Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 4, No. 2, pp. 279 – 291, May 2008

BILEVEL PROGRAMS: APPROXIMATION RESULTS INVOLVING REVERSE CONVEX PROGRAMS

Abdelmalek Aboussoror*, Zakia Ankhili and Abdelatif Mansouri

Abstract: In this paper, for a bilevel problem (S), we first give an approach by a sequence of reverse convex programs. Then, using some results on reverse convex programs, we give a second approach which reduces the problem (S) to a min-max problem with linked constraints. Finally, by considering the case where the follower's objective and constraint functions are respectively polyhedral and linear, the second approach is improved. In fact, in this last case, the problem (S) is reduced to a maximization problem of a polyhedral convex function over a compact convex set.

Key words: *two-level optimization, convex analysis, reverse convex programs, multifunctions*

Mathematics Subject Classification: *90D65, 26A51, 90C25, 90C26*

1 Introduction

We are concerned with a bilevel nonlinear programming problem relating to a two-player game in which a leader plays against a follower. For an announced strategy x by the leader, the follower reacts by playing optimally, and the aim of the two players is to minimize their own objective functions. Furthermore, it is assumed that the leader disposes of full information about the follower. More precisely, let

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad G = (G_1, ..., G_p) : \mathbb{R}^n \to \mathbb{R}^p,$$

be respectively the objective and the constraint functions of the leader, and

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad g = (g_1, ..., g_q) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q,$$

be respectively the objective and the constraint functions of the follower, with F, G, f and g being convex. As it is well known, in the optimistic case, the leader's problem (called the upper-level problem) is formulated as follows:

$$(S) \qquad \operatorname*{Min}_{\substack{x\in\mathbb{R}^n \\ G(x)\leq 0}} \inf_{y\in M(x)} F(x,y),$$

where $M(x)$ is the solution set of the follower's problem (called the lower-level problem)

$$P(x) \qquad \operatorname*{Min}_{\substack{y\in\mathbb{R}^m \\ g(x,y)\leq 0}} f(x,y).$$

*Corresponding author

The problem (S) is termed a strong Stackelberg problem [3,4,11,13,15,16]. Notice that many concrete problems can be formulated as (S), for example problems dealing with engineering and economic context (several interesting applications are given in [5]). However, when the solution set of the follower's problem is not always a singleton, the problem (S) presents difficulties for a numerical resolution (also for theoretical results) and note that the major numerical results in the literature are given in the case where the lower level has a unique solution. In order to open other ways for a possible resolution via other global optimization problems, we first give an approach of (S) by a sequence of reverse convex programs $(\hat{S}_k)$. Then, by using some results on reverse convex programs which were established in [19], we give a second approach which reduces the problem (S) to a min-max problem. More precisely, we select a sequence of feasible points of the approximating problems $(\hat{S}_k)$ that satisfy a condition involving a min-max problem. Then, under additional assumptions, we show that a certain projection of any accumulation point of such a sequence solves the problem (S). When the functions f and g are respectively polyhedral and linear, the second approach can be improved. In fact, the problem (S) is reduced to a maximization problem of a polyhedral convex function over a compact convex set. Note that in this case, similar results are given in [5].

The content of the paper is as follows. In Section 2, we introduce some notations and recall a basic result concerning the subdifferential of the lower-level's marginal function. In Section 3, after establishing some preliminary results, we give the first approach of the original problem (S) by the problems $(\hat{S}_k)$. In Section 4, in the beginning we recall some definitions and results from [8,19] about a concept of stability of optimization problems and a duality between two optimization problems, which will be used for establishing our main results. Then, we establish some stability results and optimality conditions, and finally we give the second approach. In Section 5, we give the third approach when the functions f and g are respectively polyhedral and linear.

