mathbb{R}$ and let $\varphi : C \to \mathbb{R}$ be continuous, satisfy (H^+) and (H^-) with respect to h_φ, where $h_\varphi : C \times \mathbb{R} \to \mathbb{R}$ is given for some function $\ell_\varphi : \mathbb{R} \to \mathbb{R}$ by $h_\varphi(t, r) := \ell_\varphi(t)r$ for all $(t, r) \in C \times \mathbb{R}$. Then φ is h_φ-colinfine if, and only if, either the sign of ℓ_φ is constant on the level sets of φ or (E_0) is satisfied and there exists some $t \in C$ such that $\ell_\varphi(t) = 0$.*

Proof. Suppose that φ is h_φ-colinfine. Then there exists $\lambda : C \times C \to \mathbb{P}$ such that for all $s \neq t$ in C

$$\frac{\varphi(s) - \varphi(t)}{s - t} = \lambda(s, t)\ell_\varphi(t). \tag{3.5}$$

In addition, by Proposition 3.4, φ is constant or increasing or decreasing. By relation (3.5), if φ is constant then $\ell_\varphi = 0$ and (E_0) is satisfied; otherwise the sign of ℓ_φ is constant.

For the converse, let us first suppose (E_0) is satisfied and $\ell_\varphi(t) = 0$ for some $t \in C$. Then, for any $s \in C$, $\ell_\varphi(t)(s - t) = 0$, so that, by ($E_0$), $\varphi(s) = \varphi(t)$. Hence, φ is constant and, in view of conditions (H^+) and (H^-), we must have $\ell_\varphi(r) = 0$ for all $r \in C$: then φ is h_φ-colinfine.

Now suppose that the sign of ℓ_φ is constant on the level sets of φ. Given $r \in C$ such that $\ell_\varphi(r) > 0$, we will prove that $\varphi(t) > \varphi(r)$ for all $t > r$. Let us first prove that we cannot have $\varphi(t) < \varphi(r)$. If $\varphi(t) < \varphi(r)$, let $s := \sup\{p \in [r, t] : \varphi(p) = \varphi(r)\}$. Since φ is continuous, we have $s < t$ and $\varphi(s) = \varphi(r)$. Then our assumption yields $\ell_\varphi(s) > 0$, so that, by condition (H^+) there exists some $p \in]s, t[$ with $\varphi(p) > \varphi(s)$ and the intermediate value theorem yields some $q \in [p, t]$ with $\varphi(q) = \varphi(s) = \varphi(r)$, a contradiction with the definition of s.

Now let us suppose that $\varphi(t) = \varphi(r)$. Then $\ell_\varphi(t) > 0$ by our assumption, so that $h_\varphi(t, r - t) = \ell_\varphi(t)(r - t) < 0$. Then condition ($H^-$) yields some $t' \in]r, t[$ such that $\varphi(t') < \varphi(t) = \varphi(r)$. Replacing t by t', the first part of the proof yields the expected contradiction.

A similar proof shows that $\varphi(q) < \varphi(r)$ for all $q < r$. Changing φ and ℓ_φ into their opposites, we see that when $\ell_\varphi(r) < 0$, the function φ is decreasing on C. Thus, φ is h_φ-colinfine. □

Theorem 3.15. *Let C be an open, convex subset in X, let $f : C \to \mathbb{R}$ be radially continuous and $h : VC \to \mathbb{R}$ be linear and continuous in its second variable, with $\ell(x) := h(x, \cdot)$ for all $x \in C$.*

(a) *Suppose that f and h satisfy (E_0), (H^+) and (H^-), (in particular, $D_+ f \leq h \leq D^+ f$) and $\ell(x) \neq 0$ for all $x \in C$. If $x \mapsto \dfrac{\ell(x)}{||\ell(x)||}$ is constant on the level sets of f, then f is h-colinfine.*

(b) *Assume that $C = X$, f satisfies (E_0), (H^+) and (H^-) and $\ell(x) \neq 0$ for all $x \in C$. Then f is h-colinfine if, and only if, $x \mapsto \dfrac{\ell(x)}{||\ell(x)||}$ is constant on X.*

Proof. (a) Let $w, x \in C$ and for $t \in \mathbb{R}$, $x_t := (1-t)x + tw$. Set $C_{w,x} := \{t \in \mathbb{R} : x_t \in C\}$, $\varphi(t) := f(x_t)$. For $t \in C_{w,x}$, let $\ell_\varphi(t) := \ell(x_t)(w-x)$ and $h_\varphi(t,r) := \ell_\varphi(t)r$ for $r \in \mathbb{R}$. Let us show that φ satisfies (H^+), (H^-) and (E_0) with respect to h_φ. Suppose that $h_\varphi(t, s-t) > 0$ for $s, t \in C_{w,x}$, with $s \neq t$. Then $h(x_t, x_s - x_t) = (s-t)\ell(x_t)(w-x) = h_\varphi(t, s-t) > 0$. Since f satisfies (H^+), there exists $q \in (t,s)$ such that $\varphi(q) := f(x_q) > f(x_t) := \varphi(t)$. Thus, φ satisfies (H^+). With a similar proof, we see that φ satisfies (H^-). Suppose that $h_\varphi(t, s-t) = 0$ for $s, t \in C_{w,x}$, with $s \neq t$. Then $h(x_t, x_s - x_t) = h_\varphi(t, s-t) = 0$. Since f satisfies (E_0), we get $\varphi(s) := f(x_s) = f(x_t) := \varphi(t)$, so that φ satisfies (E_0).

Suppose that $\ell_\varphi(t) \neq 0$ for all $t \in C_{w,x}$. We shall prove that the sign of ℓ_φ is constant on the level sets of φ. Let $r, s \in C_{w,x}$ be such that $\varphi(r) = \varphi(s)$. Since $f(x_r) = \varphi(r) = \varphi(s) = f(x_s)$, we have $h_\varphi(r,1) = \ell(x_r)(w-x) > 0$ and $h_\varphi(s,1) = \ell(x_s)(w-x) = \dfrac{||\ell(x_s)||}{||\ell(x_r)||}\ell(x_r)(w-x) = \dfrac{||\ell(x_s)||}{||\ell(x_r)||}h_\varphi(r,1)$. Thus, either there exists some $t \in C$ such that $\ell_\varphi(t) = 0$ or the sign of ℓ_φ is constant on level sets of φ and then φ is h_φ-colinfine, by the preceding lemma.

