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HALFSPACES AND HAHN-BANACH LIKE PROPERTIES IN $\mathbb{B}$-CONVEXITY AND MAX-PLUS CONVEXITY

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Abstract: $\mathbb{B}$-convexity was introduced in [2]. Continuing the investigation initiated in [3] we give complete geometric and analytic characterizations of closed and open $\mathbb{B}$-halfspaces, that is, $\mathbb{B}$-convex sets whose complements are also $\mathbb{B}$-convex. Combined with results of [3] this yields a Hahn-Banach like Theorem in $\mathbb{B}$-convexity which is the exact analog of the usual Hahn-Banach Theorem. Similar results are given in the context of Max-Plus convexity.

Key words: *Generalized-convexity, $\mathbb{B}$-convexity, Max-Plus convexity, halfspaces, geometric and functional separation, Hahn-Banach like theorems*

Mathematics Subject Classification: *06D50, 32F17*

1 Introduction

We will denote by $\mathbb{R}^n_+$, respectively $\mathbb{R}^n_{++}$, the set of elements of $\mathbb{R}^n$ whose coordinates are positive, respectively, strictly positive. For x and y in $\mathbb{R}^n$ we write $x \leq y$ if $y - x \in \mathbb{R}^n_+$. If $x \leq y$ then $[x, y]$ is the set of elements z of $\mathbb{R}^n$ such that $x \leq z \leq y$. If $x \leq y$ the sets $[x, y]\setminus\{y\}$, respectively $[x, y]\setminus\{x, y\}$ or $[x, y]\setminus\{x\}$, will be written $[x, y[$, respectively, $]x, y[$ or $]x, y]$. We recall that for $n \geq 1$ $z \in [x, y[$ does not imply $y - z \in \mathbb{R}^n_{++}$. Given arbitrary vectors $y_1, y_2, \cdots y_m \in \mathbb{R}^n$, we denote by $\bigvee_{i=1}^m y_i$, or by $y_1 \vee y_2 \vee \cdots \vee y_m$, the vector $(\max_{1\leq i\leq m}\{y_{i,1}\}, \cdots, \max_{1\leq i\leq m}\{y_{i,n}\})$. Given a numbered family $\{B_1, \cdots, B_m\}$ of subsets of $\mathbb{R}^n$ and a numbered family of real numbers $\{t_1, \cdots, t_m\}$ we denote by $\bigvee_{i=1}^m t_i B_i$ the set $\{\bigvee_{i=1}^m t_i x_i : \forall i \ x_i \in B_i\}$.

$\mathbb{B}$-convexity made its first appearance in [2]. A subset B of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if for all $x_1, x_2, \cdots x_m \in B$ and all $(t_1, t_2, \cdots, t_m) \in [0, 1]^m$ such that
$\max\{t_1, t_2, \cdots, t_m\} = 1$ one has $t_1x_1 \vee t_2x_2 \vee \cdots \vee t_mx_m \in B$.

As one can easily see a subset B of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if, for all $x_1, x_2 \in B$ and all $t \in [0, 1]$ one has

$$tx_1 \vee x_2 \in B.$$

Hence all $\mathbb{B}$-convex sets are contractible upper semilattices*. This property of $\mathbb{B}$-convex sets is important for matters related to fixed points and selection theorems, as shown for example in [4] or [6].

The $\mathbb{B}$-convex hulls of two or more points are depicted in the following two figures.

*A topological space X is contractible if there exists a continuous map $h : [0, 1] \times X \to X$ such that $h(0, -) : X \to X$ is a constant map and $h(1, -) : X \to X$ is the identity map. A subset X of $\mathbb{R}^n_+$ is a semilattice, more exactly, a sup-semilattice, if $X \vee X \subset X$.

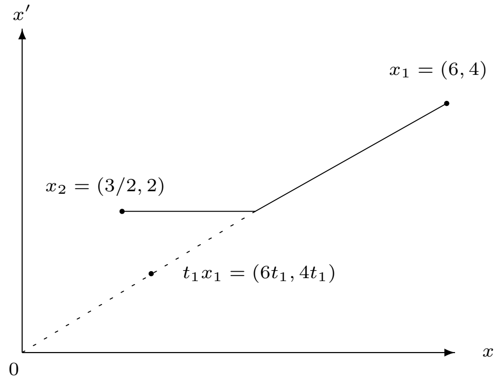

Figure 1.1 Construction of the $\mathbb{B}$-convex hull of two points.

Fix $x_1 = (6,4)$ and $x_2 = (\frac{3}{2}, 2)$ in $\mathbb{R}^2_+$ (see Figure 1.1). The $\mathbb{B}$-convex hull of x_1 and x_2 is $\mathbb{B}[\{x_1, x_2\}] = \{t_1 x \vee t_2 x_2 : t_1, t_2 \geq 0, \max\{t_1, t_2\} = 1\}$. Clearly, if $t_1 = 1$ then $t_1 x_1 \vee t_2 x_2 = (6,4) \vee t_2(\frac{3}{2}, 2) = (6,4) = x_1$. Suppose that $0 \leq t_1 \leq 1$, we have:

$$t_1 x_1 \vee x_2 = \begin{cases} (6,2) & \text{if} \quad t_1 = 1 \\ (6t_1, 4t_1) & \text{if} \quad \frac{1}{2} \leq t_1 \leq 1 \\ (6t_1, 2) & \text{if} \quad \frac{1}{4} \leq t_1 < \frac{1}{2} \\ (\frac{3}{2}, 2) & \text{if} \quad 0 \leq t_1 \leq \frac{1}{4} \end{cases} \tag{1.1}$$

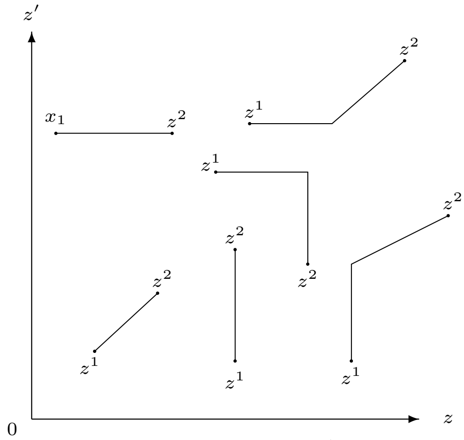

Figure 1.2 $\mathbb{B}$-convex hulls of 2 points.

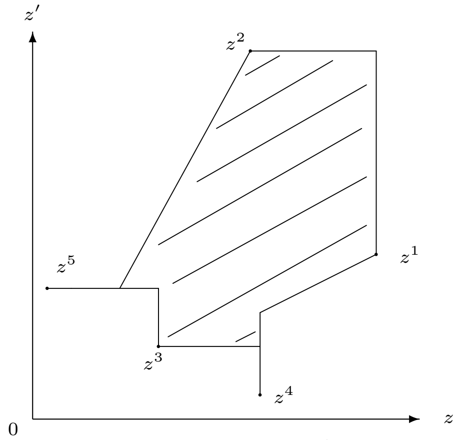

Figure 1.3 $\mathbb{B}$-convex hull of 5 points.

Separation of $\mathbb{B}$-convex sets and the study of Hahn-Banach like properties was initiated in [3] where the emphasis was on gauges and $\mathbb{B}$-halfspaces. By definition, a $\mathbb{B}$-halfspace, or simply a **halfspace**, is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ whose complement in $\mathbb{R}^n_+$ is also $\mathbb{B}$-convex. The sets $\{0\}$, $\mathbb{R}^n_+$ and their complements in $\mathbb{R}^n_+$ are the **trivial halfspaces**.

It has been shown in [3] that a closed $\mathbb{B}$-convex set is always the intersection of the closed halfspaces in which it is contained and that separation of disjoint $\mathbb{B}$-convex sets can be achieved with a class a uniformly continuous maps which are to $\mathbb{B}$-convexity what maps which are simultaneously quasiconvex and quasiconcave are to the usual linear convexity.

In [3] the structure of halspaces has only been partially elucidated and the analytic separation theorems presented there can not claim to be complete analogs of the Hahn-Banach Theorem; gauges introduce infinities and separation is done with quasiaffine like maps.

We give a complete geometric description of closed and open halfspaces and we show that all closed halfspaces $H \subset \mathbb{R}^n_+$ containing 0 and having nonempty interior are of the form

$$H = \{x \in \mathbb{R}^n_+ : \psi(x) \le 0\}$$

with $\psi(x) = \max_{i\in I}\{a_i x_i\} - \max_{j\in J}\{a_i x_i, s\}$, where I and J are disjoint subsets of $\{1, \cdots, n\}$, $(a_1, \cdots, a_n) \in \mathbb{R}^n_+$ and $s \in \mathbb{R}_+$, Theorem 5.2, 6.6 and 7.2.

If the interior of H is empty then there is a third subset K of $\{1, \cdots, n\}$ such that for all $k \in K$, $x_k = 0$.

We show that closed halfspaces with nonempty interior are equal to the closure of their interior, which is not the case for arbitrary closed $\mathbb{B}$-convex sets with nonempty interior. Combining this with the description of closed halfspaces yields the analytic description of open halfspaces which, with the separation theorem for nonproximate $\mathbb{B}$-convex sets from [3], results in a Hahn-Banach like theorem for $\mathbb{B}$-convex sets, Theorem 8.1.

From a more general standpoint, which we will not pursued here, $\mathbb{R}^n_+$ is a topological partially ordered set endowed with a continuous multiplication by positive real numbers. The topology in question in this paper is of course the induced topology. In $\mathbb{R}^2_+$ for example the vector $(0, 1)$ is in the interior of the set $B = \{(x_1, x_2) : x_1 \le x_2\}$.

Max-Plus convexity is a well known and important example of a nonlinear convexity. Let $\mathbb{M}^n = \big(\mathbb{R} \bigcup \{-\infty\}\big)^n$ and denote by $\mathbb{1}_n$ the vector of $\mathbb{R}^n$ whose coordinates are all equal to 1. A subset C of $\mathbb{M}^n$ is **Max-Plus convex** if, for all x and y in C and all $t \in [-\infty, 0]$, $\big(x + t\mathbb{1}_n\big) \vee y \in C$. Max-Plus and $\mathbb{B}$-convexity are examples of Maslov semi-modules. Idempotent analysis, or the study of Maslov semi-modules, has applications in optimization, optimal control, and game theory.

In the last section we show how $\mathbb{B}$-convexity and Max-Plus convexity are related to one another and why all the results presented in the following pages can be interpreted either in the framework of $\mathbb{B}$-convexity, that is as they are given here, or in the framework of Max-Plus convexity, after a simple translation.

2 Preliminary Material

We denote by $[n]$ the set $\{1, \cdots, n\}$ and by $\|x\|$ the l_∞ norm of $x \in \mathbb{R}^n$, that is $\|x\| = \max\{|x_1|, \cdots, |x_n|\}$; the associated metric on $\mathbb{R}^n_+$ is simply denoted by d. For $x \in \mathbb{R}^n_+$ the sets $B[x, \delta\,] = \{y \in \mathbb{R}^n_+ : \|x - y\| \le \delta\}$ and $B(x, \delta) = \{y \in \mathbb{R}^n_+ : \|x - y\| < \delta\}$ are $\mathbb{B}$-convex.

Lemma 2.1. *For all vectors x, y, x', y' in $\mathbb{R}^n_+$ and all μ, ρ in $\mathbb{R}_+$:*

$$d(\rho x' \vee \mu y', \rho x \vee \mu y) \le \max\{\rho d(x, x'), \mu d(y, y')\}$$

Proof. Since, for all $t \ge 0$ and all x and x' in $\mathbb{R}^n_+$, $d(tx, tx') = td(x, x')$ we can assume that $\rho = \mu = 1$. Also, for all $a, b, c, d \in \mathbb{R}_+$, $|\max\{a, b\} - \max\{c, d\}| \le \max\{|a - c|, |b - d|\}$. □

A map $\varphi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a $\mathbb{B}$-convex map if, for all $x, y \in \mathbb{R}^n_+$ and all $\rho, \mu \in [0, 1]$ such that $\max\{\rho, \mu\} = 1$, $\varphi(\rho x \vee \mu y) \le \max\{\rho\varphi(x), \mu\varphi(y)\}$.

The following lemma says that the distance function to a closed set $B \subset \mathbb{R}^n_+$ is a $\mathbb{B}$-convex map if and only if B is a $\mathbb{B}$-convex set.

Lemma 2.2. *If a non empty subset B of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex then, for all $x, y \in \mathbb{R}^n_+$ and for all $\rho, \mu \in [0, 1]$ such that $\max\{\rho, \mu\} = 1$, the following inequality holds*

$$(\star) \quad d(\rho x \vee \mu y, B) \le \max\{\rho d(x, B), \mu d(y, B)\}.$$

Furthermore, if $(\star)$ holds and if B is closed then it is $\mathbb{B}$-convex.

Proof. For $\varepsilon > 0$ choose x_ε and y_ε in B such that $d(x,B) > d(x,x_\varepsilon) - \varepsilon$ and $d(y,B) > d(y,y_\varepsilon) - \varepsilon$. If B is $\mathbb{B}$-convex then $\rho x_\varepsilon \vee \mu y_\varepsilon \in B$ therefore

$$d(\rho x \vee \mu y, B) \leq d(\rho x \vee \mu y, \rho x_\varepsilon \vee \mu y_\varepsilon).$$

From Lemma 2.1 we have

$$(\sharp) \quad \begin{aligned} d(\rho x \vee \mu y, \rho x_\varepsilon \vee \mu y_\varepsilon) &\leq \max\{\rho d(x,x_\varepsilon), \mu d(y,y_\varepsilon)\} \\ &\leq \max\{\rho d(x,B) + \rho\varepsilon, \mu d(y,B) + \mu\varepsilon\}. \end{aligned}$$

From $(\sharp)$ and taking into account that ε was arbitrary we obtain $(\star)$.