2 Preliminaries

Let us first introduce the following notations. Denote by

$$X = \{x \in \mathbb{R}^n / \, G(x) \leq 0\},$$

the leader's constraint set, and by

$$Y(x) = \{y \in \mathbb{R}^m / \, g(x,y) \leq 0\},$$

the follower's constraint set for an announced strategy x by the leader. Set

$$\mathcal{G} = \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} / \, G(x) \leq 0, g(x,y) \leq 0, f(x,y) \leq t\},$$

$$v(x) = \inf_{y \in Y(x)} f(x,y), \;\; \hat{F}(x,y,t) = F(x,y).$$

We will make the following assumptions.

(2.1) There exists a convex compact set $\mathcal{Z}$ of $\mathbb{R}^m$, such that $Y(x) \subset \mathcal{Z}$, for any $x \in X$,

(2.2) The Slater condition: for any $x \in X$, there exists $y \in \mathbb{R}^m$, such that $g(x,y) < 0$.

Recall that the marginal function v is convex since the functions f and g are convex (see for example [18]). We begin by the following lemma.

Lemma 2.1 ([1]). *Let $x \in X$. Suppose that assumptions (2.1) and (2.2) are satisfied. Then, there exists $y \in \mathbb{R}^m$, verifying $g(x,y) \leq 0$, and $v(x) = f(x,y)$, such that*

$$\partial v(x) \subset \partial_x f(x,y) + \sum_{j=1}^{q} \bigcup\nolimits_{\lambda_j \geq 0} \lambda_j \partial_x g_j(x,y),$$

where $\partial v(x)$ and $\partial_x f(x,y)$ are respectively the subdifferentials of v and $f(.,y)$ at x.

Consider the following bilevel programming problem

$$(\hat{S}) \qquad \underset{x \in X, y \in M(x)}{\text{Min}} F(x,y),$$

and for $\epsilon > 0$, consider the reverse convex program

$$(\hat{S}_\epsilon) \qquad \underset{\substack{(x,y,t) \in \mathcal{G}_\epsilon \\ v(x)-t \geq 0}}{\text{Min}} \hat{F}(x,y,t),$$

where

$$\mathcal{G}_\epsilon = \big\{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} / \, G(x) \leq 0, g(x,y) \leq 0, f(x,y) \leq t + \epsilon\big\}.$$

Remark 2.2. **i)** The problem $(\hat{S})$ has the same value as (S), and if (x,y) solves $(\hat{S})$, then x solves (S).

ii) If (x,y,t) is a feasible point of $(\hat{S}_\epsilon)$, then y is an ϵ-approximate solution of the problem $P(x)$.

3 The First Approach

In this section, we give an approach of (S) by a sequence of the reverse convex programs $(\hat{S}_{\epsilon_k})$, $\epsilon_k \searrow 0^+$, $k \in \mathbb{N}$. We begin by the following proposition which establishes the existence of solutions to the problem $(\hat{S}_\epsilon)$, $\epsilon > 0$.

Proposition 3.1. *Let $\epsilon > 0$. Assume that assumption (2.1) and the following assumption are satisfied*

(3.1) *The set X is bounded.*

Then, the problem $(\hat{S}_\epsilon)$ has at least one solution.

Proof. First, notice that the multifunction $Y(.) : X \rightrightarrows \mathbb{R}^m$, is upper semicontinuous and compact valued. Then, from the compactness of the set X, it follows that $\bigcup_{x \in X} Y(x)$ is a compact set (see [2]). On the other hand, for any feasible point (x,y,t) of $(\hat{S}_\epsilon)$, we have $f(x,y) - \epsilon \leq t \leq v(x)$. Since $(x,y) \in \bigcup_{x \in X} Y(x)$ (which is a compact set), and f and v are continuous, then the variable t belongs to a compact set. Hence, the set $\big\{(x,y,t) \in \mathcal{G}_\epsilon / \, v(x) \geq t\big\}$ is compact and the result follows from the continuity of $\hat{F}$. □

For $\epsilon_k > 0$, $k \in \mathbb{N}$, set $(x_{\epsilon_k}, y_{\epsilon_k}, t_{\epsilon_k}) = (x_k, y_k, t_k)$, and denote $(\hat{S}_{\epsilon_k})$ by $(\hat{S}_k)$. Then, we have following approximation result.