Since for any $w, x \in C$, φ is h_φ-colinfine and $h_\varphi(0,1) = \ell_\varphi(0) = \ell(x)(w-x) = h(x, w-x)$,

$$f(w) - f(x) = \varphi(1) - \varphi(0) = \lambda_\varphi(1,0)h_\varphi(0,1) = \lambda(w,x)h(x, w-x)$$

for $\lambda(w,x) := \lambda_\varphi(1,0) > 0$; thus f is a h-colinfine function.

(b) If $C = X$ and the mapping $x \mapsto \dfrac{\ell(x)}{||\ell(x)||}$ is constant on X, then f is h-colinfine by (a). Conversely, suppose that f is h-colinfine and there exist $w, x \in C$ such that $\dfrac{\ell(w)}{||\ell(w)||} \neq \dfrac{\ell(x)}{||\ell(x)||}$. From Proposition 3.13 (b), we have $f(w) \neq f(x)$. Let $K_w := \ker \ell(w)$ and $K_x := \ker \ell(x)$. Since $\dfrac{\ell(w)}{||\ell(w)||} \neq \dfrac{\ell(x)}{||\ell(x)||}$, there exists $\overline{z} \in (w + K_w) \cap (x + K_x)$. One has $\overline{z} - w \in K_w$, $\overline{z} - x \in K_x$, hence $\ell(w)(\overline{z} - w) = 0$ and $\ell(x)(\overline{z} - x) = 0$. Since f is h-colinfine, we have $f(x) = f(\overline{z}) = f(w)$: a contradiction. □

The following example illustrates the preceding theorem; note that since f is not differentiable, [8, Theorems 4.13, 4.14], [31], [34] cannot be applied.

Example 9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = x_2$ for $x_2 \leq 0$, $f(x_1, x_2) = x_2^2$ for $x_2 > 0$, and let $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by $h((x_1, x_2), (u_1, u_2)) = u_2$ for $x_2 \leq 0$, $u := (u_1, u_2) \in \mathbb{R}^2$, and $h((x_1, x_2), (u_1, u_2)) = x_2 u_2$ for $x_2 > 0$, $u \in \mathbb{R}^2$. Then, as easily seen, f satisfies (E_0), (H^+) and (H^-), and $x \mapsto \dfrac{\ell(x)}{||\ell(x)||}$ is constant with $l(x) := h(x, \cdot)$. Thus, f is h-colinfine.

The next proposition is related to the preceding results; however, the assumptions are different.

Proposition 3.16. *Let C be open, let f be radially continuous and $\ell(x) := h(x, \cdot) \in X^* \backslash \{0\}$ for $x \in C$.*

If f is h-quasiconvex then f is h-pseudoconvex.

If f is h-protoaffine then f is h-colinfine and condition (E) is satisfied.

Proof. Let $w, x \in C$ be such that $f(w) < f(x)$. Then $\ell(x)(w - x) \leq 0$ by h-quasiconvexity of f. Suppose that $\ell(x)(w - x) = 0$. Since $\ell(x) \neq 0$, we can find $u \in X$ such that $\ell(x)(u) = 1$. By the radial continuity of f, there exists $\varepsilon > 0$ such that $y := w + \varepsilon u \in C$ and $f(y) < f(x)$. By h-quasiconvexity of f, $0 \geq \ell(x)(y - x) = \varepsilon \ell(x)(u) = \varepsilon > 0$, a contradiction. Hence $\ell(x)(w - x) < 0$. Thus f is h-pseudoconvex.

Since h-protoconvexity implies h-quasiconvexity, if f is h-protoconvex and $\ell(x) \neq 0$, then, by what precedes, f is h-pseudoconvex. Similarly, if f is h-protoaffine then f is h-pseudoaffine, hence h-semiaffine and h-colinfine by Proposition 3.2. The last assertion follows from Remark 3.9. □

We deduce from [42] a characterization of continuous h-colinfine functions; see also [51]. We suppose that X is finite dimensional.

Proposition 3.17. *A continuous (resp. lower semicontinuous), nonconstant function $f : X \to \mathbb{R}$ is h-colinfine for some $h : X \times X \to \mathbb{R}$ if, and only if, there exist a continuous linear form ℓ on X, $\ell \neq 0$ and a continuous (resp. lower semicontinuous) increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g \circ \ell$.*

Proof. The condition is clearly sufficient: setting, for $u, w, x \in X$, $h(x, u) := \ell(u)$ and

$$\lambda(w, x) := \frac{g(\ell(w)) - g(\ell(x))}{\ell(w) - \ell(x)} \text{ for } \ell(w) \neq \ell(x), \quad \lambda(w, x) := 1 \text{ for } \ell(w) = \ell(x)$$

one has $f(w) - f(x) = \lambda(w, x) h(x, w - x)$.