If inequality $(\star)$ holds and if the right hand side is zero then the left hand side is also zero. This shows that B is $\mathbb{B}$-convex if it closed and $(\star)$ holds. □

Proposition 2.3. *The interior and the closure of a $\mathbb{B}$-convex set are $\mathbb{B}$-convex sets, the interior and the closure of a halfspace are halfspaces.*

Proof. If B is a $\mathbb{B}$-convex set then inequality $(\star)$ of Lemma 2.2 holds, the distance of a point to a set is identical to the distance to the closure of the set. The second part of Lemma 2.2 implies that $\overline{B}$ is a $\mathbb{B}$-convex set. One could have reached the same conclusion using the continuity of the map $(x,y) \mapsto x \vee y$.

If x and y are in the interior of B we can choose $\delta > 0$ such that $B(x,\delta) \subset B$ and $B(y,\delta) \subset B$. For a fixed, but arbitrary $t \in]0,1]$ we have to show that $tx \vee y$ belongs to the interior of B. We have $tB(x,\delta) \vee B(y,\delta) \subset B$, since B is $\mathbb{B}$-convex. We show that there exists $\eta > 0$ such that $B(tx \vee y, \eta) \subset tB(x,\delta) \vee B(y,\delta)$. Let

$$\begin{aligned} I_+ &= \{i \in [n] : tx_i < y_i\} \\ I_- &= \{i \in [n] : tx_i > y_i\} \\ I_0 &= \{i \in [n] : tx_i = y_i\} \end{aligned}$$

Choose $\eta > 0$ such that:

$$\begin{array}{c} \eta < t\delta \\ \forall i \in I_+ \quad y_i - \eta > tx_i \\ \forall i \in I_- \quad tx_i - \eta > y_i \end{array}$$

Given $w \in B(tx \vee y, \eta)$ define two elements u and v of $\mathbb{R}^n_+$ as follows:

$$u_i = \begin{cases} x_i & i \in I_+ \\ t^{-1}w_i & i \in I_- \cup I_0 \end{cases} \qquad v_i = \begin{cases} y_i & i \in I_- \\ w_i & i \in I_+ \cup I_0 \end{cases}$$

If $i \in I_+$ then $|u_i - x_i| = 0$, if $i \in I_- \cup I_0$ then $\max\{tx_i, y_i\} = tx_i$ and, from $w \in B(tx \vee y, \eta)$, $|u_i - x_i| < t^{-1}\eta$. This shows that $u \in B(x,\delta)$.

If $i \in I_-$ then $|v_i - y_i| = 0$, if $i \in I_+ \cup I_0$ then $\max\{tx_i, y_i\} = y_i$ and, from $w \in B(tx \vee y, \eta)$, $|v_i - y_i| < \eta$. This shows that $v \in B(y,\delta)$.

To conclude this part of the proof let us see that $w = tu \vee v$. We have

$$(tu \vee v)_i = \begin{cases} \max\{tx_i, w_i\} & \text{if} \quad i \in I_+ \\ \max\{y_i, w_i\} & \text{if} \quad i \in I_- \\ w_i & \text{if} \quad i \in I_0 \end{cases}$$

For $i \in I_+$ we have $y_i - \eta > tx_i$ and $\max\{tx_i, y_i\} = y_i$. Furthermore, from $w \in B(tx \vee y, \eta)$, $w_i > \max\{tx_i, y_i\} - \eta$, and therefore, for $i \in I_+$, $\max\{tx_i, w_i\} = w_i$. Similarly, for $i \in I_-$, $\max\{y_i, w_i\} = w_i$.

If $H \subset \mathbb{R}^n_+$ is a $\mathbb{B}$-halfspace then $\overline{H}$ as well as $\mathrm{int}(\mathbb{R}^n_+ \setminus H)$ are $\mathbb{B}$-convex. This proves that $\overline{H}$ is a halfspace since $\mathrm{int}(\mathbb{R}^n_+ \setminus H) = \mathbb{R}^n_+ \setminus \overline{H}$. Similarly, $\mathrm{int}H$ and $\overline{(\mathbb{R}^n_+ \setminus H)} = \mathbb{R}^n_+ \setminus \mathrm{int}H$ are $\mathbb{B}$-convex. □

Proposition 2.3, or at least the results it contains, can also be found, with a somewhat different proof in [2].

Lemma 2.4. *A subset B of $\mathbb{R}^n_+$ containing 0 is $\mathbb{B}$-convex if and only if it is a semilattice which is starshaped* † *at* 0 .

Proof. If B is starshaped at 0 then, for all $t \in [0,1]$ and all $x \in B$, $tx \in B$; if B is also a semillatice then one also has, for all $y \in B$, $tx \vee y \in B$. This shows that a semilattice which is starshaped at 0 is $\mathbb{B}$-convex.

Reciprocally, if B is $\mathbb{B}$-convex and contains 0 then, for all $t \in [0,1]$ and all $x \in B$, $0 \vee (tx) \in B$; but $0 \vee (tx) = tx$. This shows that B is starshaped at zero. Furthermore, taking $t = 1$ in the definition of a $\mathbb{B}$-convex set we have, for all x and y in B, $x \vee y \in B$. □

We recall that an element $x^\star$ of a set $B \subset \mathbb{R}^n_+$ is a **maximal element** of B if for all $y \in B \backslash \{x^\star\}$ it is false that $x^\star \leq y$. An element $x^\star$ of B is a **largest element** if, for all $y \in B$, $y \leq x^\star$. A set B has at most one largest element but it can have many maximal elements.

Lemma 2.5. *A nonempty compact $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ has a largest element.*

Proof. Since $B \subset \mathbb{R}^n_+$ is compact there is a point $x^\star \in B$ such that $\sum_{i=1}^n x_i^\star = \max \Big\{ \sum_{i=1}^n x_i : x \in B \Big\}$. For all $x \in B$ we have $x^\star \vee x \in B$ and therefore $\sum_{i=1}^n x_i^\star \geq \sum_{i=1}^n \max\{x_i^\star, x_i\}$. From $x^\star \vee x \geq x^\star$ we also have $\sum_{i=1}^n \max\{x_i^\star, x_i\} \geq \sum_{i=1}^n x_i^\star$ and consequently $\sum_{i=1}^n \max\{x_i^\star, x_i\} = \sum_{i=1}^n x_i^\star$. Since all the quantities involved are non negative we must have, for all $i \in [n]$, $x_i^\star = \max\{x_i^\star, x_i\}$. □

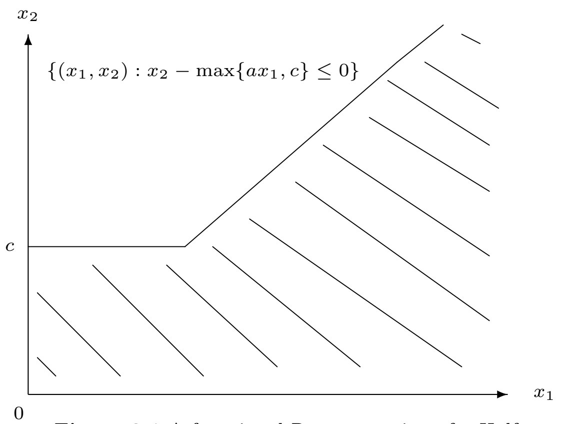

Figure 2.1 A functional Representation of a Halfspace.

Lemma 2.6. *A halfspace B of $\mathbb{R}^n_+$ has nonempty interior if and only if $B \cap \mathbb{R}^n_{++} \neq \emptyset$.*

Proof. One implication is obvious, since $\mathbb{R}^n_+ \backslash \mathbb{R}^n_{++}$ has empty interior. To prove the nontrivial part, let $u = (u_1, \cdots, u_n) \in B \cap \mathbb{R}^n_{++}$. Firts we show that :

$$(\star) \quad \begin{cases} (1) & \text{either, for all } t > 0,\ (u_1 + t, u_2, \cdots, u_n) \in B \quad \text{or} \\ (2) & \text{for all } s \in]0, u_1[,\ (u_1 - s, u_2, \cdots, u_n) \in B \end{cases}$$

If neither (1) nor (2) is the case then there exist $t_0 > 0$ and $s_0 \in]0, u_1[$ such that $x = (u_1 + t_0, u_2, \cdots, u_n) \in \mathbb{R}^n_+ \backslash B$ and $y = (u_1 - s_0, u_2, \cdots, u_n) \in \mathbb{R}^n_+ \backslash B$. Let $\rho = u_1(u_1 + t_0)^{-1}$,

† B is starshaped at 0 if, for all $x \in B$ and all $t \in [0,1]$, $tx \in B$.

it belongs to $[0,1]$, and since $\mathbb{R}^n_+ \setminus B$ is $\mathbb{B}$-convex we must have $\rho x \vee y \in \mathbb{R}^n_+ \setminus B$. But $\rho x \vee y = (u_1, u_2, \cdots, u_n)$ which is in B. This proves $(\star)$.

If (1) holds put $v_1 = 2u_1$, if (2) holds put $v_1 = \dfrac{u_1}{2}$ and set $\delta_1 = \dfrac{u_1}{4}$. Then, in either case, and for all $r \in [0, \delta_1]$, $(v_1 \pm r, u_2, \cdots, u_n) \in B$. We apply the same procedure to $(v_1, u_2, \cdots, u_n)$, but with respect to the second coordinate, to find $v_2 > 0$ and $\delta_2 > 0$ such that, for all $r \in [0, \delta_2]$, $(v_1, v_2 + r, u_3, \cdots, u_n) \in B$. Iterating this procedure we obtain $(v_1, \cdots, v_n) \in B$ and $\delta > 0$ such that, for all $i \in [n]$ and for all $r \in [0, \delta]$, $(v_1, \cdots, v_i + r, \cdots, v_n) \in B$. From the $\mathbb{B}$-convexity of B, if $(r_1, \cdots, r_n) \in]0, \delta[^n$ then $\bigvee_{i=1}^n (v_1, \cdots, v_i + r_i, \cdots, v_n) \in B$. Let

$$U = \Big\{(v_1 + r_1, \cdots, v_i + r_i, \cdots, v_n + r_n) : (r_1, \cdots, r_n) \in]0, \delta[^n\Big\}.$$

We have shown that U is contained in B and U is clearly open. □

Given a subset $I = \{i_1 < \cdots < i_k\}$ of $[n]$ let $I' = [n] \setminus I$, and $\mathbb{R}^{(n,I)}_+ = \{x \in \mathbb{R}^n_+ : x_j = 0 \text{ if } j \in I'\}$. We have $\mathbb{R}^{(n,\emptyset)}_+ = \{0\}$ and $\mathbb{R}^{(n,[n])}_+ = \mathbb{R}^n_+$. With $|I|$ equal to the cardinality of I one can naturally identify $\mathbb{R}^{(n,I)}_+$ with $\mathbb{R}^{|I|}_+$ and $\mathbb{R}^n_+$ with $\mathbb{R}^{(n,I)}_+ \times \mathbb{R}^{(n,I')}_+$. We will be somewhat careless about the difference, unless there is a risk of confusion and since, in most discussions, the dimension n will be fixed, we will use $\mathbb{R}^I_+$ instead of $\mathbb{R}^{(n,I)}_+$; $\mathbb{R}^I_{++}$ is $\{x \in \mathbb{R}^I_+ : \min_{i\in I} x_i > 0\}$. For all $x \in \mathbb{R}^n_+$ we have a unique decomposition $x = x_{I'} + x_I$, where x_I is the projection of x on $\mathbb{R}^I_+$ and $x_{I'}$ is the projection of x on $\mathbb{R}^{I'}_+$. Obviously, $x = x_{I'} \vee x_I$ and $\|x\| = \max\{\|x_{I'}\|, \|x_I\|\}$. For all $J \subset [n]$ and for all $w \in \mathbb{R}^J_+$ we have $w = w_J$, consequently, whenever it might help in keeping track of the ongoing calculations, arbitrary elements of $\mathbb{R}^J_+$ will be, indifferently denoted by small case letters $u, v, w, \cdots$ without subscripts or with small case letters with the appropriate subscripts $u_J, v_J, w_J, \cdots$. For example, if $n = 5$ and $I = \{1, 3\}$ then $\mathbb{R}^{(5,I)}_+ = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_+ : (x_2, x_4, x_5) = (0, 0, 0)\}$. And, for arbitrary $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_+$, $x_I = (x_1, 0, x_3, 0, 0)$ and $x_{I'} = (0, x_2, 0, x_4, x_5)$.