Theorem 3.2. *Let assumptions (2.1) and (3.1) hold. Let $\epsilon_k \searrow 0^+$, (x_k, y_k, t_k) be a solution of $(\hat{S}_k)$, and let $(\hat{x}, \hat{y}, \hat{t})$ be any accumulation point of the sequence $(x_k, y_k, t_k)_k$. Then, $\hat{x}$ solves (S).*

Proof. First, let us show the existence of accumulation points. As mentioned in the proof of Proposition 3.1, the set $\bigcup_{x\in X} Y(x)$ is compact. On the other hand, we have

$$f(x_k, y_k) - \epsilon_k \leq t_k \leq v(x_k).$$

Since $X \times \bigcup_{x\in X} Y(x)$ is a compact set, f and v are continuous and $\epsilon_k \searrow 0^+$, it follows that there exists $(a, b) \in \mathbb{R}^2$, such that

$$a \leq f(x_k, y_k) - \epsilon_k \leq t_k \leq v(x_k) \leq b.$$

Hence, $(x_k, y_k, t_k) \in X \times \bigcup_{x\in X} Y(x) \times [a, b]$, which is a compact set. Hence such an accumulation point exists. Let $\mathcal{N} \subset \mathbb{N}$, such that $\lim_{\substack{k\to+\infty\\ k\in\mathcal{N}}} (x_k, y_k, t_k) = (\hat{x}, \hat{y}, \hat{t})$. It is easy to see that

$$G(\hat{x}) \leq 0,\ g(\hat{x}, \hat{y}) \leq 0,\ f(\hat{x}, \hat{y}) \leq \hat{t}, \quad \text{and} \quad v(\hat{x}) \geq \hat{t}.$$

Then,

$$G(\hat{x}) \leq 0,\ g(\hat{x}, \hat{y}) \leq 0, \quad \text{and} \quad v(\hat{x}) = f(\hat{x}, \hat{y}).$$

Hence, $(\hat{x}, \hat{y})$ is a feasible point of $(\hat{S})$. Let (x, y) be any feasible point of $(\hat{S})$. Then, $(x, y, v(x))$ is feasible for $(\hat{S}_k)$. It follows that

$$F(x_k, y_k) \leq F(x, y).$$

By passing to the limit, we obtain $F(\hat{x}, \hat{y}) \leq F(x, y)$. Then, $(\hat{x}, \hat{y})$ solves $(\hat{S})$, and hence $\hat{x}$ solves (S) (Remark 2.2). $\square$

Remark 3.3. The applicability of Theorem 3.2 mainly depends on the applicability of the numerical methods that exist in the literature of reverse convex programs.

4 The Second Approach

4.1 Preliminaries

In order to establish our main result in this section (Theorem 4.10), we first need to show some stability results for every problem $(\hat{S}_\epsilon)$, $\epsilon > 0$. For this, we recall the following definitions and results from [8].

Let $\hat{f}, \hat{g} : \mathbb{R}^n \to \mathbb{R}$, α, $\beta \in \mathbb{R}$, and $\hat{D}$ is a nonempty subset of $\mathbb{R}^n$. Consider the following problems which are in duality

$$(\mathcal{P}_\beta) \qquad \operatorname*{Min}_{\substack{x\in\hat{D}\\ \hat{g}(x)\geq\beta}} \hat{f}(x), \qquad\qquad (\mathcal{Q}_\alpha) \qquad \operatorname*{Max}_{\substack{x\in\hat{D}\\ \hat{f}(x)\leq\alpha}} \hat{g}(x).$$

Definition 4.1. i) The problem $(\mathcal{P}_\beta)$ is stable if $\lim_{\beta'\to\beta^+} \inf \mathcal{P}_{\beta'} = \inf \mathcal{P}_\beta$.

ii) A feasible point x of $(\mathcal{P}_\beta)$ is said to be regular for $(\mathcal{P}_\beta)$, if there exists a sequence (x_k) converging to x such that $x_k \in \hat{D}$, and $\hat{g}(x_k) > \beta$, for large k.