Conversely, let $f : X \to \mathbb{R}$ be l.s.c. (resp. continuous) and h-colinfine, hence quasiaffine. According to [42], [51], there exist a continuous linear form ℓ on X and a l.s.c. (resp. continuous) nondecreasing function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g \circ \ell$. Since f is nonconstant, $\ell \neq 0$. Let us prove that g is increasing. Suppose that there exist $r_1 < r_2$ in $\mathbb{R}$ such that for all $r \in [r_1, r_2]$, $g(r) = g(r_1) = g(r_2)$. Since f is nonconstant, g is nonconstant and then for $t > 1$ large enough, $g(r_1 + t(r_2 - r_1)) > g(r_2)$ or $g(r_2 + t(r_1 - r_2)) < g(r_1)$. Without loss of generality, we can assume that $g(r_1 + t(r_2 - r_1)) > g(r_2)$ for t large enough. Since ℓ is linear and continuous with $\ell \neq 0$, there exist $w, x \in X$ such that $\ell(w) = r_1 < \ell(x) = r_2$. Taking t large enough $\ell(w + t(x - w)) > \ell(x) = r_2$ and then $f(w + t(x - w)) = g(\ell(w + t(x - w))) = g(r_1 + t(r_2 - r_1)) > g(r_1) = f(w) = f(x)$. By Proposition 3.4, f is not h-colinfine. Therefore g is increasing. □

Note that if f is constant, then one can take g constant and $\ell = 0$. Conversely, if g is constant or if $\ell = 0$ then $g \circ l$ is constant.

4 Vector-valued Maps and Preservation Properties

Extensions of the preceding concepts to vector-valued maps can be given. The cases of quasiconvex and pseudoconvex maps have been studied in detail by several authors ([8], [17], [40]...). The cases of colinfine and semiaffine maps seem to be new. Our purpose here is limited. We just intend to use vector-valued concepts in order to get composition properties, and thus some more means for constructing examples. In the sequel, Y is another n.v.s., with dual space Y^*.

Definition 4.1. Given a map $H : VC \to Y$ which is positively homogeneous in its second variable, a map $F : C \to Y$ is said to be H-colinfine at $x \in C$ if there exists some function $\lambda : C \times C \to \mathbb{P}$ such that for all $w \in C$

$$F(w) - F(x) = H(x, \lambda(w, x)(w - x)).$$

It is H-colinfine on C if it is H-colinfine at every $x \in C$.

An immediate composition result can be stated.

Proposition 4.2. *Let X, Y, Z be n.v.s., let C (resp. D) be a convex subset of X (resp. Y) and let $H : VC \to Y$, $K : VD \to Z$. If $F : C \to Y$ and $G : D \to Z$ are H-colinfine on C and K-colinfine on D respectively and if $F(C) \subset D$, then $G \circ F$ is L-colinfine on C, for $L : VC \to Z$ given by $L(x, u) := K(F(x), H(x, u))$ for $(x, u) \in VC$.*

To get an extension of the concepts of h-pseudoconvexity and h-colinvexity, one needs an order structure on Y. Thus we assume that Y is ordered by a closed convex cone Y_+. We denote by Y_+^* the dual cone of Y_+: $Y_+^* := \{y^* \in Y^* : \forall y \in Y_+ \ \langle y^*, y\rangle \geq 0\}$.

Definition 4.3. Given a map $F : C \to Y$.

(a) F is said to be H-pseudoconvex at $x \in C$ if

$$w \in C, \ H(x, w - x) \geq 0 \Longrightarrow F(w) \geq F(x).$$

It is H-pseudoconvex on C if it is H-pseudoconvex at every $x \in C$.

(b) F is said to be H-protoconvex at $x \in C$ if

$$w \in C, \ F(w) \leq F(x) \Longrightarrow H(x, w - x) \leq 0.$$

It is H-protoconvex on C if it is H-protoconvex at every $x \in C$.

(c) F is said to be H-semiconvex at $x \in C$ if it is both H-pseudoconvex and H-protoconvex at x. It is said to be H-protoaffine (resp. H-pseudoaffine) at x if it is H-protoconvex (resp. H-pseudoconvex) at x and if $-F$ is $-H$-protoconvex (resp. $-H$-pseudoconvex) at x.

(d) F is said to be H-colinvex on C if there exists some function $\lambda : C \times C \to \mathbb{P}$ such that for all $w, x \in C$

$$F(w) - F(x) \geq H(x, \lambda(w, x)(w - x)).$$

If $Y_+ := \{0\}$, F is H-colinfine on C if, and only if, F is H-colinvex on C. If F is H-colinvex on C then F is clearly H-pseudoconvex on C.

Let us give another means to extend the previous notions to the vectorial case.

Definition 4.4. A map $F : C \to Y$ is said to be scalarly H-pseudoconvex (resp. scalarly H-protoconvex, scalarly H-semiconvex, scalarly H-pseudoaffine, scalarly H-semiaffine) on C if, for all $y^* \in Y_+^*$, the function $y^* \circ F$ is $y^* \circ H$-pseudoconvex (resp. $y^* \circ H$-protoconvex, $y^* \circ H$-semiconvex, $y^* \circ H$-pseudoaffine, $y^* \circ H$-semiaffine).

Proposition 4.5. *A map $F : C \to Y$ is H-pseudoconvex (resp. H-protoconvex, H-semiconvex, H-pseudoaffine, H-semiaffine) if, and only if, it is scalarly H-pseudoconvex (resp. scalarly H-protoconvex, scalarly H-semiconvex, scalarly H-pseudoaffine, scalarly H-semiaffine).*

Proof. Let us prove that if F is scalarly H-pseudoconvex then it is H-pseudoconvex; the converse is obvious. Let $w, x \in C$ be such that $H(x, w-x) \geq 0$. Then, for all $y^* \in Y_+^*$ we have $(y^* \circ H)(x, w-x) \geq 0$, hence $(y^* \circ F)(w) \geq (y^* \circ F)(x)$. Then, the bipolar theorem ensures that $F(w) \geq F(x)$. The proofs of the other cases are similar. □

Given some n.v.s. X, Y, a subset C of X, we say that a map $F : C \to Y$ is *directionally differentiable* (resp. radially differentiable) at $x \in C$ if for any $u \in T(C, x)$ (resp. $u \in T^r(C, x)$) the quotient $(1/t)(F(x+tv) - F(x))$ (resp. $(1/t)(F(x+tu) - F(x))$) has a limit as $(t, v) \to (0_+, u)$ with $x + tv \in C$ (resp. $t \to 0_+$). The preceding limit is then called the *directional derivative* (resp. the radial derivative) of F at x in the direction u and is denoted by $F'(x, u)$ (resp. $DF(x, u)$). Such a concept is a natural extension of the notion described above for real-valued functions.