Proposition 2.7. *If $B \subset \mathbb{R}^n_+$ is a nonempty halfspace then there is a unique subset $I \subset [n]$ such that $B \subset \mathbb{R}^I_+$ and B is a halfspace in $\mathbb{R}^I_+$ with (relative) nonempty interior.*

Proof. If B has nonempty interior then, by Lemma 2.6 $B \cap \mathbb{R}^n_{++} \neq \emptyset$ and therefore $I = [n]$. If B has empty interior then, for all $y \in B$, the set $I(y) = \{i \in [n] : y_i > 0\}$ is not $[n]$. Let $I = \bigcup_{y\in B} I(y)$. For all $y \in B$ we have $y \in \mathbb{R}^{I(y)}_+$ and therefore $B \subset \mathbb{R}^I_+$. For all $i \in I$ there exists $y^{(i)} \in B$ such that $y^{(i)}_i > 0$, using the $\mathbb{B}$-convexity of B we obtain $\bigvee_{i\in I} y^{(i)} \in B$ from which it follows that $B \cap \mathbb{R}^I_{++} \neq \emptyset$ and, as a subset of $\mathbb{R}^I_+$, B is $\mathbb{B}$-convex. To complete the proof we have to see that $\mathbb{R}^I_+ \setminus B$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^I_+$, which is a consequence of the $\mathbb{B}$-convexity of $\mathbb{R}^I_+$ and of $\mathbb{R}^n_+ \setminus B$. □

By definition, the cardinality of the set I from Proposition 2.7 is the **dimension of the halfspace**.

The following proposition shows that the sets described in the introduction are indeed halfspaces. Much of the remaining work will aim at showing that closed halfspaces containing 0 are of that form, this is the content of Theorems 7.2.

Proposition 2.8. *Given $u, v \in \mathbb{R}^n_+$, and $r, s \in \mathbb{R}_+$, let, for $x \in \mathbb{R}^n_+$,*

$$\theta(x) = \max_{j\in[n]} \{u_j x_j, r\} - \max_{i\in[n]} \{v_i x_i, s\}.$$

Then, for all positive real number λ the sets $\{x \in \mathbb{R}^n_+ : \theta(x) \leq \lambda\}$ and $\{x \in \mathbb{R}^n_+ : \theta(x) < \lambda\}$ are, respectively, closed and open $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$. For $\lambda = 0$ they are halfspaces.

Proof. That those sets are either closed or open follows from the continuity of θ.

Since an intersection $\mathbb{B}$-convex sets is $\mathbb{B}$-convex and each u_j is either 0 or strictly positive showing that sets of the form $\{x \in \mathbb{R}^n_+ : x_j \leq \max_{i\in[n]} \{v_i x_i, s\} + \lambda\}$ or $\{x \in \mathbb{R}^n_+ : r \leq \max_{i\in[n]} \{v_i x_i, s\} + \lambda\}$ are $\mathbb{B}$-convex will prove that $\{x \in \mathbb{R}^n_+ : \theta(x) \leq \lambda\}$ is $\mathbb{B}$-convex.

This can be done using three properties, namely, associativity of the max operation, distributivity of multiplication by of positive scalar ρ over the max operation and $\max\{a + \lambda, b + \lambda\} = \max\{a, b\} + \lambda$. Details are left to the reader.

Similarly, $\{x \in \mathbb{R}^n_+ : \theta(x) < \lambda\}$ is $\mathbb{B}$-convex since sets of the form $\{x \in \mathbb{R}^n_+ : x_j < \max_{i\in[n]} \{v_i x_i, s\} + \lambda\}$ or $\{x \in \mathbb{R}^n_+ : r < \max_{i\in[n]} \{v_i x_i, s\} + \lambda\}$ are $\mathbb{B}$-convex. □

For $\lambda > 0$ the set $\{x \in \mathbb{R}^n_+ : \theta(x) \leq \lambda\}$ is generally not a halfspace. For example, $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 - x_1 \leq 1\}$ is $\mathbb{B}$-convex but is not a halfspace, since $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 - x_1 > 1\}$ is not $\mathbb{B}$-convex as one can see with $x = (1, 3)$, $x' = (10, 12)$, for which we have $x_2 - x_1 > 1$ but $x \vee (1/4)x' = (5/2, 3)$ for which we have $x_2 - x_1 = (1/2)$.

This does not mean that, for $\lambda > 0$, the set $\{x \in \mathbb{R}^n_+ : \theta(x) \leq \lambda\}$ is never a halfspace. Indeed, as one can easily check, $\{(x_1, x_2) \in \mathbb{R}^2_+ : \max\{x_1, x_2\} \leq 1\}$ is a halfspace.

If $r = s = 0$ then the sets $\{x \in \mathbb{R}^n_+ : \theta(x) \leq 0\}$ from Proposition 2.8 are cones ‡. The set $\{(x_1, x_2) \in \mathbb{R}^2_+ : \max\{x_1, x_2\} \leq 1\}$ is a two dimensional cube. As we will see, closed halfspaces containing 0 can be decomposed in two parts, both halfspaces, one of which is a cone the other one being either a cube or the cartesian product of a lower dimensional cube with halflines.

3 Bounded Halfspaces

For $x \in \mathbb{R}^n_+$ we denote by $\mathcal{R}x$ the set $\{tx : t \geq 1\}$; if $x \neq 0$ we call $\mathcal{R}x$ the **ray with vertex *x***. A bounded set cannot contain a ray, but a set, even a $\mathbb{B}$-convex set can be unbounded and be without ray (consider in $\mathbb{R}^2_+$ the vertical line through the point $(1, 0)$). We will see that a halfspace that does not contain a ray is bounded.

Lemma 3.1. (1) *Suppose $B \subset \mathbb{R}^n_+$ is $\mathbb{B}$-convex and $x \in \mathbb{R}^n_+$. Then $x \in B$ if and only if $B \bigcap \mathcal{R}x \neq \emptyset$ and $B \bigcap [0, x] \neq \emptyset$.*

(2) *If $A \subset \mathbb{R}^n_+$ has complement that is $\mathbb{B}$-convex then, for all $x \in A$, either $\mathcal{R}x \subset A$ or $[0, x] \subset A$.*

Proof. (1) One implication is trivial, if $x \in B$ then $x \in B \bigcap \mathcal{R}x \bigcap [0, x]$. Assume that $B \bigcap \mathcal{R}x \neq \emptyset$ and $B \bigcap [0, x] \neq \emptyset$. There exists $t \geq 1$ such that $tx \in B$ and $y \leq x$ with $y \in B$. Therefore, $x = x \vee y = t^{-1}(tx) \vee y$ and $t^{-1}(tx) \vee y \in B$ since B is $\mathbb{B}$-convex.
(2) Let $x \in A$ such that $\mathcal{R}x$ is not contained in A. Since $[\mathbb{R}^n_+ \setminus A] \bigcap \mathcal{R}x \neq \emptyset$ and $\mathbb{R}^n_+ \setminus A$ is $\mathbb{B}$-convex we cannot have, from (1), $[\mathbb{R}^n_+ \setminus A] \bigcap [0, x] \neq \emptyset$. □

Proposition 3.2. *A halfspace $B \subset \mathbb{R}^n_+$ is bounded if and only if it contains no ray.*

Proof. We prove that a halfspace which contains no ray is bounded. First, a nonempty halfspace which contains no ray contains 0. Indeed, if $0 \in \mathbb{R}^n_+ \setminus B$ take an arbitrary x in B and a $t > 1$ such that $tx \in \mathbb{R}^n_+ \setminus B$; then $x = 0 \vee t^{-1}(tx)$ would be in $\mathbb{R}^n_+ \setminus B$.

If B is a halfspace which contains no ray then, for all $i \in [n]$, there exists $\beta_i \geq 1$ such that $\beta_i e_i \in \mathbb{R}^n_+ \setminus B$. Notice that if $t \geq \beta_i$ then te_i is also in $\mathbb{R}^n_+ \setminus B$, otherwise, from $0 \in B$, $te_i \in B$ and $0 < \beta_i t^{-1} \leq 1$ we would have $0 \vee (\beta_i t^{-1})(te_i) \in B$. Let $x^\star = \bigvee_{i\in[n]} \beta_i e_i$;

‡A set $B \subset \mathbb{R}^n_+$ is a cone if $tB \subset B$ for all $t > 0$.

We have $x^\star \in \mathbb{R}^n_+ \setminus B$. Let us see that $B \subset [0, x^\star]$. If $x \notin [0, x^\star]$ then, for at least one index $i_0 \in [n]$, $x_{i_0} > \beta_{i_0}$ and therefore $x_{i_0} e_{i_0} \in \mathbb{R}^n_+ \setminus B$. If x were also in B, we would have $[0, x] \subset B$, since $\mathcal{R}x$ is not contained in B by hypothesis, and, from $x_{i_0} e_{i_0} \leq x$, we would also have $x_{i_0} e_{i_0} \in B$. □

Proposition 3.3. *$B \subset \mathbb{R}^n_+$ is a closed nonempty bounded halfspace if and only if there exists $u^\star \in \mathbb{R}^n_+$ such that $B = [0, u^\star]$. Furthermore, $intB \neq \emptyset$ if and only if $u^\star \in \mathbb{R}^n_{++}$*

Proof. Since $[0, u^\star]$ is a semilattice which is starshaped at 0 it is $\mathbb{B}$-convex. If $y \in \mathbb{R}^n_+ \setminus [0, u^\star]$ then $y_k > u^\star_k$ for at least one index $k \in [n]$. Since, for all $t \in [0, 1]$ and $x \in \mathbb{R}^n_+$, $\max\{tx_k, y_k\} \geq y_k$ we conclude that $\mathbb{R}^n_+ \setminus [0, u^\star]$ is also $\mathbb{B}$-convex.

Now, let $B \subset \mathbb{R}^n_+$ be a closed and bounded halfspace. By (2) of Lemma 3.1 and by Proposition 3.2 we have, $0 \in B$ and, for all $x \in B$, $[0, x] \subset B$. By Lemma 2.5 has a largest element $u^\star$. We have seen that $[0, u^\star] \subset B$ and we have $B \subset [0, u^\star]$ from the definition of a largest element. The last part is trivial. □

4 Downward and Conical Halfspaces

A subset A of $\mathbb{R}^n_+$ is **downward** if, for all $x \in A$, $[0, x] \subset A$. As we have seen in the proof of Proposition 3.2, a halfspace containing no ray is downward. For an arbitrary subset A of $\mathbb{R}^n_+$ the set $\bigcup_{a \in A} [0, a]$ is the smallest downward set containing A, it is denoted by $\downarrow A$. From the definition we have

$$\downarrow A = \{x \in \mathbb{R}^n_+ : \exists a \in A \text{ s.t. } x \leq a\} = \bigcup_{a \in A} \downarrow \{a\}.$$

Lemma 4.1. *The closure of a downward set of $\mathbb{R}^n_+$ is a downward set.*

Proof. Assume that $A = \downarrow A$ and let u be an element of the closure of A. Let $I = \{i \in [n] : u_i > 0\}$ and $I' = [n] \setminus I$. Let y be an arbitrary element of $\downarrow \{u\}$. We show that $y \in \overline{A}$. Let $I_1 = \{i \in I : y_i < u_i\}$ and $I_2 = \{i \in I : y_i = u_i\}$. Notice that for $i \in I'$ $y_i = u_i = 0$.

Let $\varepsilon_0 = \min\{y_i : 0 < y_i\}$. For all $\varepsilon \in]0, \varepsilon_0[$ there exists $a^{[\varepsilon]} \in A$ such that

$$\begin{cases} 0 \leq a_i^{[\varepsilon]} < \varepsilon & \text{if} \quad i \in I' \\ y_i < a_i^{[\varepsilon]} - \varepsilon < u_i < a_i^{[\varepsilon]} + \varepsilon & \text{if} \quad i \in I_1 \\ y_i - \varepsilon < a_i^{[\varepsilon]} < y_i + \varepsilon & \text{if} \quad i \in I_2 \end{cases}$$

If, for all $\varepsilon \in]0, \varepsilon_0[$, one defines $y^{[\varepsilon]}$: as follows:

$$\begin{cases} y_i^{[\varepsilon]} = 0 & \text{if} \quad i \in I' \\ y_i^{[\varepsilon]} = y_i - \varepsilon & \text{if} \quad i \in I \end{cases}$$

then $y^{[\varepsilon]} \in \downarrow \{a^{[\varepsilon]}\}$. From $a^{[\varepsilon]} \in A$ and $\downarrow A = A$ we have $y^{[\varepsilon]} \in A$. This shows that $y \in \overline{A}$. □

A subset A of $\mathbb{R}^n_+$ is **radial** if, for all $a \in A$, $\mathcal{R}a \subset A$; it is **conical** if, for all $a \in A$, $\{ta : t \in \mathbb{R}_{++}\} \subset A$. Clearly, a set A which is starshaped at 0 is conical if and only if it is radial and a closed conical set always contains 0. Also, if A is conical then its closure $\overline{A}$ in $\mathbb{R}^n_+$ and its complement $\mathbb{R}^n_+ \setminus A$ are conical. From Lemma 2.4, a $\mathbb{B}$-convex set containing 0 is conical if and only if it is radial.

Given a subset A of $\mathbb{R}^n_+$, let

$$A_\infty = \{a \in A : \mathcal{R}a \subset A\} \text{ and } A_0 = A \setminus A_\infty.$$

The description of the closed halfspaces of $\mathbb{R}^n_+$ containing 0 will start from the following lemma, which gives a first and very crude description of their structure.