We recall the following fundamental results.

Proposition 4.2 ([20]). *If* $\inf \mathcal{P}_\beta > -\infty$*,* $\hat{f}$ *is upper semicontinuous on* $\hat{D}$*, and there exists at least one solution of* $(\mathcal{P}_\beta)$ *that is regular for* $(\mathcal{P}_\beta)$*, then* $(\mathcal{P}_\beta)$ *is stable.*

Proposition 4.3 ([8]). *Assume that* $(\mathcal{P}_\beta)$ *is stable. Then,* $\beta \geq \sup \mathcal{Q}_\alpha$*, implies* $\alpha \leq \inf \mathcal{P}_\beta$*.*

Before introducing our additional assumptions, we need some notations and definitions. For $i \in \{1,...,p\}$, let $G_i^{'}(x;d)$ denote the directional derivative of G_i at x, in the direction $d \in \mathbb{R}^n \setminus \{0\}$, i.e.,

$$G_i^{'}(x;d) = \lim_{t \searrow 0^+} \frac{G_i(x+td) - G_i(x)}{t}$$

and let $I(x)$ denote the index set of active constraints G_i at x, i.e., $I(x) = \big\{i \in \{1,...,p\}/\, G_i(x) = 0\big\}$. For $k \in \{1,...,p\}$, set $I_k = \big\{1,...,k\big\}$, $J = \{1,...,q\}$,

$$\begin{cases} \mathcal{D}_{\epsilon,k} = \bigcup_{i \in I_k} \big\{(x,y,t) \in \mathcal{G}_\epsilon /\, G_i(x) = 0\big\} \bigcup \bigcup_{j \in J} \big\{(x,y,t) \in \mathcal{G}_\epsilon /\, g_j(x,y) = 0\big\} \\ \qquad\quad \bigcup \big\{(x,y,t) \in \mathcal{G}_\epsilon /\, f(x,y) - t = \epsilon\big\}, \\ \mathcal{G}_{\epsilon,k} = \mathcal{G}_\epsilon \setminus \mathcal{D}_{\epsilon,k}, \end{cases}$$

$$f_{\max} = \sup_{\substack{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \\ G(x) \le 0 \\ g(x,y) \le 0}} f(x,y), \quad \text{and} \quad f_{\min} = \inf_{\substack{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \\ G(x) \le 0 \\ g(x,y) \le 0}} f(x,y).$$

We will make the following assumptions which were introduced in [1].

(4.1) $\exists l \in I_p$, $\exists f_l \ge f_{\max}$, $\exists x(l) \in X$, $y(l) \in \operatorname{int} Y(x(l))$, and $t(l) \le f_{\min}$, with $f(x(l), y(l)) < t(l) + \epsilon$, such that

1) $\hat{F}(x(l), y(l), t(l)) < \inf_{\substack{(x,y,t) \in \mathcal{D}_{\epsilon,l} \\ t \le f_l}} \hat{F}(x,y,t)$,

and for any $(x,y) \in X \times \mathbb{R}^m$, such that $g(x,y) < 0$, we have

2) $0 \notin \bigcup_{i \in I(x)} \partial G_i(x)$,

3) $\begin{cases} \partial_x f(x,y) \subset \bigcap_{i \in I(x)} \big\{d \in \mathbb{R}^n /\, G_i^{'}(x;d) < 0\big\}, \\ \bigcup_{j \in J} \partial_x g_j(x,y) \subset \bigcap_{i \in I(x)} \big\{d \in \mathbb{R}^n /\, G_i^{'}(x;d) \le 0\big\}. \end{cases}$