If $F : C \to Y$ has a radial or a directional derivative, we say that F is colinfine if H is the radial derivative and F is H-colinfine. We use a similar convention for the other concepts introduced above.

Example 10. Given n.v.s. X, Y, $A \in L(X, Y)$, $y \in Y$, $b \in X^*$, $\beta \in \mathbb{R}$, $C := \{x \in X : b(x) + \beta > 0\}$, the map $F : C \to Y$ given by

$$F(x) := (A(x) + y)/(b(x) + \beta)$$

is a colinfine map on C, as a computation similar to the one of Example 6 shows.

Example 11. Given $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Y_+ := \mathbb{R}_+^2$, let $F : X \to Y$ be given by

$$F(x) := (x, x^3).$$

Then F is quasiconvex but it is not scalarly pseudoconvex as $y^* \circ F$ is not pseudoconvex when y^* is the second projection. Note that F is of class C^∞ and its derivative is never 0.

Remark 4.6. Here, the concepts of H-colinfine map and H-scalarly semiaffine map are different from the one of vector pseudolinear in [61], where the components of the map are supposed to be colinfine. Also, our definition of a pseudoaffine map differs from the one in [8, Definition 4.13], even when one takes for H the derivative of F.

Let us give some preservation properties. The first one is immediate.

Proposition 4.7. *Given $H : VC \to Y$ and H-colinfine (resp. H-scalarly semiaffine) maps $F : C \to Y$, $G : C \to Y$ and $r \in \mathbb{R}$, the map $F + G$ is H-colinfine (resp. H-scalarly semiaffine) and rF is rH-colinfine (resp. rH-scalarly semiaffine).*

Now let us consider composition properties.

Proposition 4.8. *Let X, Y, Z be n.v.s., Y and Z being ordered by closed convex cones Y_+ and Z_+ respectively. Let C (resp. D) be a convex subset of X (resp. Y) and let $F : C \to Y$, $G : D \to Z$, $H : VC \to Y$, $K : VD \to Z$. Assume that $F(C) + Y_+ \subset D$, F is H-colinvex on C and that for every $(y, y') \in VD$, $y'' \in Y_+$ one has $K(y, y' + y'') \geq K(y, y')$. Let $L : VC \to Z$ be given by $L(x, u) := K(F(x), H(x, u))$ for $(x, u) \in VC$.*

(a) *If G is K-pseudoconvex on D, then $G \circ F$ is L-pseudoconvex on C.*

(b) *If G is K-protoconvex on D, then $G \circ F$ is L-protoconvex on C.*

(c) *If G is K-colinvex on D, then $G \circ F$ is L-colinvex on C.*

Proof. (a) Since F is H-colinvex on C, given $w, x \in C$ one can find $\lambda(w,x) \in \mathbb{P}$ such that $F(w) - F(x) = \lambda(w,x)H(x, w-x) + y''$ with $y'' \in Y_+$. Since G is K-pseudoconvex on D, one has

$$\begin{aligned} L(x, w-x) \geq 0 \Rightarrow \quad & K(F(x), H(x, w-x)) \geq 0 \Rightarrow K(F(x), \lambda(w,x)H(x, w-x) + y'') \geq 0 \\ \Rightarrow \quad & K(F(x), F(w) - F(x)) \geq 0 \Rightarrow G(F(w)) \geq G(F(x)). \end{aligned}$$

so that $G \circ F$ is L-pseudoconvex on C.

(b) Given $w, x \in C$ such that $G(F(w)) \leq G(F(x))$ one has $K(F(x), F(w) - F(x)) \leq 0$ by K-protoconvexity of G, hence $K(F(x), H(x, w-x)) \leq 0$ since F is H-colinvex and $K(F(x), \cdot)$ is nondecreasing.

(c) Let $y := F(w)$, $y' := F(x)$. Then, by K-colinvexity of G, there exists some $\mu(y, y') \in \mathbb{P}$ such that $G(y) - G(y') \geq \mu(y, y')K(y, y - y') \geq \mu(y, y')K(y, \lambda(w,x)H(x, w-x))$. □

Remark 4.9. Note that assertion (c) yields Proposition 4.2 by taking $Y_+ := \{0\}$, $Z_+ := \{0\}$.

Taking $Z := \mathbb{R}$, the preceding proposition yields a means to generate more examples of generalized convex functions. Observing that affine mappings being colinfine, the same can be said for precompositions with affine maps.

Now let us give a chain rule in which the first map is not supposed to be colinvex.

Proposition 4.10. *Let $f : C \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and let $h : VC \to \mathbb{R}$, $k : \mathbb{R}^2 \to \mathbb{R}$ be positively homogeneous in their second variables and let $\ell : VC \to \mathbb{R}$ be given by $\ell(x, u) := k(f(x), h(x, u))$.*

(a) *If f is h-pseudoconvex, if g is nondecreasing and if $k(r, -1) < 0$ for all $r \in f(C)$, then $g \circ f$ is ℓ-pseudoconvex.*

(b) *If f is h-protoconvex, if g is increasing and if $D^+g \geq k$, then $g \circ f$ is ℓ-protoconvex;*

(c) *If f is h-semiconvex, if g is increasing and if $k(r, -1) < 0$ for all $r \in f(C)$, then $g \circ f$ is ℓ-semiconvex.*

Note that when f and g have radial derivatives and $h = Df$, $k = Dg$, then ℓ is the radial derivative of $g \circ f$ (observe that the radial derivative of g is also the directional derivative of g).

Proof. (a) Let $w, x \in C$ be such that $\ell(x, w-x) = k(f(x), h(x, w-x)) \geq 0$. Since $k(f(x), -1) < 0$, we have $h(x, w-x) \geq 0$. Since f is h-pseudoconvex we have $f(w) \geq f(x)$ and since g is nondecreasing, we get $(g \circ f)(w) \geq (g \circ f)(x)$. Thus $g \circ f$ is ℓ-pseudoconvex.