Lemma 4.2. *If B is a halfspace of $\mathbb{R}^n_+$ containing 0 then B_∞ is a conical halfspace containing 0, which could be $\{0\}$ or B itself; B_0 is a $\mathbb{B}$-convex subset of B and $(\downarrow B_0)$ is a halfspace contained in B. Furthermore, if B is closed then B_∞ is also closed and, clearly, $B = \overline{(\downarrow B_0)} \bigcup B_\infty$.*

Proof. Since B is a halfspace containing 0 we have $B_\infty = \{x \in \mathbb{R}^n_+ : \mathbb{R}_+ x \subset B\}$. For $s \in [0,1]$ and $t \in \mathbb{R}_+$, and arbitrary x and y, $t(sx \vee y) = s(tx) \vee ty$; this shows that B_∞ is $\mathbb{B}$-convex. If x and y do not belong to B_∞ then there exists $t > 1$ such that neither tx nor ty belong to B (since $0 \in B$ and B is $\mathbb{B}$-convex); $\mathbb{R}^n_+ \setminus B$ is also $\mathbb{B}$-convex, therefore, for all $s \in [0,1]$, $t(sx \vee y) \in \mathbb{R}^n_+ \setminus B$. This shows that $\mathbb{R}^n_+ \setminus B_\infty$ is $\mathbb{B}$-convex. The $\mathbb{B}$-convexity of B_0 follows from its definition $B_0 = B \bigcap [\mathbb{R}^n_+ \setminus B_\infty]$ and this easily implies that $(\downarrow B_0)$ is $\mathbb{B}$-convex. Also, $\mathbb{R}^n_+ \setminus (\downarrow B_0) = \bigcap_{x \in B_0} \mathbb{R}^n_+ \setminus [0,x]$ and, since each $[0,x]$ is a halfspace, $\mathbb{R}^n_+ \setminus (\downarrow B_0)$ is an intersection of $\mathbb{B}$-convex sets, and is therefore $\mathbb{B}$-convex.

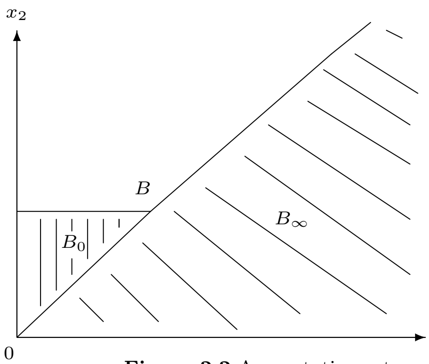

Figure 2.2 Asymptotic part.

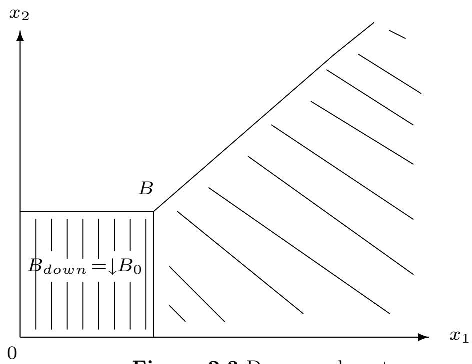

Figure 2.3 Downward part.

To see that B_∞ is closed if B is closed consider a converging sequence $(x_m)_{m \in \mathbb{N}}$ of points of B_∞ and let $\hat{x}$ be its limit. From $B_\infty \subset B$ we have $\hat{x} \in B$. Let $t > 0$ be an arbitrary positive real number. For all $m \in \mathbb{N}$ we have $tx_m \in B$ and therefore $t\hat{x} \in B$. This shows that $\hat{x} \in B_\infty$. □

For a closed halfspace B containing 0 we call B_∞ the **asymptotic part of** B and $B_{\text{down}} = \overline{(\downarrow B_0)}$ the **downward part of** B.

For example, if $B = \{(x_1, x_2) \in \mathbb{R}^2_+ : 0 \leq x_1 \leq 1\}$ then $B_\infty = \{(0,t) : t \in \mathbb{R}_+\}$ and $B_{\text{down}} = B$. If $B = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 \leq x_2 \text{ or } x_2 \leq 1\}$ then $B_\infty = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 \leq x_2\}$ and $B_{\text{down}} = [0,1] \times [0,1] = \{x \in \mathbb{R}^2_+ : x \leq (1,1)\}$.

A closed halfspace B containing 0 for which $B = B_{\text{down}}$ is a **downward halfspace**.

A (nonempty) halfspace B which is not reduced to $\{0\}$ and for which $B = B_\infty$ is a **conical halfspace**.

Given a subset B of $\mathbb{R}^n_+$ let

$$\Lambda(B) = \{i \in [n] : e_i \in B\} \text{ and } \Lambda'(B) = [n] \setminus \Lambda(B).$$

The canonical basis of $\mathbb{R}^I_+$, where I is a subset of $[n]$, is the subset $\{e_i : i \in I\}$ of the canonical basis of $\mathbb{R}^n_+$ and for $B \subset \mathbb{R}^I_+$ the set $\Lambda(B)$ can be interpreted either as a subset of $[n]$ or as a subset of I, but for $\Lambda'(B)$ there is a potential ambiguity since $[n] \setminus \Lambda(B)$ and

$I \backslash \Lambda(B)$ are different sets. Using the notation $\Lambda_I(B)$ and $\Lambda'_I(B)$ would avoid this ambiguity. Since it should be clear from the context which sets we are working with we will use the simpler notation, without subscript.

A non trivial halfspace[§] for which the normality condition below holds is a **normal halfspace**.

(Normality) $\forall i \in \Lambda(B)\ e_i \in \operatorname{int}(B)$

Lemma 4.3. *If $B \subset \mathbb{R}^n_+$ is a closed normal conical halfspace with non empty interior then $\overline{\mathbb{R}^n_+ \backslash B}$ is also a closed normal conical halfspace with non empty interior. Furthermore, $\Lambda(B) = \Lambda'\left(\overline{\mathbb{R}^n_+ \backslash B}\right)$ and $\Lambda'(B) = \Lambda\left(\overline{\mathbb{R}^n_+ \backslash B}\right)$.*

Proof. If $B \subset \mathbb{R}^n_+$ is a conical halfspace then $\mathbb{R}^n_+ \backslash B$ is also a conical halfspace, and so is its closure. Il the interior of $\overline{\mathbb{R}^n_+ \backslash B}$ is empty then $\overline{\mathbb{R}^n_+ \backslash B} \cap \mathbb{R}^n_{++} = \emptyset$ and therefore $\mathbb{R}^n_{++} \subset B$, and since B is closed, $B = \mathbb{R}^n_+$.

If $e_i \in \partial\left(\overline{\mathbb{R}^n_+ \backslash B}\right)$ then $e_i \in \partial B$, since $\partial\left(\overline{\mathbb{R}^n_+ \backslash B}\right) = \partial B$. This shows that $\left(\overline{\mathbb{R}^n_+ \backslash B}\right)$ is normal if B is.

If $e_i \in \left(\overline{\mathbb{R}^n_+ \backslash B}\right)$ then $e_i \notin B$, by the normality of B, and therefore $\Lambda\left(\overline{\mathbb{R}^n_+ \backslash B}\right) \subset \Lambda'(B)$. If $e_i \in \Lambda'(B)$ then $e_i \in \mathbb{R}^n_+ \backslash B \subset \overline{\mathbb{R}^n_+ \backslash B}$. We have shown that $\Lambda\left(\overline{\mathbb{R}^n_+ \backslash B}\right) = \Lambda'(B)$. □

The next lemma tells us how to decompose a halfspace as a supremum of halfspaces of smaller dimensions.

Given a nontrivial halfspace $B \subset \mathbb{R}^n_+$ let, for all $j \in \Lambda'(B)$ and $x \in \mathbb{R}^n_+$,

$$x_{[j]} = \bigvee_{i \in \Lambda(B)} x_i e_i \vee x_j e_j = x_{\Lambda(B)} \vee x_j e_j \text{ and } B_{[j]} = \{x_{[j]} : x \in B\}$$

Lemma 4.4 (Decomposition Lemma). *If $B \subset \mathbb{R}^n_+$ is a nontrivial halfspace then*

(1) *$B_{[j]}$ is a $\mathbb{B}$-convex subset of B, $B = \bigvee_{j \in \Lambda'(B)} B_{[j]}$ and, consequently, for all $x \in \mathbb{R}^n_+$, $x \in B$ if and only if, for all $j \in \Lambda'(B)$, $x_{[j]} \in B_{[j]}$.*

(2) *$B_{[j]}$ is a halfspace in $\mathbb{R}^{\Lambda(B) \cup \{j\}}_+$ for which $\Lambda(B_{[j]}) = \Lambda(B)$ and $\Lambda'(B_{[j]}) = \{j\}$.*

(3) (a) *If B is a closed conical halfspace in $\mathbb{R}^n_+$ then, for all $j \in \Lambda'(B)$, $B_{[j]}$ is a closed conical halfspace in $\mathbb{R}^{\Lambda(B) \cup \{j\}}_+$;*

(b) *an arbitrary point $x \in \mathbb{R}^n_+$ is in the (relative) interior of B in $\mathbb{R}^n_+$ if and only if, for all $j \in \Lambda'(B)$, $x_{[j]}$ is in the (relative) interior of $B_{[j]}$ in $\mathbb{R}^{\Lambda(B) \cup \{j\}}_+$.*

(c) *if B is a closed normal conical halfspace with nonempty interior in $\mathbb{R}^n_+$ then, for all $j \in \Lambda'(B)$, $B_{[j]}$ a closed normal conical halfspace with nonempty interior in $\mathbb{R}^{\Lambda(B) \cup \{j\}}_+$.*

Proof. (1) The $\mathbb{B}$-convexity of $B_{[j]}$ follows from $(tx \vee y)_{[j]} = tx_{[j]} \vee y_{[j]}$ and the $\mathbb{B}$-convexity of B. From $x = \bigvee_{j \in \Lambda'(B)} x_{[j]}$ we have $B \subset \bigvee_{j \in \Lambda'(B)} B_{[j]}$. For all $x \in \mathbb{R}^n_+$, $x_{\Lambda'(B)} \in \mathbb{R}^n_+ \backslash B$ (since $\mathbb{R}^n_+ \backslash B$ is $\mathbb{B}$-convex) , and $x = x_{[j]} \vee x_{\Lambda'(B)}$; therefore, if for a given x there exists $j \in \Lambda'(B)$ such that $x_{[j]} \notin B$ then $x \notin B$, since $\mathbb{R}^n_+ \backslash B$ is $\mathbb{B}$-convex set. This shows that, for all $j \in \Lambda'(B)$, $B_{[j]} \subset B$. The inclusion $\bigvee_{j \in \Lambda'(B)} B_{[j]} \subset B$ follows from the $\mathbb{B}$-convexity of B.

[§]We recall that the trivial halfspaces are $\mathbb{R}^n_+$ and $\{0\}$ and their complements in $\mathbb{R}^n_+$.

(2) If u and v belong to $\mathbb{R}_+^{\Lambda(B)\cup\{j\}} \setminus B_{[j]}$ then, by (1), they belong to $\mathbb{R}_+^n \setminus B$, which is $\mathbb{B}$-convex; consequently, for $t \in [0,1]$, we have $tu \vee v \in (\mathbb{R}_+^n \setminus B) \cap (\mathbb{R}_+^{\Lambda(B)\cup\{j\}})$, since $\mathbb{R}_+^{\Lambda(B)\cup\{j\}}$ is $\mathbb{B}$-convex in $\mathbb{R}_+^n$. This shows that $tu \vee v \in \mathbb{R}_+^{\Lambda(B)\cup\{j\}} \setminus B_{[j]}$.

The last part of (2) trivially follows from $B_{[j]} \subset B$ and $e_j \notin B$.

(3) The closedness of $B_{[j]}$, if B is closed, is a consequence of $B_{[j]} = B \cap \mathbb{R}_+^{\Lambda(B)\cup\{j\}}$.

If x is in the interior of B then for each $i \in [n]$ we can find an interval $[a_i, b_i]$ with $0 \le a_i < b_i$ such that $a_i = 0$ if $x_i = 0$, $a_i < x_i < b_i$ if $x_i > 0$ and $\prod_{i\in[n]}[a_i, b_i] \subset B$. For $i \in \Lambda(B) \cup \{j\}$ let $[c_i, d_i] = [a_i, b_i]$ and, otherwise, let $[c_i, d_i] = \{0\}$. Then $\prod_{i\in[n]}[c_i, d_i]$ has nonempty interior in $\mathbb{R}_+^{\Lambda(B)\cup\{j\}}$, $x_{[j]} \in \prod_{i\in[n]}[c_i, d_i]$. If $y \in \prod_{i\in[n]}[c_i, d_i]$ then $y_i = 0$ if $i \notin \Lambda(B) \cup \{j\}$ therefore $\prod_{i\in[n]}[c_i, d_i] \subset B_{[j]}$.