(4.2) $\exists l \in I_p$, $\exists f_l \ge f_{\max}$, $\exists x(l) \in X$, $y(l) \in \operatorname{int} Y(x(l))$, and $t(l) \le f_{\min}$, with $f(x(l), y(l)) < t(l) + \epsilon$, such that

1) $\hat{F}(x(l), y(l), t(l)) < \inf_{\substack{(x,y,t) \in \mathcal{D}_{\epsilon,l} \\ t \le f_l}} \hat{F}(x,y,t)$,

and for any $(x,y) \in X \times \mathbb{R}^m$, such that $g(x,y) < 0$, and any $(u, v_j) \in \partial_x f(x,y) \times \partial_x g_j(x,y)$, $j = 1,...,q$, we have

2) $0 \notin \bigcup_{i \in I(x)} \partial G_i(x)$,

3) $\langle w_i, u \rangle \ge 0$, $\langle w_i, v_j \rangle \ge 0$, for any descent direction w_i of G_i at x, $i \in I(x)$, where $\langle .,. \rangle$ denotes the inner product of two vectors.

(4.3) $\exists l \in I_p$, $\exists f_l \ge f_{\max}$, $\exists x(l) \in X$, $y(l) \in \operatorname{int} Y(x(l))$, and $t(l) \le f_{\min}$, with $f(x(l), y(l)) < t(l) + \epsilon$, such that

1) $\hat{F}(x(l), y(l), t(l)) < \inf_{\substack{(x,y,t) \in \mathcal{D}_{\epsilon,l} \\ t \le f_l}} \hat{F}(x,y,t)$,

and for any $(x,y) \in X \times \mathbb{R}^m$, such that $g(x,y) < 0$, we have

2) the functions f and g are differentiable at (x,y), and for any $i \in I(x)$, the function G_i is differentiable at x, and satisfies

i) $\langle \nabla G_i(x), \nabla_x f(x,y)\rangle < 0$, $\langle \nabla G_i(x), \nabla_x g_j(x,y)\rangle \le 0$, $\forall j$, where ∇ stands for the gradient,

ii) $0 \notin \{\nabla G_i(x), i \in I(x)\}$.

Let us give in the following remark some explanations and properties that result from the above definitions and assumptions. Note that assumption (4.3) expresses the assumption (4.1) in the differentiable case.

Remark 4.4. **i)** Set $\mathcal{G}^*_{\epsilon,l} = \{x \in \mathbb{R}^n / \exists (y,t) \text{ such that } (x,y,t) \in \mathcal{G}_{\epsilon,l}\}$, i.e., the projection onto X of the set $\mathcal{G}_{\epsilon,l}$. Then, from the definition of $\mathcal{G}_{\epsilon,l}$, we have $\bigcup_{x\in\mathcal{G}^*_{\epsilon,l}} I(x) \subset \{l+1,...,p\}$.

ii) Assumption (4.1) implies that $(x(l),y(l),t(l))$ is a feasible point of $(\hat{S}_\epsilon)$. Furthermore, we have $t(l) \le f_l$, $(x(l),y(l),t(l)) \notin \mathcal{D}_{\epsilon,l}$, and

$$\hat{F}(x(l),y(l),t(l)) < \inf_{\substack{(x,y,t)\in\mathcal{D}_{\epsilon,l}\\ v(x)\ge t}} \hat{F}(x,y,t).$$

iii) If $(\hat{x},\hat{y},\hat{t})$ solves $(\hat{S}_\epsilon)$, then $(\hat{x},\hat{y},\hat{t}) \in \mathcal{G}_{\epsilon,l} \setminus \{(x,y,t) \in \mathcal{D}_{\epsilon,l} / v(x) \ge t\}$.

iv) We have $\mathcal{D}_{\epsilon,k} \subset \operatorname{bd}\mathcal{G}_\epsilon$, the boundary of $\mathcal{G}_\epsilon$.