(b) Let $w, x \in C$ be such that $\ell(x, w-x) = k(f(x), h(x, w-x)) > 0$. Since g is nondecreasing and $k \leq D^+g$, we have $k(f(x), -1) \leq D^+g(f(x), -1) \leq 0$; thus, we must have $h(x, w-x) > 0$ (recall that $k(f(x), 0) = 0$). Since f is h-protoconvex, we have $f(w) > f(x)$ and since g is increasing, we get $(g \circ f)(w) > (g \circ f)(x)$.

(c) We already know that $g \circ f$ is ℓ-pseudoconvex. Let us prove that $g \circ f$ is ℓ-protoconvex by slightly modifying (b). Let $w, x \in C$ be such that $\ell(x, w-x) = k(f(x), h(x, w-x)) > 0$. Since $k(f(x), -1) < 0$, we have $h(x, w-x) > 0$; thus, as above, we get $(g \circ f)(w) > (g \circ f)(x)$. □

5 Characterizations of Solution Sets

In the present section, we apply the previous concepts to the constrained minimization problem

$$(\mathcal{C}) \qquad \text{minimize } f(x) \text{ subject to } x \in C,$$

where C is a subset of X and $f : C \to \mathbb{R}$. We denote by S the solution of the constrained problem $(\mathcal{C})$. Let $h : VC \to \mathbb{R}$ be a generalized directional derivative of f which is positively homogeneous in the second variable and such that $h(\cdot, 0) = 0$.

We say that f satisfies (H$^-$) at $a \in C$ (relatively to h) if $h(a, x - a) < 0$ for some $x \in C$, then there exists $z \in]a, x[$ such that $f(z) < f(a)$. Thus, f satisfies (H$^-$) if, and only if, f satisfies (H$^-$) at every $a \in C$.

Proposition 5.1. *Let f be h-pseudoconvex. If $a \in C$ is such that $h(a, x - a) \geq 0$ for all $x \in C$, then $a \in S$.*

Conversely, if $a \in S$, if C is starshaped at a, and if f satisfies (H$^-$) at a (in particular, if $h(a, \cdot) \geq D_+ f(a, \cdot)$), then $h(a, x - a) \geq 0$ for all $x \in C$.

Proof. The first assertion follows from the definition of h-pseudoconvexity of f at a.

Conversely, let us suppose f satisfies (H$^-$) at $a \in S$. We must prove $h(a, x - a) \geq 0$, for all $x \in C$. If there exists some $x \in C$ such that $h(a, x - a) < 0$, then by (H$^-$) and starshapedness of C, there exists $z \in]a, x[\subset C$ such that $f(z) < f(a)$: this is a contradiction with the assumption that a is a solution to $(\mathcal{C})$. □

The following variant is a direct consequence of the definition of an h-semiaffine function (or of the fact that f is h-colinfine).

Proposition 5.2. *Suppose that f is h-semiaffine on C and C is starshaped at a. Then*

(a) *$a \in S$ if and only if $h(a, x - a) \geq 0$ for all $x \in C$ and if and only if $h(x, a - x) \leq 0$ for all $x \in C$;*

(b) *If a, b are in S, then $(a + \mathbb{R}(b - a)) \cap C$ is included in S;*

(c) *If a is a solution to $(\mathcal{C})$ then the set S of solutions to $(\mathcal{C})$ is given by $S = S_h$, where $S_h := \{x \in C : h(a, x - a) = 0\}$;*

(d) *If the feasible set C is convex, then a local minimizer to $(\mathcal{C})$ is a global minimizer to $(\mathcal{C})$.*

Let us consider now the case in which the constraint set C is defined by a finite family of inequalities, so that problem $(\mathcal{C})$ turns into the mathematical programming problem

$$(\mathcal{M}) \qquad \text{minimize } f(x) \text{ subject to } x \in C := \{x \in W : g_1(x) \leq 0, ..., g_m(x) \leq 0\},$$

where $f : W \to \mathbb{R}$, $g_i : W \to \mathbb{R}$ and W is an open convex subset of X.

We assume that $h : W \times X \to \mathbb{R}$, $h_i : W \times X \to \mathbb{R}$ ($i \in \mathbb{N}_m := \{1, ..., m\}$) are generalized directional derivatives of f and g_i, respectively, which are positively homogeneous in their second variables and such that $h(\cdot, 0) = h_i(\cdot, 0) = 0$. Given $a \in C$, let $I(a) := \{i \in \mathbb{N}_m : g_i(a) = 0\}$ and let $P(a)$ stand for a set of $i \in I(a)$ such that g_i is h_i-pseudoconcave at a, while $N(a) := I(a) \backslash P(a)$.

Each continuous linear functional x^* on X satisfying $\langle x^*, \cdot \rangle \leq h(x, \cdot)$ is said to be a subderivative of f with respect to h at x. The set $\partial^h f(x)$ of all subderivatives at x is called the *h-subdifferential of f at x*; it is a weak* closed convex subset of X^*. Obviously, for $a \in X$,

$$h(a, u) \geq 0, \ \forall u \in X \quad \Leftrightarrow \quad 0 \in \partial^h f(a).$$

The subdifferential of a convex function $k : X \to \mathbb{R}_\infty$ at $x \in \operatorname{dom} k := k^{-1}(\mathbb{R})$ is the set

$$\partial k(x) := \{x^* : \ k(w) - k(x) \geq \langle x^*, w - x \rangle, \ \forall w \in X\}.$$

The sufficient optimality criteria which follows is an easy consequence of the preceding proposition. If f and g_i $(i \in \mathbb{N}_m)$ are differentiable, then the sufficient optimality condition is the same as the classical result in [41] and [8, Thm. 4.5].