Reciprocally, if, for all $j \in \Lambda'(B)$, $x_{[j]}$ is in the relative interior in $\mathbb{R}_+^{\Lambda(B)\cup\{j\}}$ of $B_{[j]}$ then there exists for all $i \in \Lambda(B) \cup \{j\}$ real numbers $0 \le a_i^{[j]} < b_i^{[j]}$, with $a_i^{[j]} = 0$ if $x_i = 0$ and $a_i^{[j]} < x_i < b_i^{[j]}$ if $x_i \ne 0$, such that

$$x_{[j]} \in \prod_{i\in\Lambda(B)\cup\{j\}} [a_i^{[j]}, b_i^{[j]}] \subset B_{[j]}.$$

For $i \in \Lambda(B)$ let $a_i = \max_{j\in\Lambda'(B)} a_i^{[j]}$ and $b_i = \min_{j\in\Lambda'(B)} b_i^{[j]}$ and for $j \in \Lambda'(B)$ let $a_j = a_j^{[j]}$ and $b_j = b_j^{[j]}$. Then, for all $j \in \Lambda'(B)$,

$$x_{[j]} \in \prod_{i\in\Lambda(B)\cup\{j\}} [a_i, b_i] \subset \prod_{i\in\Lambda(B)\cup\{j\}} [a_i^{[j]}, b_i^{[j]}] .$$

If $y \in \prod_{i\in[n]}[a_i, b_i]$ then, $y_{[j]} \in \prod_{i\in\Lambda(B)\cup\{j\}}[a_i, b_i]$ and therefore, from $\bigvee_{j\in\Lambda'(B)} y_{[j]} = y$, we obtain $y \in B$. This shows that $\prod_{i\in[n]}[a_i, b_i] \subset B$ and therefore that x is in the interior of B.

Part (c) of (3) follows from (a) and (b). We only have to see that $B_{[j]}$ is a nontrivial halfspace in $\mathbb{R}_+^{\Lambda(B)\cup\{j\}}$. From $B_{[j]} = B \cap \mathbb{R}_+^{\Lambda(B)\cup\{j\}}$ and $e_j \notin B$, but $e_j \in \mathbb{R}_+^{\Lambda(B)\cup\{j\}}$, we have $B_{[j]} \ne \mathbb{R}_+^{\Lambda(B)\cup\{j\}}$. □

In the following pages we will first give a complete description of closed downward and conical halfspaces containing 0. We will then conclude with the description of general closed halspaces containing 0.

5 Downward Halfspaces

For $0 \le a_i \le \infty$, let $B_i = [0, a_i]$ if $a_i < \infty$ and $B_i = \mathbb{R}_+$ if $a_i = \infty$. Then, $B = \prod_{i=1}^n B_i$ is a closed halfspace which, by definition, is a **closed rectangular halfspace**. If $a_i \ne \infty$ for all $i \in [n]$ then B is a **bounded closed rectangular halfspace**. A closed rectangular halfspace has nonempty interior if and only if, for all $i \in [n]$, $a_i > 0$.

Proposition 5.1. *The following assertions are equivalent:*

(1) $B \subset \mathbb{R}_+^n$ *is a closed rectangular halfspace with nonempty interior;*

(2) *there exists* $u \in \mathbb{R}_+^n$ *such that* $B = \{x \in \mathbb{R}_+^n : \max_{i\in[n]} u_i x_i \le 1\}$*;*

Proof. If B is a closed rectangular halfspace with nonempty interior let $u_i = a_i^{-1}$ if $a_i \neq \infty$ and $u_i = 0$ otherwise. Then $x \in B = \prod_{i=1}^n B_i$ if and only if $u_i x_i \leq 1$ for all $i \in [n]$.

Let $B = \{x \in \mathbb{R}_+^n : \max_{i\in[n]} u_i x_i \leq 1\}$ where $u \in \mathbb{R}_+^n$. Let $a_i = u_i^{-1}$ if $u_i > 0$ and $a_i = \infty$ otherwise. Then $x \in B$ if and only if $0 \leq x_i \leq a_i$ if $a_i \neq \infty$. This shows that (1) and (2) are equivalent. □

Theorem 5.2. *If $B \subset \mathbb{R}_+^n$ is a closed halfspace containing* 0 *then B_{down} is a rectangular halfspace.*

Proof. If B_{down} is bounded then, by Proposition 3.3, $B_{\text{down}} = [0, x^\star]$ for some $x^\star$ in $\mathbb{R}_+^n$. Otherwise let, for all $i \in [n]$, $a_i = \sup\{x_i : x \in B_{\text{down}}\}$ and let $I = \{i \in [n] : 0 < a_i < \infty\}$ and $I' = \{i \in [n] : a_i = \infty\}$.
Let $R = \{x \in \mathbb{R}_+^n : \forall i \in I \ x_i \leq a_i\}$; it is a rectangular space. We show that $B_{\text{down}} = R$.

By Lemma 4.1 B_{down} is a downward set therefore

$$\bigcup_{x\in B_{\text{down}}} [0, x] = B_{\text{down}}.$$

For all $x \in B_{\text{down}}$ we have $[0, x] \subset R$ and therefore $B_{\text{down}} \subset R$.

To see that $R \subset B_{\text{down}}$ choose, for all $\varepsilon > 0$ and all $i \in [n]$, a point $z^{[\varepsilon,i]} \in B_{\text{down}}$ such that:

$$\begin{cases} a_i - \varepsilon < z_i^{[\varepsilon,i]} \leq a_i & \text{if} \quad 0 < a_i < \infty \\ z_i^{[\varepsilon,i]} = 0 & \text{if} \quad a_i = 0 \\ z_i^{[\varepsilon,i]} > \varepsilon^{-1} & \text{if} \quad a_i = \infty \end{cases}$$

and let $z^{[\varepsilon]} = \bigvee_{i\in[n]} z^{[\varepsilon,i]}$. The set B_{down} is $\mathbb{B}$-convex therefore $z^{[\varepsilon]} \in B_{\text{down}}$ and since B_{down} is also downward we have

$$\bigcup_{\varepsilon>0} [0, z^{[\varepsilon]}] \subset B_{\text{down}}.$$

To complete the proof recall that B_{down} is closed and notice that

$$\begin{cases} a_i - \varepsilon < z_i^{[\varepsilon]} \leq a_i & \text{if} \quad 0 < a_i < \infty \\ z_i^{[\varepsilon]} = 0 & \text{if} \quad a_i = 0 \\ z_i^{[\varepsilon]} > \varepsilon^{-1} & \text{if} \quad a_i = \infty \end{cases}$$

from which one sees that the set $\bigcup_{\varepsilon>0} [0, z^{[\varepsilon]}]$ is dense in R. □

From Proposition 2.7, Proposition 5.1 and Theorem 5.2 we obtain the analytic description of closed downward halfspaces containing 0.

Corollary 5.3. *A closed halfspace $B \subset \mathbb{R}_+^n$ containing* 0 *is downward if and only if there exist $u \in \mathbb{R}_+^n \setminus \{0\}$ and a subset K of $[n]$, which is empty if the interior of B is nonempty, such that*

$$B = \{x \in \mathbb{R}_+^n : \max_{i\in[n]} u_i x_i \leq 1 \ \textit{and} \ \max_{k\in K} x_k = 0\}.$$

Let B be a closed downward halfspace of $\mathbb{R}^n_+$ which contains 0 and assume that its interior is not empty. With the notation from Corollary 5.3 let $J = \{j \in [n] : u_j > 0\}$, $x^\star_j = u_j^{-1}$ for $j \in J$ and $x^\star_i = 0$ for $i \in [n]\backslash J$. Then, $x^\star \in B$ and

$$B = \left\{ x \in \mathbb{R}^n_+ : \max_{j \in J} \left(\frac{x_j}{x^\star_j} \right) \leq 1 \right\}$$

from which one can see that

$$B_\infty = \left\{ x \in \mathbb{R}^n_+ : x_J = 0 \right\} \text{ and } B_0 = \{ x \in B : x_J \neq 0 \}.$$

6 On the Structure of Conical Halfspaces

As the title indicates we will give in this section a full description of conical halfspaces. We will start with normal conical halfspaces for which we will proceed in two steps. First, we will describe normal conical halfspaces of dimension two. Then we will give the description of normal halfspaces for which Λ' has cardinality one. From the Decomposition Lemma, Lemma 4.4, we will obtain the general form of conical halfspaces.

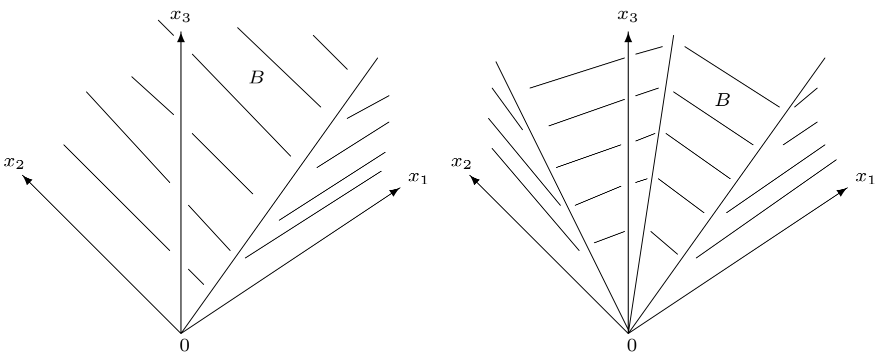

Figure 6.1 Examples of conical halfspaces

6.1 Normal Conical Halfspaces

The following Lemma tells us how to push a strictly positive element of a conical halfspace in the interior of that halfspace.

Lemma 6.1 (Push Lemma). *Let $B \subset \mathbb{R}^n_+$ be a conical halfspace with nonempty interior and $u = \vee_{i \in \Lambda(B)} r_i e_i \in \mathbb{R}^{\Lambda(B)}_{++}$. Then, for all $x \in B \bigcap \mathbb{R}^n_{++}$, $x + u$ belongs to the interior of B.*

Proof. Obviously, we can assume that B is not $\mathbb{R}^n_+$. The proof is done in two steps.
(A) We assume that $\Lambda(B) = [n-1]$, and therefore $\Lambda'(B) = \{n\}$. For each $i \in [n-1]$ define two elements $z^{[i]}_-$ and $z^{[i]}_+$ of $\mathbb{R}^n_+$ as follows:

$$(z^{[i]}_-)_j = \begin{cases} x_i + \frac{r_i}{2} & \text{if } i = j \\ x_j & \text{otherwise} \end{cases} \text{ and } (z^{[i]}_+)_j = \begin{cases} x_i + \frac{3r_i}{2} & \text{if } i = j \\ x_j & \text{otherwise.} \end{cases}$$

In other words, $z^{[i]}_- = x \vee \left(x_i + \dfrac{r_i}{2} \right) e_i$ and $z^{[i]}_+ = x \vee \left(x_i + \dfrac{3r_i}{2} \right) e_i$.

From $x \in B$ and $e_i \in B$, and taking into account that B is conical, we have $z^{[i]}_- \in B$ and

$z_+^{[i]} \in B$. Consequently, if s_i, t_i, for $i \in [n-1]$, are arbitrary positive numbers we also have $(\bigvee_{i\in[n-1]} s_i z_-^{[i]}) \vee (\bigvee_{i\in[n-1]} t_i z_+^{[i]}) \in B$. Call this element of B, which depends on the numbers s_i and t_i, ω. As one can see the coordinates of ω are

$$\omega_k = \begin{cases} \max_{i\in[n-1]}\{s_k(x_k + \frac{r_k}{2}), t_k(x_k + \frac{3r_k}{2}), s_i x_k, t_i x_k\} \text{ for } k \in [n-1] \\ \\ \max_{i\in[n-1]}\{s_i, t_i\} x_n \text{ for } k = n. \end{cases}$$

Let Ω be the set of all the ω as s_i and t_i takes all the possible values in $\mathbb{R}_+$. With

$$t_i^\star = \frac{x_i + r_i}{x_i + (3/2)r_i} \text{ and } s_i^\star = 1$$

one obtains $\omega = x + u$.

To complete this part the proof we show that there is an open set U contained in Ω and containing $x + u$. Let

$$\delta = \max_{k\in[n-1]} \left\{ \frac{x_k + (5/4)r_k}{x_k + (3/2)r_k} \right\} \text{ and } \eta = \min_{k\in[n-1]} \left\{ \frac{x_k + (3/4)r_k}{x_k + (1/2)r_k} \right\}$$

and notice that $0 < \delta < 1 < \eta$. Let

$$U = \prod_{i\in[n-1]} \left] x_i + \frac{3r_i}{4}, x_i + \frac{5r_i}{4} \right[\times]\delta x_n, \eta x_n[$$

Let us see that $U \subset \Omega$.

With $y_i \in \left] x_i + \frac{3r_i}{4}, x_i + \frac{5r_i}{4} \right[$ let $t_i = \frac{y_i}{x_i + (3/2)r_i}$ and notice that

$$t_i < \frac{x_i + (5/4)r_i}{x_i + (3/2)r_i} \leq \delta.$$

If $y_n \in]\delta x_n, \eta x_n[$ let, for all $i \in [n-1]$, $s_i = (y_n/x_n)$. Then $\max_{i\in[n-1]}\{s_i, t_i\} = (y_n/x_n)$ and, for $k \in [n-1]$,

$$\begin{aligned} &\max_{i\in[n-1]}\{(y_n/x_n)(x_k + \frac{r_k}{2}), t_k(x_k + \frac{3r_k}{2}), (y_n/x_n)x_k, t_i x_k\} \\ &= \max_{i\in[n-1]}\{(y_n/x_n)(x_k + \frac{r_k}{2}), y_k, (y_n/x_n)x_k, t_i x_k\} = y_k. \end{aligned}$$

(B) If $\Lambda(B)$ is of cardinality strictly smaller than $n-1$ then, for each $j \in \Lambda'(B)$, we apply the first part of the proof to $B_{[j]}$, $x_{[j]}$ and u, which makes sense since $\Lambda(B) = \Lambda(B_{[j]})$. We conclude that $x_{[j]} + u$ is in the interior of $B_{[j]}$ (relative to $\mathbb{R}_+^{\Lambda(B)\cup\{j\}}$). Notice that $(x+u)_{[j]} = x_{[j]} + u$, therefore, by (3) of Lemma 4.4, $x + u$ belongs to the interior of B. □

Lemma 6.2. *A closed conical halfspace with nonempty interior is the closure of its interior.*

Proof. If $B \subset \mathbb{R}_+^n$ is a halfspace with nonempty interior then, by Lemma 2.6 we can find $z \in B \bigcap \mathbb{R}_{++}^n$. If x is an arbitrary element of B then, for all $t \in]0,1]$, $x \vee tz \in B \bigcap \mathbb{R}_{++}^n$. By Lemma 6.1 $(x \vee tz) + tz_{\Lambda(B)}$ belongs to the interior of B. From $x = \lim_{t\to 0}[\,(x \vee tz) + tz_{\Lambda(B)}\,]$ we have $B \subset \overline{\text{int } B}$. □

For a more general statement see Corollary 7.4.