v) If $(x,y,t) \notin \mathcal{D}_{\epsilon,k}$, then, $G_i(x) < 0$, $\forall i \in \{1,...,k\}$, $f(x,y) < t+\epsilon$, and $g_j(x,y) < 0$, $\forall j \in \{1,...,q\}$, but it is possible to have $G_i(x) = 0$, for some $i \in \{k+1,...,p\}$.

vi) Let $(x,y) \in X \times \mathbb{R}^m$, such that $g(x,y) < 0$. The property 3) of assumption (4.1) (with the convexity of the function $G_i^{'}(x;.)$) implies that for any $(u^*, v_j^*) \in \partial_x f(x,y) \times \partial_x g_j(x,y)$, $j \in J$, and any $(\alpha,\beta_j) \in \mathbb{R}^*_+ \times \mathbb{R}_+$, with $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the vector $\alpha u^* + \sum_{j\in J} \beta_j v_j^*$ is a descent direction of the function G_i at x, for $i \in I(x)$. This property will be used for establishing our stability results.

vii) The properties i) and ii) of 2) in assumption (4.3) will play the same role as property 3) of assumption (4.1).

$\boxed{4.2}$ The Second Approach

In order to give the second approach, we proceed in two steps. Firstly, we select a feasible point (x_k,y_k,t_k) of $(\hat{S}_k)$ that satisfies a condition that uses a min-max problem. Then, we show that the projection onto X of any accumulation points of the sequence $(x_k,y_k,t_k)_k$ solves (S).

Remark 4.5. Let $\epsilon > 0$. Let assumptions (2.1), (3.1) and (4.1) be satisfied, and let $(\hat{x},\hat{y},\hat{t})$ be a solution of $(\hat{S}_\epsilon)$. Assume that $I(\hat{x}) = \emptyset$. Then

$$G(\hat{x}) < 0,\ g(\hat{x},\hat{y}) \le 0,\ f(\hat{x},\hat{y}) \le \hat{t}+\epsilon, \text{ and } v(\hat{x}) \ge \hat{t}.$$

Since $(\hat{x},\hat{y},\hat{t})$ solves $(\hat{S}_\epsilon)$, it follows that

$$g(\hat{x},\hat{y}) < 0,\ f(\hat{x},\hat{y}) < \hat{t}+\epsilon.$$

Otherwise, $(\hat{x},\hat{y},\hat{t}) \in \mathcal{D}_{\epsilon,l}$, and the contradiction [see iii) of Remark 4.4]. Define $x_k = \hat{x}$, $y_k = \hat{y}$, and $t_k = \hat{t} - 1/k$, for all $k \ge 1$. Then $(x_k,y_k,t_k) \to (\hat{x},\hat{y},\hat{t})$, as $k \to +\infty$. Hence,

$$G(x_k) < 0,\ g(x_k,y_k) < 0,\ f(x_k,y_k) < t_k+\epsilon,\ v(x_k) > t_k, \quad \text{for large } k.$$

That is $(\hat{x},\hat{y},\hat{t})$ is regular for $(\hat{S}_\epsilon)$. Then, in the following propositions 4.6–4.8, without loss of generality, we will prove that any solution (x,y,t) of $(\hat{S}_\epsilon)$ is regular by assuming that $I(x) \ne \emptyset$.

Proposition 4.6. *Let $\epsilon > 0$. Let assumptions (2.1), (2.2), (3.1) and (4.1) hold. Then, the problem $(\hat{S}_\epsilon)$ is stable.*