Proposition 5.3. *Let f be h-pseudoconvex at $a \in W$ and let g_i be h_i-protoconvex at a for $i = 1, ..., m$. If a is such that the following conditions are satisfied for some $y_i \in \mathbb{R}_+$, $i = 1, ..., m$, then a is a solution to problem $(\mathcal{M})$:*

$$h(a, x-a) + \sum_{i=1}^{m} y_i h_i(a, x-a) \geq 0, \qquad \forall x \in C, \tag{5.1}$$

$$g_1(a) \leq 0, ..., \; g_m(a) \leq 0, \qquad y_1 g_1(a) = 0, ..., \; y_m g_m(a) = 0. \tag{5.2}$$

Proof. Suppose on the contrary that there exists some $x \in C$ such that $f(x) < f(a)$. By h-pseudoconvexity of f at a, we have $h(a, x-a) < 0$.

Then, for $i \in I(a)$, by h_i-protoconvexity of g_i at a, we have $h_i(a, x-a) \leq 0$. Consequently, we have

$$\begin{aligned} h(a, x-a) &< 0, \\ h_i(a, x-a) &\leq 0, \qquad \forall i \in I(a). \end{aligned}$$

Multiplying each side of the last inequalities by y_i and adding the obtained sides to the ones of the preceding relation, since $y_i = 0$ for $i \notin I(a)$, we get

$$0 \leq h(a, x-a) + \sum_{i=1}^{m} y_i h_i(a, x-a) < 0,$$

a contradiction. □

Corollary 5.4. *Let f be h-pseudoconvex at $a \in X$ and let g_i be h_i-protoconvex at a for $i = 1, ..., m$. If a is such that the following conditions are satisfied for some $y_i \in \mathbb{R}_+$, $i = 1, ..., m$, then a is a solution to problem $(\mathcal{M})$:*

$$\begin{aligned} 0 &\in \partial^h f(a) + y_1 \partial^{h_1} g_1(a) + ... + y_m \partial^{h_m} g_m(a), \\ g_1(a) &\leq 0, ..., g_m(a) \leq 0, \quad y_1 g_1(a) = 0, ..., y_m g_m(a) = 0. \end{aligned}$$

Proof. Let $a^* \in \partial^h f(a)$, $a_i^* \in \partial^{h_i} g_i(a)$ for $i = 1, ..., m$ be such that

$$a^* + y_1 a_1^* + ... + y_m a_m^* = 0.$$

Since for every $x \in C$ and $i \in \mathbb{N}_m$ we have

$$\begin{aligned} \langle a^*, x-a \rangle &\leq h(a, x-a), \\ \langle a_i^*, x-a \rangle &\leq h_i(a, x-a), \end{aligned}$$

multiplying each side of the last inequalities by y_i and adding the obtained sides to the ones of the preceding relation, since $y_i = 0$ for $i \notin I(a)$, we get

$$0 = \langle a^*, x-a \rangle + \sum_{i=1}^{m} y_i \langle a_i^*, x-a \rangle \leq h(a, x-a) + \sum_{i=1}^{m} y_i h_i(a, x-a)$$

and the preceding proposition applies. □

Now let us give a necessary optimality condition in a limiting form. We assume now that X is a Banach space.

Proposition 5.5. *Let $a \in S$. Assume that h, h_i $(i \in I(a))$ are finite, sublinear and l.s.c. in their second variables. Assume that f satisfies condition (H^-) at a (relatively to h), g_i are u.s.c. at a for $i \notin I(a)$. Suppose that for $i \in N(a)$, g_i satisfies condition (H^-) at a (relatively to h_i) and g_i is quasiconvex at a. Then there exists $(y_0, (y_n)) \in \mathbb{R}_+ \times \mathbb{R}_+^{N(a)}$, with $(y_0, (y_n)) \neq 0$ such that*

$$0 \in \mathrm{cl}\{y_0 \partial^h f(a) + \sum_{n \in N(a)} y_n \partial^{h_n} g_n(a) + \sum_{p \in P(a)} y_p \partial^{h_p} g_p(a) : \ (y_p) \in \mathbb{R}_+^{P(a)}\}. \tag{5.3}$$

In particular, if $\partial^{h_i} g_i(a) \neq \varnothing$ for $i \in \mathbb{N}_m \backslash I(a)$ then

$$0 \in \mathrm{cl}\{y_0 \partial^h f(a) + \sum_{i=1}^{m} y_i \partial^{h_i} g_i(a) : y_i \in \mathbb{R}_+, \ y_i g_i(a) = 0, \ y_0 + \sum_{n \in N(a)} y_n = 1\}.$$

Proof. Let $a \in S$. In a first step, we prove that the system

$$\begin{pmatrix} h(a, x-a) & < 0 & \\ h_i(a, x-a) & < 0 & \forall i \in N(a) \\ h_i(a, x-a) & \leq 0 & \forall i \in P(a) \end{pmatrix}$$

has no solution x in X.

Let us suppose, on the contrary, that x satisfy these inequalities. Then, since W is open, there exists $\delta > 0$ such that $x_t := a + t(x-a) \in W$ for $t \in [0, \delta]$. Thus, for $i \in P(a)$ one has $h_i(a, x_t - a) \leq 0$, so that $g_i(x_t) \leq g_i(a)$ for $t \in]0, \delta]$ by h_i-pseudoconcavity of g_i at a. For $i \in N(a)$, and $t \in]0, \delta]$, since $h_i(a, x_t - a) < 0$, by condition (H$^-$) there exists $z_t := x_s \in]x_t, a[\cap W$ such that $g_i(z_t) < g_i(a)$. In addition, as g_i is quasiconvex, one has $g_i(x_r) \leq g_i(a)$ for all $r \in [0, s]$. Moreover, since for $i \notin I(a)$, $g_i(a) < 0$ and g_i is u.s.c, we can take s small so that $g_i(x_r) < 0$ for $r \in [0, s]$. Since $h(a, x_s - a) < 0$, by condition (H$^-$) we get $f(x_r) < f(a)$ for some $r \in [0, s]$, a contradiction with $a \in S$. Our claim is proved.