Lemma 6.3. *If B is a closed normal conical halfspace of $\mathbb{R}^2_+$ for which $\Lambda(B) = \{j\}$ and $\Lambda'(B) = \{i\}$ then there exists $x^\star \in \mathbb{R}^2_{++}$ such that*

$$B = \left\{ x \in \mathbb{R}^2_+ : \left(\frac{x_i}{x_i^\star}\right) - \left(\frac{x_j}{x_j^\star}\right) \leq 0 \right\}.$$

And, obviously, we can choose $x^\star$ such that either $x_i^\star = 1$ or $x_j^\star = 1$.

Proof. We can assume that $\Lambda(B) = \{1\}$, that is, $(1, 0) \in B$. The interior of B is not empty, since B is normal. Let (x_1, x_2) be an arbitrary point in $B \cap \mathbb{R}^2_{++}$. Since B is conical we also have $(1, x_1^{-1} x_2) \in B$.
Let $t^\star = \sup\{t \in \mathbb{R}_+ : (1, t) \in B\}$. From $(1, x_1^{-1} x_2) \in B$ we have $t^\star > 0$. Also, $t^\star < \infty$, otherwise there would exists an increasing sequence of real numbers t_n converging to ∞ such that, for all n, $(1, t_n) \in B$, and therefore also $(t_n^{-1}, 1) \in B$. But B is closed, taking the limit as n goes to ∞ would yield $(0, 1) \in B$, contrary to the hypothesis.

If $(x_1, x_2) \in B$ and $x_1 = 0$ then $x_2 = 0$, since B is radial and $(0, 1) \notin B$. If $x_1 > 0$ then $(1, x_1^{-1} x_2) \in B$ and therefore $x_1^{-1} x_2 \leq t^\star$. This shows that $B \subset \{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 \leq t^\star x_1\}$.

Notice that for all $t \in \mathbb{R}_+$, $(t, t^\star t) \in B$; since B is conical and $(1, t^\star) \in B$, from the closedness of B.

If $(x_1, x_2) \in \mathbb{R}^2_+$ is such that $x_2 \leq t^\star x_1$ and $x_1 \neq 0$ then $0 < (t^\star x_1)^{-1} x_2 \leq 1$ and

$$(x_1, x_2) = (t^\star x_1)^{-1} x_2 (x_1, t^\star x_1) \vee (x_1, 0).$$

Both $(x_1, t^\star x_1)$ and $(x_1, 0)$ are in B which is $\mathbb{B}$-convex. This shows that $(x_1, x_2) \in B$ and therefore that $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 \leq t^\star x_1\} \subset B$. □

Lemma 6.4. *Let B be a closed normal conical halfspace of $\mathbb{R}^n_+$ for which $\Lambda'(B) = \{i\}$. Then there exists $x^\star \in \mathbb{R}^n_{++}$ such that*

$$B = \left\{ x \in \mathbb{R}^n_+ : \left(\frac{x_i}{x_i^\star}\right) - \max_{j \neq i} \left(\frac{x_j}{x_j^\star}\right) \leq 0 \right\}.$$

Proof. Let $A = \overline{\mathbb{R}^n_+ \backslash B}$. It is a normal closed halfspace for which $\Lambda(A) = \{i\}$ and $\Lambda'(A) = [n] \backslash \{i\}$, Lemma 4.3. For all $j \in \Lambda'(A)$ the set $A_{[j]}$ is a closed normal conical halfspace in $\mathbb{R}^{\{i,j\}}_+$ such that $\Lambda(A_{[j]}) = \{i\}$ and $\Lambda'(A_{[j]}) = \{j\}$. By Lemma 6.3, for all $j \neq i$, there exists $x_j^{\star(i,j)} \in \mathbb{R}_{++}$ such that

$$(\sharp) \quad A_{[j]} = \left\{ x \in \mathbb{R}^{\{i,j\}}_+ : \left(x_j / x_j^{\star(i,j)}\right) - x_i \leq 0 \right\}.$$

By Lemma 4.4, for all $x \in \mathbb{R}^n_+$, $x \in A$ if and only if, for all $j \neq i$, $x_{[j]} \in A_{[j]}$, where $x_{[j]} = x_j e_j \vee x_i e_i$. From $(\sharp)$ we obtain

$$(\sharp\sharp) \quad A = \left\{ x \in \mathbb{R}^n_+ : \forall j \in [n] \backslash \{i\} \ \left(x_j / x_j^{\star(i,j)}\right) - x_i \leq 0 \right\}.$$

Define an element $x^\star$ of $\mathbb{R}^n_{++}$ as follows

$$\begin{cases} x_i^\star = 1 \\ \\ x_j^\star = x_j^{\star(i,j)} \ \text{ if } j \neq i. \end{cases}$$

From the definition of A and from $(\sharp\sharp)$ we obtain,

$$\text{int}B = \left\{ x \in \mathbb{R}^n_+ : \left(\frac{x_i}{x_i^\star}\right) - \max_{j \neq i} \left(\frac{x_j}{x_j^\star}\right) < 0 \right\}.$$

By Lemma 6.2 B is the closure of its interior.

To complete the proof we have to see that $\left\{ x \in \mathbb{R}^n_+ : \left(\frac{x_i}{x_i^\star}\right) - \max_{j \neq i} \left(\frac{x_j}{x_j^\star}\right) \leq 0 \right\}$ is a subset of the closure of $\left\{ x \in \mathbb{R}^n_+ : \left(\frac{x_i}{x_i^\star}\right) - \max_{j \neq i} \left(\frac{x_j}{x_j^\star}\right) < 0 \right\}$.

In other words, assuming that the point $x \in \mathbb{R}^n_+$ is such that

$$\left(\frac{x_i}{x_i^\star}\right) = \max_{j \neq i} \left(\frac{x_j}{x_j^\star}\right)$$

we have to produce a sequence $(x^{(m)})_{m \in \mathbb{N}}$ in $\mathbb{R}^n_+$ which converges to x and such that, for all $m \in \mathbb{N}$, $\left(\frac{x_i^{(m)}}{x_i^\star}\right) - \max_{j \neq i} \left(\frac{x_j^{(m)}}{x_j^\star}\right) < 0$.

Let $x^{(m)} = x \vee \left(\bigvee_{j \neq i} (x_j + 2^{-m}) e_j\right)$ and notice that, for all $m \in \mathbb{N}$, $\bigvee_{j \neq i} (x_j + 2^{-m}) e_j \in B$. From $x \in B$ we have $x^{(m)} \in B$.

The sequence $(x^{(m)})_{m \in \mathbb{N}}$ clearly converges to x furthermore $x_i^{(m)} = x_i$ and

$$\max_{j \neq i} \left(\frac{x_j^{(m)}}{x_j^\star}\right) = \max_{j \neq i} \left(\frac{x_j + 2^{-m}}{x_j^\star}\right) > \max_{j \neq i} \left(\frac{x_j}{x_j^\star}\right)$$

The sequence $(x^{(m)})_{m \in \mathbb{N}}$ has all the required properties. □

We can now state and prove the main result of this section.

Theorem 6.5. *A nonempty subset B of $\mathbb{R}^n_+$ is a closed normal conical halfspace if and only if there exists $x^\star \in \mathbb{R}^n_{++}$ such that*

$$B = \left\{ x \in \mathbb{R}^n_+ : \max_{i \in \Lambda'(B)} \left(\frac{x_i}{x_i^\star}\right) - \max_{j \in \Lambda(B)} \left(\frac{x_j}{x_j^\star}\right) \leq 0 \right\}.$$

Proof. Assume that B is a closed normal conical halfspace in $\mathbb{R}^n_+$. By Lemma 4.4 and Lemma 6.4 there exists for all $i \in \Lambda'(B)$ an $x^{\star(i)} \in \mathbb{R}^{\Lambda(B)}_{++}$ such that

$$B_{[i]} = \left\{ x \in \mathbb{R}^{\Lambda(B) \cup \{i\}}_+ : x_i - \max_{j \in \Lambda(B)} \left(\frac{x_j}{x_j^{\star(i)}}\right) \leq 0 \right\}.$$

There is a halfline in $\mathbb{R}^{\Lambda(B)}_{++}$ on which all the $x^{\star(i)}$, $i \in \Lambda'(B)$, are situated. Indeed, if it were not so then there would exist two indices i and k in $\Lambda'(B)$, two indices j_1 and j_2 in $\Lambda(B)$ and a real number $r > 0$ such that

$$\frac{x_{j_1}^{\star(i)}}{x_{j_2}^{\star(i)}} > r > \frac{x_{j_1}^{\star(k)}}{x_{j_2}^{\star(k)}}.$$

We define two elements x and y of $\mathbb{R}^n_+$ as follows:

$x_i = \dfrac{1}{x^{\star(i)}_{j_2}}$, $x_{j_1} = r$ and $x_l = 0$ otherwise;

$y_k = \dfrac{r}{x^{\star(k)}_{j_1}}$, $y_{j_2} = 1$ and $y_l = 0$ otherwise.

From $x_i - \max_{j\in\Lambda(B)} \left(\dfrac{x_j}{x^{\star(i)}_j}\right) = \dfrac{1}{x^{\star(i)}_{j_2}} - \dfrac{r}{x^{\star(i)}_{j_1}} > 0$ and

$y_k - \max_{j\in\Lambda(B)} \left(\dfrac{y_j}{x^{\star(k)}_j}\right) = \dfrac{r}{x^{\star(k)}_{j_1}} - \dfrac{1}{x^{\star(k)}_{j_2}} > 0$ we have $x_{[i]} \notin B_{[i]}$ and $y_{[k]} \notin B_{[k]}$

and therefore, $x \notin B$ and $y \notin B$. Since B is a halfspace we also have $x \vee y \notin B$. If $l \in \Lambda'(B)\backslash\{i,k\}$ then $\max\{x_l, y_l\} = 0$ and consequently $(x\vee y)_{[l]} \in B_{[l]}$. We must therefore have either $(x \vee y)_{[i]} \notin B_{[i]}$ or $(x\vee y)_{[k]} \notin B_{[k]}$.

If $(x \vee y)_{[i]} \notin B_{[i]}$ then $\max\{x_i, y_i\} > \max_{j\in\Lambda(B)} \left(\dfrac{\max\{x_j, y_j\}}{x^{\star(i)}_j}\right) > 0$ that is

$\dfrac{1}{x^{\star(i)}_{j_2}} > \max\left\{\dfrac{r}{x^{\star(i)}_{j_1}}, \dfrac{1}{x^{\star(i)}_{j_2}}\right\}$; similarly, the condition $(x\vee y)_{[k]} \notin B_{[k]}$ leads to

$\dfrac{r}{x^{\star(k)}_{j_1}} > \max\left\{\dfrac{r}{x^{\star(k)}_{j_1}}, \dfrac{1}{x^{\star(k)}_{j_2}}\right\}$. We have reached a contradiction.

In conclusion, there exist $x^\star_{\Lambda(B)} \in \mathbb{R}^{\Lambda(B)}_{++}$ and, for all $i \in \Lambda'(B)$, $x^\star_i \in \mathbb{R}_{++}$ such that $x^\star_{\Lambda(B)} = x^\star_i x^{\star(i)}$. From this we have, for all $i \in \Lambda'(B)$,

$$B_{[i]} = \left\{x \in \mathbb{R}^{\Lambda(B)\cup\{i\}}_+ : \frac{x_i}{x^\star_i} - \max_{j\in\Lambda(B)} \left(\frac{x_j}{x^\star_j}\right) \le 0\right\}$$

and since, for all $x \in \mathbb{R}^n_+$, $x \in B$ if and only if, for all $i \in \Lambda'(B)$, $x_{[i]} \in B_{[i]}$ we have

$$B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in\Lambda'(B)} \left(\frac{x_i}{x^\star_i}\right) - \max_{j\in\Lambda(B)} \left(\frac{x_j}{x^\star_j}\right) \le 0\right\}.$$

This completes this part of the proof.

The proof that a subset of $\mathbb{R}^n_+$ of the form described above is a normal closed conical halfspace is easy. □

6.2 Characterization of General Conical Halfspaces

The characterization of arbitrary closed conical halfspaces is obtained from Proposition 2.7 and Theorem 6.5 .