Proof. First, note that from Proposition 3.1, we have $\inf \hat{S}_\epsilon > -\infty$. On the other hand, as mentioned above in Remark 4.5, let $(\hat{x}, \hat{y}, \hat{t})$ be an arbitrary solution of $(\hat{S}_\epsilon)$ such that $I(\hat{x}) \neq \emptyset$, and let us prove that $(\hat{x}, \hat{y}, \hat{t})$ is regular for $(\hat{S}_\epsilon)$. We have $(\hat{x}, \hat{y}, \hat{t}) \in \mathcal{G}_{\epsilon,l}$, and $v(\hat{x}) \geq \hat{t}$ [see iii) of Remark 4.4]. Let $x^* \in \partial v(\hat{x})$. From Lemma 2.1, there exist $(u^*, v_j^*) \in \partial_x f(\hat{x}, y) \times \partial_x g_j(\hat{x}, y)$, and $\lambda_j \geq 0$, $j = 1, ..., q$, such that

$$x^* = u^* + \sum_{j=1}^{q} \lambda_j v_j^*,$$

where $y \in \mathbb{R}^m$ satisfies $g(\hat{x}, y) \leq 0$, and $v(\hat{x}) = f(\hat{x}, y)$. Let $i \in I(\hat{x}) \subset \{l+1, ..., p\}$ [see i) of Remark 4.4]. We have

$$G_i^{'}(\hat{x}; x^*) \leq G_i^{'}(\hat{x}; u^*) + \sum_{j=1}^{q} \lambda_j G_i^{'}(\hat{x}; v_j^*) < 0, \tag{2}$$

where the last strict inequality follows from assumption (4.1). Then, from (2) we deduce that x^* is a descent direction of G_i at $\hat{x}$, and hence $x^* \neq 0$. Let $x_k = \hat{x} + \alpha_k x^*$, $y_k = \hat{y}$, $t_k = \hat{t}$, for all $k \in \mathbb{N}$, with $\alpha_k \searrow 0^+$. Then, $(x_k, y_k, t_k) \to (\hat{x}, \hat{y}, \hat{t})$, as $k \to +\infty$. It follows that

$$G_i(\hat{x} + \alpha_k x^*) < G_i(\hat{x}) = 0, \qquad \text{for large } k.$$

Besides, we have $g(\hat{x}, \hat{y}) < 0$, and $f(\hat{x}, \hat{y}) < \hat{t} + \epsilon$. Hence,

$$g(x_k, y_k) < 0, \quad \text{and} \quad f(x_k, y_k) < t_k + \epsilon, \quad \text{for large } k.$$

Finally, for $i \notin I(\hat{x})$, it is easy to see that $G_i(x_k) < 0$, for large k. On the other hand, since $x^* \in \partial v(\hat{x})$, we have

$$v(x_k) \geq v(\hat{x}) + \alpha_k \|x^*\|^2 > t_k, \qquad \text{for all } k,$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Hence $(\hat{x}, \hat{y}, \hat{t})$ is regular and the result follows from Proposition 4.2. □

Proposition 4.7. *Proposition 4.6 holds with the qualification condition (4.1) is replaced by (4.2).*

Proof. The majority of the proof is identical to the one of Proposition 4.6. Let us give the arguments concerning the modifications. Let $i \in I(\hat{x}) \subset \{l+1, ..., p\}$ and $x^* \in \partial v(\hat{x})$. Let w^* be a descent direction of G_i at $\hat{x}$. Define $x_k = \hat{x} + \alpha_k w^*$, $\alpha_k \searrow 0^+$, $y_k = \hat{y}$, and $t_k = \hat{t} - 1/k$, for all $k \geq 1$. Then, $(x_k, y_k, t_k) \to (\hat{x}, \hat{y}, \hat{t})$, as $k \to +\infty$. From Lemma 2.1, there exist $u^* \in \partial_x f(\hat{x}, y)$, $v_j^* \in \partial_x g_j(\hat{x}, y)$, and $\lambda_j \geq 0$, such that

$$x^* = u^* + \sum_{j=1}^{q} \lambda_j v_j^*,$$

where y satisfies $g(\hat{x}, y) \leq 0$, and $f(\hat{x}, y) = v(\hat{x})$. Since $x^* \in \partial v(\hat{x})$, it follows that

$$v(x_k) \geq v(\hat{x}) + \alpha_k \langle x^*, w^* \rangle = v(\hat{x}) + \alpha_k \langle u^*, w^* \rangle + \alpha_k \sum_{j=1}^{q} \lambda_j \langle v_j, w^* \rangle.$$