In a second step, we prove that relation (5.3) is satisfied. According to the first step, the system

$$\begin{pmatrix} -h(a, x-a) & > 0 & \\ -h_n(a, x-a) & > 0 & \forall n \in N(a) \\ -h_p(a, x-a) & \geq 0 & \forall p \in P(a) \end{pmatrix}$$

has no solution x in X. Since $h(a, \cdot)$ and $h_i(a, \cdot)$ are sublinear and l.s.c., using the generalized Farkas lemma [21, Theorem 3], there exists $0 \neq (y_0, (y_n)) \in \mathbb{R}_+ \times \mathbb{R}_+^{N(a)}$ such that

$$0 \in \mathrm{cl}\{\partial\left(y_0 h(a, \cdot) + \sum_{n \in N(a)} y_n h_n(a, \cdot)\right)(0) + \partial\left(\sum_{p \in P(a)} y_p h_p(a, \cdot)\right)(0) : (y_p) \in \mathbb{R}_+^{P(a)}\}.$$

Since the Attouch-Brézis qualification condition for a sum of convex functions is fulfilled, this condition can be rewritten as in (5.3), taking into account the notation $\partial^h f(a) = \partial h(a, \cdot)(0)$ and $\partial^{h_i} g_i(a) = \partial h_i(a, \cdot)(0)$ for $i \in I(a)$. The last assertion is obtained by taking $y_i = 0$ for $i \in \mathbb{N}_m \backslash I(a)$ and by using a normalization, replacing $(y_0, (y_i))$ by $(t^{-1} y_0, (t^{-1} y_i))$, where $t := y_0 + \sum_{n \in N(a)} y_n$. □

In the case we can take for $P(a)$ the empty set, we get the following consequence.

Corollary 5.6. *Let $a \in S$. Assume that h, h_i $(i \in I(a))$ be finite, sublinear and l.s.c. in their second variables. Assume that f satisfies condition (H^-) at a (relatively to h), g_i are u.s.c. at a for $i \notin I(a)$. Suppose that for $i \in I(a)$, g_i satisfy condition (H^-) at a (relatively to h_i) and g_i is quasiconvex at a. Then there exists $(y_0, (y_i)) \in \mathbb{R}_+ \times \mathbb{R}_+^{I(a)}$ such that $(y_0, (y_i)) \neq 0$ and*

$$0 \in \mathrm{cl}\{y_0 \partial^h f(a) + \sum_{i \in I(a)} y_i \partial^{h_i} g_i(a)\}.$$

Moreover, if the qualification condition

$$0 \notin \mathrm{clco}(\bigcup_{i \in I(a)} \partial^{h_i} g_i(a)) \tag{5.4}$$

holds, then there exists $(y_i) \in \mathbb{R}_+^{I(a)}$ such that

$$0 \in \partial^h f(a) + \sum_{i \in I(a)} y_i \partial^{h_i} g_i(a).$$

Proof. The first assertion is a direct consequence of the preceding proposition, with $N(a) = I(a)$, $P(a) = \varnothing$. The second assertion stems from the fact that if $y_0 = 0$, replacing (y_i) by $(t^{-1} y_i)$ with $t := \sum_{i \in I(a)} y_i$ we get $0 \in \mathrm{cl}(\sum_{i \in I(a)} y_i \partial^{h_i} g_i(a)) \subset \mathrm{clco}(\bigcup_{i \in I(a)} \partial^{h_i} g_i(a))$, a contradiction with (5.4). □

The case $P(a) = I(a)$, $N(a) = \varnothing$ is considered in the following statement; then we have $y_0 \neq 0$, or, equivalently, $y_0 = 1$ after normalization.

Corollary 5.7. *Let $a \in S$. Assume that h, h_i $(i \in I(a))$ are sublinear and l.s.c. in their second variables. Assume that f satisfies condition (H^-) at a (relatively to h), g_i are h_i-pseudoconcave at a for $i \in I(a)$ and g_i is u.s.c at a for $i \notin I(a)$. Then*

$$0 \in \mathrm{cl}(\{\partial^h f(a) + y_1 \partial^{h_1} g_1(a) + ... + y_m \partial^{h_m} g_m(a) : y_i \geq 0,\ y_i g_i(a) = 0\}). \tag{5.5}$$

Let us gather a necessary condition deduced from the preceding statement with the sufficient condition of Corollary 5.4.

Corollary 5.8. *Let $a \in W$. Assume that f is h-pseudoconvex at a and g_i is h_i-colinfine at a for some bifunctions $h : W \times X \to \mathbb{R}$, $h_i : W \times X \to \mathbb{R}$ $(i \in \{1, ..., m\})$ which are such that $h(a, \cdot)$ and $h_i(a, \cdot)$ are positively homogeneous and $h(a, \cdot) \geq D_+ f(a, \cdot)$. Assume that g_i is u.s.c for $i \notin I(a)$. Then $a \in C$ is a solution to $(\mathcal{M})$ if there exists $y := (y_1, ..., y_m) \in \mathbb{R}_+^m$ such that $y_i g_i(a) = 0$ for $i = 1, ..., m$ and*

$$0 \in \partial^h f(a) + y_1 \partial^{h_1} g_1(a) + ... + y_m \partial^{h_m} g_m(a). \tag{5.6}$$

If $h(a, \cdot)$ and $h_i(a, \cdot)$ $(i \in I(a))$ are sublinear and continuous and if the following qualification condition holds, then condition (5.6) is necessary for a to be in S :

$$0 \notin \mathrm{co}(\bigcup_{i \in I(a)} \partial^{h_i} g_i(a)). \tag{5.7}$$

Proof. The sufficient condition is obvious from Corollary 5.4. For the necessary condition, since $h(a, \cdot) \geq D_+ f(a, \cdot)$, f satisfies condition (H^-) at a. Given a sequence (ε_n) in $\mathbb{P}_+$ with limit 0, relation (5.5) yields for each $i \in I(a)$ some sequences $(y_{i,n})$ in $\mathbb{R}_+$, (y_n^*), $(z_{i,n}^*)$ in X^* such that $y_n^* \in \partial^h f(a)$, $z_{i,n}^* \in \partial^{h_i} g_i(a)$ for all $n \in \mathbb{N}$ and

$$\left\| y_n^* + \sum_{i \in I(a)} y_{i,n} z_{i,n}^* \right\| \leq \varepsilon_n. \tag{5.8}$$