Theorem 6.6. *If $B \subset \mathbb{R}^n_+$ is a closed conical halfspace with nonempty interior then there exist two disjoint subsets I and J of $[n]$ and an $x^\star \in B$ with $x^\star_l > 0$ for all $l \in I \cup J$ such that*

$$B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in I} \left(\frac{x_i}{x^\star_i}\right) - \max_{j\in J} \left(\frac{x_j}{x^\star_j}\right) \le 0\right\}.$$

If the interior of B is empty then there is a third subset $K \subset [n]$ such that

$$B = \left\{ x \in \mathbb{R}^n_+ : \max_{i \in I} \left(\frac{x_i}{x_i^\star} \right) - \max_{j \in J} \left(\frac{x_j}{x_j^\star} \right) \le 0 \text{ and } \max_{k \in K} x_k = 0 \right\}.$$

Proof. (A) Let us assume that B is a closed conical halfspace with nonempty interior. If $B = \mathbb{R}^n_+$ take $I = \emptyset$, $J = [n]$ and $x^\star$ an arbitrary element of $\mathbb{R}^n_{++}$. We assume that $B \neq \mathbb{R}^n_+$.

Let $I = \{i \in [n] : e_i \in \text{int}B\}$ and $J = \Lambda'(B)$. If I were empty then e_i would be in $\overline{\mathbb{R}^n_+ \setminus B}$ for all $i \in [n]$ and this would imply $\overline{\mathbb{R}^n_+ \setminus B} = \mathbb{R}^n_+$ and therefore $\text{int}B = \emptyset$. Also $J \neq \emptyset$, since $B \neq \mathbb{R}^n_+$, and $I \cap J = \emptyset$ is obvious. Let

$$\Lambda^\partial(B) = \{i \in \Lambda(B) : e_i \in \partial B\}$$

and, for all $x \in \mathbb{R}^n_+$ let

$$x_\sharp = \bigvee_{k \in I \cup J} x_k e_k \text{ and } B_\sharp = \{x_\sharp : x \in B\}.$$

We show that $B_\sharp$ is a closed conical and normal halfspace with nonempty (relative) interior in $\mathbb{R}^{I \cup J}_+$.

(a) For all $x \in \mathbb{R}^n_+$ we have $x = x_\sharp \vee x_{\Lambda^\partial(B)}$. If, for a given x, $x_\sharp \notin B$ then, from $x_{\Lambda^\partial(B)} \in \partial B \subset \overline{\mathbb{R}^n_+ \setminus B}$, we have $x \in \overline{\mathbb{R}^n_+ \setminus B}$. This shows that, for all $x \in \text{int}B$, $x_\sharp \in B$.

If x is an arbitrary element of B then, by Lemma 6.2, there exists a sequence $(x^{(m)})_{m \in \mathbb{N}}$ of elements of $\text{int}B$ which converges to x. The sequence $(x_\sharp^{(m)})_{m \in \mathbb{N}}$ converges to $x_\sharp$. We have shown that, for all $x \in B$, $x_\sharp \in B$ and therefore

$$(\sharp_1) \quad B_\sharp = B \cap \mathbb{R}^{I \cup J}_+$$

and, from $x = x_\sharp \vee x_{\Lambda^\partial(B)}$ and $x_{\Lambda^\partial(B)} \in \partial B \subset B$,

$$(\sharp_2) \quad B = \{x \in \mathbb{R}^n_+ : x_\sharp \in B_\sharp\}.$$

From $(\sharp_1)$ one can see that $B_\sharp$ is $\mathbb{B}$-convex, closed and conical.

(b) If $u, v \in \mathbb{R}^{I \cup J}_+ \setminus B_\sharp$ then, by $(\sharp_1)$, u and v are in $\mathbb{R}^n_+ \setminus B$, which is $\mathbb{B}$-convex, and therefore, for all $t \in [0,1]$, $tu \vee v \in \left[\mathbb{R}^n_+ \setminus B\right] \cap \mathbb{R}^{I \cup J}_+$. From $(\sharp_1)$ we conclude that $tu \vee v \in \mathbb{R}^{I \cup J}_+ \setminus B_\sharp$. This shows that $B_\sharp$ is a halfspace in $\mathbb{R}^{I \cup J}_+$.

(c) If x is an arbitrary element of $B \cap \mathbb{R}^n_{++}$, which is not empty by Lemma 2.6, then $x_\sharp \in B_\sharp \cap \mathbb{R}^{I \cup J}_{++}$. This shows that the relative interior of $B_\sharp$ in $\mathbb{R}^{I \cup J}_+$ is not empty.

(d) To complete this part of the proof we have to see that the normality condition holds for $B_\sharp$, as a subset of $\mathbb{R}^{I \cup J}_+$. By construction, $\Lambda(B_\sharp) = I$ (where, of course, $\Lambda(B_\sharp) = \{k \in I \cup J : e_k \in B_\sharp\}$). If $i \in I$ then e_i is in the interior of B in $\mathbb{R}^n_+$ and therefore, by $(\sharp_1)$, in the relative interior of $B_\sharp$ in $\mathbb{R}^{I \cup J}_+$.

In conclusion, we have verified that $B_\sharp$ is a closed normal conical halfspace in $\mathbb{R}^{I \cup J}_+$.

By Theorem 6.5 there exists an element $x^\star$ of $\mathbb{R}^{I \cup J}_{++}$ such that

$$B_\sharp = \left\{ x \in \mathbb{R}^{I \cup J}_+ : \max_{i \in I} \left(\frac{x_i}{x_i^\star} \right) - \max_{j \in J} \left(\frac{x_j}{x_j^\star} \right) \le 0 \right\}$$

and, by ($\sharp_2$),

$$B = \left\{ x \in \mathbb{R}^n_+ : \max_{i \in I} \left(\frac{x_i}{x_i^\star} \right) - \max_{j \in J} \left(\frac{x_j}{x_j^\star} \right) \le 0 \right\}.$$

(B) If the interior of B is empty then the argument above and Proposition 2.7 conclude the proof. □

7 The General Case

First, we show that the caracterization of closed halfspaces with nonempty interior can be reduced to the caracterization of conical halfspaces. The general caracterization then follows from Theorem 6.6.

Given a subset B of $\mathbb{R}^n_+$ let $\mathcal{K}(B) = \{tx : t \in \mathbb{R}_+ \text{ and } x \in B\}$. If B is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ then $\mathcal{K}(B)$ and $\overline{\mathcal{K}(B)}$ are $\mathbb{B}$-convex and conical. We set $B_1 = B \times \{1\}$. If B is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ then B_1 is a $\mathbb{B}$-convex subset of $\mathbb{R}^{n+1}_+$ and therefore $\overline{\mathcal{K}(B_1)}$ is a $\mathbb{B}$-convex and conical subset of $\mathbb{R}^{n+1}_+$.

Lemma 7.1. *If B is a closed halfspace of $\mathbb{R}^n_+$ (with nonempty interior) then $\overline{\mathcal{K}(B_1)}$ is a nonempty closed conical halfspace of $\mathbb{R}^{n+1}_+$ (with nonempty interior).*

Proof. If B is a closed halfspace of $\mathbb{R}^n_+$ with nonempty interior then there is a strictly positive element $(x_1, \cdots, x_n)$ in B and therefore $(x_1, \cdots, x_n, 1)$ is in $\mathcal{K}(B_1)$. We only have to see that $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$ is $\mathbb{B}$-convex.

We have seen that $\overline{\mathcal{K}(B_1)}$ is conical. Therefore $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$ is also conical. To complete the proof we have to show that $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$ is a semilattice, that is, $x \vee y \in \mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$ whenever x and y are in $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$.

Let us write $B = B_{\text{down}} \cup B_\infty$ where B_{down} is the downward part of B and B_∞ is its asymptotic part, they are both closed halfspaces containing 0. From $\mathcal{K}(B_1) = \mathcal{K}(B_{\text{down } 1}) \cup \mathcal{K}(B_{\infty\ 1})$ we have

$$\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)} = \mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_{\text{down } 1})} \cap \mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_{\infty\ 1})}.$$

To show that $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$ is $\mathbb{B}$-convex it is enough to show that $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_{\text{down } 1})}$ and $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_{\infty\ 1})}$ are $\mathbb{B}$-convex. We therefore split the proof of the $\mathbb{B}$-convexity of $\mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{K}(B_1)}$ in two parts: in the first part we assume that B is downward, in the second part we assume that B is conical.

(A) We assume that B is downward and that the interior of B is not empty. There exists $a \in \mathbb{R}^n_+$ such that

$$B = \{x \in \mathbb{R}^n_+ : \max_{i \in [n]} \{a_i x_i\} \le 1\}.$$

Then, $(x_1, \cdots, x_{n+1}) \in \mathcal{K}(B_1)$ if and only if either $(x_1, \cdots, x_{n+1}) = (0, \cdots, 0)$ or $x_{n+1} > 0$ and $(x_1/x_{n+1}, \cdots, x_n/x_{n+1}) \in B$ and therefore

$$\mathcal{K}(B_1) \backslash \{0\} = \{(x_{[n]}, x_{n+1}) \in \mathbb{R}^{n+1}_+ : x_{n+1} > 0 \text{ and } \max_{i \in [n]} \{a_i x_i\} \le x_{n+1}\}.$$

From $\overline{\mathcal{K}(B_1)} = \overline{\mathcal{K}(B_1) \backslash \{0\}}$ we obtain

$$\overline{\mathcal{K}(B_1)} = \{(x_{[n]}, x_{n+1}) \in \mathbb{R}^{n+1}_+ : \max_{i\in[n]}\{a_i x_i\} \leq x_{n+1}\}.$$

which is a halfspace.

If the interior of B is empty then there is a nonempty subset K of $[n]$ for which we have to add to the previous condition the constraints $x_k = 0$ for all $k \in K$.

(B) We assume that B is conical with nonempty interior.

There is a vector $a \in \mathbb{R}^n_+$ and two disjoint subsets I and J of $[n]$ such that

$$B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in I}\{a_i x_i\} \leq \max_{j\in I}\{a_j x_j\}\right\}$$

and therefore

$$\mathcal{K}(B_1)\backslash\{0\} = \left\{(x_{[n]}, x_{n+1}) \in \mathbb{R}^{n+1}_+ : x_{n+1} > 0 \text{ and } \max_{i\in I}\{a_i x_i\} \leq \max_{j\in I}\{a_j x_j\}\right\}$$

and finally

$$\overline{\mathcal{K}(B_1)} = \left\{(x_{[n]}, x_{n+1}) \in \mathbb{R}^{n+1}_+ : \text{ and } \max_{i\in I}\{a_i x_i\} \leq \max_{j\in I}\{a_j x_j\}\right\}$$

which is a halfspace in $\mathbb{R}^{n+1}_+$.

If the interior of B is empty we proceed as in (A). □

Theorem 7.2. *If B is closed halfspace of $\mathbb{R}^n_+$ such that $0 \in B$ and $intB \neq \emptyset$, then there exist a vector $a \in \mathbb{R}^n_+$, a real number $s \in \mathbb{R}_+$ (which we can take to be either 0 or 1), and two disjoint subsets I and J of $[n]$ such that*

$$B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in I}\{a_i x_i\} \leq \max_{j\in J}\{a_j x_j, s\}\right\}.$$

If the interior of B is empty then there is a third subset K of $[n]$ such that,

$$B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in I}\{a_i x_i\} \leq \max_{j\in J}\{a_j x_j, s\} \text{ and } \max_{k\in K}\{x_k\} = 0\right\}.$$

Proof. $\overline{\mathcal{K}(B_1)}$ is a closed conical halfspace of $\mathbb{R}^{n+1}_+$. There are subsets U, V and W of $[n+1]$, with $W = \emptyset$ if the interior of B is not empty, and a vector $(a_1, \cdots, a_{n+1}) \in \mathbb{R}^{n+1}_+$ such that $(x_1, \cdots, x_{n+1}) \in \overline{\mathcal{K}(B_1)}$ if and only if $\max_{i\in U}\{a_i x_i\} \leq \max_{j\in V}\{a_j x_j\}$ and $x_k = 0$ for $k \in W$.
Without loss of generality we can assume that $a_i = 0$ if $i \notin U \cup V$ and $a_i > 0$ if $i \in U \cup V$. Notice that $n+1 \notin W$, since $B \times \{1\} \subset \mathcal{K}(B_1)$; therefore $W \subset [n]$. Set $K = W$. From $(0, \cdots, 0, 1) \in \mathcal{K}(B_1)$ we see that $n+1 \notin U$. Let $I = U$ and $J = V \setminus \{n+1\}$.

Notice also that $(x_1, \cdots, x_n) \in B$ if and only if $(x_1, \cdots, x_n, 1) \in \overline{\mathcal{K}(B_1)}$ (consider a sequence in $\mathcal{K}(B_1)$ converging to $(x_1, \cdots, x_n, 1)$ with the last term of each element of the sequence strictly positive and recall that B is closed). Therefore, $(x_1, \cdots, x_n) \in B$ if and only if, $x_k = 0$ for all $k \in K$ and $\max_{i\in I}\{a_i x_i\} \leq \max_{j\in J}\{a_j x_j, a_{n+1}\}$. □

Proposition 7.3. *Let I and J be nonempty and disjoint subsets of $[n]$ and $a \in \mathbb{R}^{I\cup J}_{++}$. If $B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in I}\{a_i x_i\} \leq \max_{j\in J}\{a_j x_j, s\}\right\}$, then the interior of B is $\left\{x \in \mathbb{R}^n_+ : \max_{i\in I}\{a_i x_i\} < \max_{j\in J}\{a_j x_j, s\}\right\}$ and the boundary of B is $\left\{x \in \mathbb{R}^n_+ : \max_{i\in I}\{a_i x_i\} = \max_{j\in J}\{a_j x_j, s\}\right\}$.*

Proof. Since $B = \bigcap_{i\in I} \left\{x \in \mathbb{R}^n_+ : a_i x_i \le \max_{j\in J}\{a_j x_j, s\}\right\}$ we have to see that the interior of $\left\{x \in \mathbb{R}^n_+ : a_i x_i \le \max_{j\in J}\{a_j x_j, s\}\right\}$ is $\left\{x \in \mathbb{R}^n_+ : a_i x_i < \max_{j\in J}\{a_j x_j, s\}\right\}$ and we can assume that s is either 0 or 1. The details are left to the reader. The second part follows from the first since, B being closed, $\partial B = B \backslash \mathrm{int} B$. □

If B is an open halfspace containing 0 then $\overline{B}$ is a closed halfspace with nonempty interior containing 0. The formula $\mathrm{int} A = \mathbb{R}^n_+ \setminus \overline{\left(\mathbb{R}^n_+ \setminus A\right)}$ holds for arbitrary subsets of $\mathbb{R}^n_+$ and $\mathbb{R}^n_+ \setminus B$ is a closed halfspace which, by Lemma 6.2, is the closure of its interior; taking into account that B is open, we have $B = \mathrm{int}\overline{B}$. From Proposition 7.3 we obtain the analytic description of open halfspaces containing 0; the analytic description of closed halfspaces whose complement contains 0 follows easily.

A closed $\mathbb{B}$-convex set with nonempty interior does not have to be the closure of its interior; an example can be found in [2]. For halfspaces the situation is somewhat better as is shown by the result below which completes Lemma 6.2.

Corollary 7.4. *A closed halfspace with nonempty interior is the closure of its interior.*

Proof. By Theorem 7.2 and Proposition 7.3. □

So far the description of halfspaces has been asymmetrical, the analytic representation of a halfspace to which 0 belongs is somewhat different from the analytic representation of a halfspace to which 0 does not belong. Using Theorem 7.2 one can see that a halfspace with nonempty interior, whether it contains 0 or not, can be written

$$B = \left\{x \in \mathbb{R}^n_+ : \max_{i\in[n]} \{u_i x_i, r\} \le \max_{i\in[n]} \{v_i x_i, s\}\right\}$$

with $u, v \in \mathbb{R}^n_+$, and $r, s \in \mathbb{R}_+$. It has been shown in Proposition 2.8 that such a set is always a closed halfspace, possibly with empty interior, or even empty. Using Lemma 2.6, necessary and sufficient conditions on the parameters (r, s, u, v) for the interior of B to be nonempty are not hard to find.

8 Functional Separation of $\mathbb{B}$-convex Sets

Two subsets A and B of a metric space (X, d), $\mathbb{R}^n_+$ for example, are nonproximate if $\inf_{(x,y)\in A\times B} d(x, y) > 0$. If C_1 and C_2 are nonproximate $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ then, according to Theorem 7.2 in [3], there is a closed halfspace B such that C_1 is contained in the interior of B and C_2 is contained in $\mathbb{R}^n_+ \backslash B$. We can now use Theorem 7.2 and Proposition 7.3 to obtain a functional Hahn-Banach like separation Theorem in $\mathbb{B}$-convexity.

Theorem 8.1. *Let C_1 and C_2 be two nonproximate $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ then, there exist $u, v \in \mathbb{R}^n_+$ and $r, s \ge 0$ such that for all $x \in C_1$ and all $y \in C_2$*

$$\max_{i\in[n]}\{u_i x_i, r\} - \max_{i\in[i]}\{v_i x_i, s\} < \max_{i\in[n]}\{u_i y_i, r\} - \max_{i\in[n]}\{v_i y_i, s\}.$$

We have given Theorem 8.1 its symmetric form, but of course, we can assume that $\{i : u_i \ne 0\} \cap \{i : v_i \ne 0\} = \emptyset$ and that r and s are either 0 or 1.

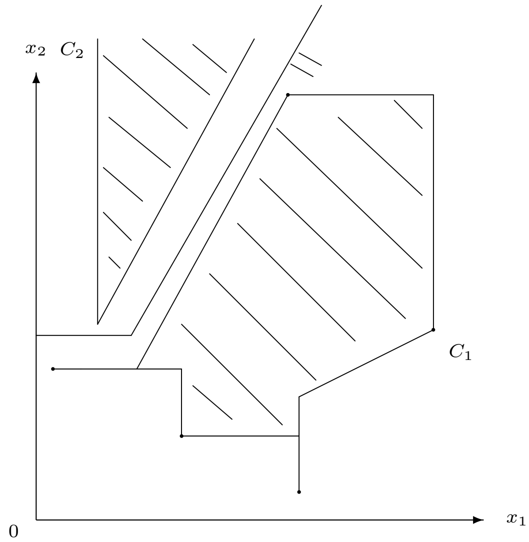

Figure 8 Separation in $\mathbb{B}$-convexity.

9 Max-Plus Convexity

The basic algebraic structures of semi-rings and of Maslov semi-modules over a semi-ring are presented in [9], applications can be found in [7] and[10] and also [5] for Max-Plus. We do not want to go here in the full generality of Maslov's semi-modules. We only wish to point to the fact that Max-Plus convexity - in its continuous as opposed to its discrete version - and $\mathbb{B}$-convexity - in the finite dimensional version presented here - are isomorphic topological Maslov's semi-modules and, consequently, a proposition that is true in the framework of $\mathbb{B}$-convexity holds, with obvious lexical modifications, in Max-Plus convexity.

To be more precise, for x and y in $\mathbb{M}^n$ let $d_{\mathrm{M}_+}(x,y) =\| \mathbf{e}^x - \mathbf{e}^y \|_\infty$ where $\mathbf{e}^x = (e^{x_1}, \cdots, e^{x_n})$, with the convention $e^{-\infty} = 0$, and, for $u \in \mathbb{R}^n_+$, $\| u \|= \max_{1\le i\le n} x_i$. The map $x \mapsto \mathbf{e}^x$ is a homeomorphism from $\mathbb{M}^n$ with the metric d_{M_+} to $\mathbb{R}^n_+$ endowed with the metric induced by the norm $\|\cdot\|_\infty$; its inverse is the map $\mathbf{ln}(x) = (\ln(x_1), \cdots, \ln(x_n))$ from $\mathbb{R}^n_+$ to $\mathbb{M}^n$, with the convention $\ln(0) = -\infty$. One can easily show that a subset C of $\mathbb{M}^n$ is Max-Plus convex if an only if, for all $n \in \mathbb{N}\setminus\{0\}$, for all $(x_1, \cdots, x_n) \in C^n$ and for all $(t_1, \cdots, t_n) \in [-\infty, 0]^n$ such that $\max\{t_1, \cdots, t_n\} = 0$ one has $\bigvee_{i=1}^n \big(x_i + t_i \mathbb{1}_n\big) \in C$. The Max-Plus convex hull of a subset of $\mathbb{M}^n$ is the smallest Max-Plus convex set which contains it.

The following two assertions hold, and are equivalent:

(1) A subset C of $\mathbb{M}^n$ is Max-Plus convex if and only if the set $\{\mathbf{e}^x : x \in C\}$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$.

(2) A subset C of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if the set $\{\mathbf{ln}(x) : x \in C\}$ is a Max-Plus convex subset of $\mathbb{M}^n$. In other words, the map $x \mapsto \mathbf{e}^x$ is an order and distance preserving map, therefore an homeomorphism, which sends Max-Plus convex sets to $\mathbb{B}$-convex sets. Max-Plus convexity and $\mathbb{B}$-convexity are isomorphic and homeomorphic structures; to be more precise, they are isometric topological Maslov semi-modules over isomorphic semi-rings, respectively $\mathbb{R}\bigcup\{-\infty\}$ and $\mathbb{R}_+$. The translation of a statement from the language of $\mathbb{B}$-convexity to that of Max-Plus convexity is done with the following simple dictionary.

$\mathbb{B}$-convexity	Max-Plus convexity
$\mathbb{R}_+$	$\mathbb{R}\cup\{-\infty\}$
$[0,1]$	$[-\infty,0]$
0	$-\infty$
1	0
max	max
$\max\{u_1,\cdots,u_n\}=1$	$\max\{v_1,\cdots,v_n\}=0$
$u_1.u_2$	v_1+v_2

For example, in Max-Plus convexity halfspaces are described by Theorem 9.1, which comes from Theorem 7.2. The Hahn-Banach Theorem in Max-Plus convexity is Theorem 9.2 below.

For $x \in \mathbb{R}\cup\{-\infty\}$ and $S \subset \mathbb{R}\cup\{-\infty\}$ let $I_\infty(x) = \{i \in [n] : x_i = -\infty\}$, $I_\Im(x) = [n]\setminus I_\infty(x)$, $I_\infty(S) = \bigcap_{x\in S} I_\infty(x)$ and $I_\Im(S) = [n]\setminus I_\infty(S)$. The smallest element of $(\mathbb{R}\cup\{-\infty\})^n$, that is $(-\infty,\cdots,-\infty)$ is denoted by $-\infty_{\mathbf{n}}$.

Theorem 9.1. *A closed subset M of $(\mathbb{R}\cup\{-\infty\})^n$ which contains $-\infty_{\mathbf{n}}$ is a halfspace if and only if there exists $a \in (\mathbb{R}\cup\{-\infty\})^n$, $s \in \mathbb{R}\cup\{-\infty\}$ and two disjoint subsets I and J of $I_\Im(M)$ such that*

$$M = \Big\{x \in (\mathbb{R}\cup\{-\infty\})^n : \forall k \in I_\infty(M)\ x_k = -\infty \quad \textit{and} \quad \max_{i\in I}\{x_i + a_i\} \le \max_{j\in J}\{x_j + a_j, s\}\Big\}.$$

Furthermore, the interior of M is nonempty if and only if $I_\infty(M) = \emptyset$ and, in that case,

$$\textit{int}(M) = \Big\{x \in (\mathbb{R}\cup\{-\infty\})^n : \max_{i\in I}\{x_i + a_i\} < \max_{j\in J}\{x_j + a_j, s\}\Big\}.$$

Also, the interior and the closure of a halfspace are halfspaces.

Theorem 9.2 (Analytic Hahn-Banach in Max-Plus). *Let C_1 and C_2 be two nonproximate Max-Plus convex sets. Then, there exists $s, t \in \mathbb{R}\cup\{-\infty\}$ and two disjoint subsets I and J of $[n]$ such*

$$\forall x \in C_1 \quad \max_{i\in I}\{x_i + a_i, t\} < \max_{j\in J}\{x_j + a_j, s\}$$
$$\textit{and}$$
$$\forall x \in C_2 \quad \max_{j\in J}\{x_j + a_j, s\} < \max_{i\in I}\{x_i + a_i, t\}.$$

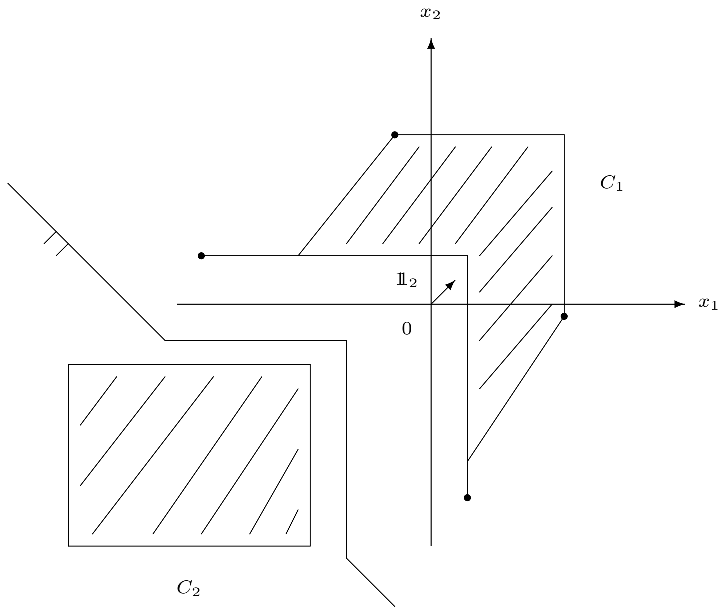

Figure 9 Separation in Max-Plus convexity.

The metric d_{M_+} might seem artificial and *ad hoc*, defined with the obvious intention to make our statements true. There are two things about the metric d_{M_+} that one could notice, beside that it serves our pourpose well. Firstly, the max operation $(x, y) \mapsto x \vee y$ from $\big(\mathbb{R} \cup \{-\infty\}\big)^n \times \big(\mathbb{R} \cup \{-\infty\}\big)^n$ to $\big(\mathbb{R} \cup \{-\infty\}\big)^n$ is continuous and secondly, for all $x \in \big(\mathbb{R} \cup \{-\infty\}\big)^n$, the set $\downarrow\{x\}$, that is $\{y \in \big(\mathbb{R} \cup \{-\infty\}\big)^n : y \leq x\}$, is compact. Using a hard and classical result of J.D. Lawson, [8], one could show that there is only one topology on $\big(\mathbb{R} \cup \{-\infty\}\big)^n$ with the two properties above. It is therefore the topology associated with the metric d_{M_+}. One can also notice that the induced topology on $\mathbb{R}^n$ is the usual euclidean topology.

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