Let us show that the sequences $(y_{i,n})$ are bounded. If one of them is unbounded, taking a subsequence if necessary, we may assume that $(\|y_n\|) \to +\infty$, where $y_n = (y_{i,n})_{i \in I(a)} \in \mathbb{R}_+^{I(a)}$, the space $\mathbb{R}^{I(a)}$ being endowed with the norm given by $\|y\| := \sum_{i \in I(a)} |y_i|$. Since for $i \in I(a)$ the functions $h_i(a, \cdot)$ are sublinear and continuous, the sets $\partial^{h_i} g_i(a)$ are weak* compact; $\partial^h f(a)$ is also weak* compact. It follows that we can find a limit point (u, z^*) of the sequence (u_n, z_n^*), where $u_n := y_n / \|y_n\| \in \mathbb{R}_+^{I(a)}$. Dividing by $\|y_n\|$ both sides of relation (5.8) and taking limits we get

$$\left\| \sum_{i \in I(a)} u_i z_i^* \right\| = 0.$$

Since $z_i^* \in \partial^{h_i} g_i(a)$ and $u := (u_i)_{i \in I(a)}$ is in the canonical simplex of $\mathbb{R}^{I(a)}$, we get a contradiction with (5.7).

Since the sequence (y_n) is bounded, and since (y_n^*), $(z_{i,n}^*)$ are also bounded in X^*, we can take weak* limit points y, y^*, z_i^* and pass to the limit in (5.8). Thus, we get $y \in \mathbb{R}_+^{I(a)}$, $y^* \in \partial^h f(a)$, $z_i^* \in \partial^{h_i} g_i(a)$ such that

$$y^* + \sum_{i \in I(a)} y_i z_i^* = 0.$$

Hence condition (5.6) is satisfied. □

The next example illustrates the corollary; note that since f is not differentiable at $a = 0$, [41, page 153] cannot be applied. Since f is not a Plastria function at a, [38, Theorem 10] cannot be applied either. Moreover, in Example 13 f is not locally Lipschitz at a; thus [43, Prop. 6.1] and [44, Prop. 6.3] cannot be applied. Finally, let us note the usefulness of admitting that the bifunction h may differ from the Dini derivatives of f.

Example 12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for $x > 0$, $f(x) = -x^2$ for $x \leq 0$ and let $g_1 : \mathbb{R} \to \mathbb{R}$ be given by $g_1(x) = -x$. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $h(0, u) = |u|$ for $u \in \mathbb{R}$ and let $h_1 := g_1'$. Then f is h-pseudoconvex and $h(a, \cdot) \geq D_+ f(a, \cdot)$ at $a = 0$ and g_1 is colinfine. We can take $y_1 = 1$ as a multiplier since $\partial^h f(a) = [-1, 1]$, $\partial g_1(a) = \{-1\}$.

Example 13. One may take for f a continuous function, for instance f given by $f(x) = x$ for $x > 0$, $f(x) = -\sqrt{-x}$ for $x \leq 0$, with g_1 and $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as in the preceding example.

In convex mathematical programming, the multipliers are the same for all solutions. A related result has been given in [16] in the case the function f is pseudoaffine, differentiable, $h := f'$ and the functions g_i $(i \in \{1, ..., m\})$ are linear. Here f and g_i are nonlinear and even nonsmooth.

Proposition 5.9. *Let $a \in S$ be such that there exists $y = (y_i) \in \mathbb{R}_+^m$ satisfying the optimality conditions (5.1) and (5.2).*

(a) *If f is h-colinvex at a and g_i is h_i-protoconvex at a, then $y_i h_i(a, b - a) = 0$ for all $i \in I(a)$ and every $b \in S$.*

(b) *Let f be h-colinvex at a and g_i be h_i-colinvex at a. Let L be the Lagrangian given by $L(x,z) := f(x) + zg(x)$ for $(x,z) \in C \times \mathbb{R}^m$ and let $b \in C$. Then, $b \in S$ if, and only if, $yg(b) = 0$ and $L(a,y) = L(b,y)$.*

Proof. (a) Let $b \in S$. Since f is h-colinvex at a, there exists $\lambda(b,a) \in \mathbb{P}$ such that $0 = f(b) - f(a) \geq \lambda(b,a)h(a, b-a)$. By condition (5.1), one has

$$\sum_{i \in I(a)} y_i h_i(a, b-a) \geq -h(a, b-a) \geq 0.$$

On the other hand, since g_i is h_i-protoconvex at a and for all $i \in I(a)$, $g_i(b) \leq g_i(a) = 0$, one has $h_i(a, b-a) \leq 0$ for all $i \in I(a)$. Hence $\sum_{i \in I(a)} y_i h_i(a, b-a) = 0$ and since each term of the sum is non positive, one has $y_i h_i(a, b-a) = 0$ for all $i \in I(a)$.

(b) Let $b \in S$. Since g_i is h_i-colinvex at a then there exists $\mu_i(b,a) \in \mathbb{P}$ such that for any $i \in I(a)$,

$$g_i(b) = g_i(b) - g_i(a) \geq \mu_i(b,a) h_i(a, b-a).$$

By (a), $y \in \mathbb{R}^m_+$ and $b \in C$, $0 \geq yg(b) \geq \sum_{i \in I(a)} \mu_i(b,a) y_i h_i(a, b-a) = 0$ and thus $yg(b) = 0$ and $L(a,y) := f(a) + yg(a) = f(a) = f(b) = f(b) + yg(b) =: L(b,y)$.

Conversely, let $b \in C$ be such that $yg(b) = 0$ and $L(a,y) = L(b,y)$. Then $f(a) = L(a,y) = L(b,y) = f(b) + yg(b) = f(b)$, hence $b \in S$. $\square$

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