Then, by assumption (4.2), we have $v(x_k) \geq v(\hat{x}) > t_k$, for large k. On the other hand, from the definition of w^*, it follows that $G_i(x_k) < G_i(\hat{x}) = 0$, for large k. The end of the proof is identical to the end of the proof of Proposition 4.6. □

Proposition 4.8. *Proposition 4.6 holds with the qualification condition (4.1) is replaced by (4.3).*

Proof. Let $x^* \in \partial v(\hat{x})$. Then, from [18, Theorem 2.1], we have

$$\partial v(\hat{x}) = \Big\{\nabla_x f(\hat{x}, y) + \sum_{j=1}^{q} \lambda_j \nabla_x g_j(\hat{x}, y) / \nabla_y f(\hat{x}, y) + \sum_{j=1}^{q} \lambda_j \nabla_y g_j(\hat{x}, y) = 0, \lambda_j \geq 0, \text{ and } \lambda_j = 0, \text{ if } g_j(\hat{x}, y) < 0\Big\},$$

where $y \in \mathbb{R}^m$ satisfies $g(\hat{x}, y) \leq 0$, and $v(\hat{x}) = f(\hat{x}, y)$. Hence, there exists $\lambda_j \geq 0$, $j = 1, ..., q$, with $\lambda_j = 0$, if $g_j(\hat{x}, y) < 0$, such that

$$x^* = \nabla_x f(\hat{x}, y) + \sum_{j=1}^{q} \lambda_j \nabla_x g_j(\hat{x}, y).$$

Let $i \in I(\hat{x})$. Assumption (4.3) implies that

$$\langle \nabla G_i(\hat{x}), x^* \rangle < 0. \tag{3}$$

Then, x^* is a descent direction of G_i at $\hat{x}$, and hence $x^* \neq 0$. Let $(x_k, y_k, t_k) \to (\hat{x}, \hat{y}, \hat{t})$, as $k \to +\infty$, be the sequence defined in Proposition 4.6, $x_k = \hat{x} + \alpha_k x^*$, $\alpha_k \searrow 0^+$, $y_k = \hat{y}$, and $t_k = \hat{t}$, for all k, that satisfies $v(x_k) > t_k$, for large k. Since the function G_i is differentiable at $\hat{x}$, it follows that

$$G_i(x_k) = G_i(\hat{x}) + \alpha_k \langle \nabla G_i(\hat{x}), x^* \rangle + \alpha_k \|x^*\| \beta(\hat{x}, \alpha_k x^*), \tag{4}$$

where $\beta(\hat{x}, \alpha_k x^*) \to 0$ as $k \to +\infty$. Then, using (3) and (4), we get

$$G_i(x_k) < G_i(\hat{x}) = 0, \quad \text{for large } k.$$

On the other, we have $g(\hat{x}, \hat{y}) < 0$, and $f(\hat{x}, \hat{y}) < \hat{t} + \epsilon$. Hence

$$g(x_k, y_k) < 0, \quad \text{and} \quad f(x_k, y_k) < t_k + \epsilon, \quad \text{for large } k.$$

Finally, for $i \notin I(\hat{x})$, we have $G_i(x_k) < 0$, for large k. Hence, $(\hat{x}, \hat{y}, \hat{t})$ is regular and the result is deduced from Proposition 4.2. □

The following theorem gives sufficient optimality conditions for the problem $(\hat{S}_\epsilon)$.

Theorem 4.9. *Let $\epsilon > 0$. Let assumptions of one of Propositions 4.6 to 4.8 hold. Let $(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)$ be a feasible point of $(\hat{S}_\epsilon)$ that satisfies the following condition*

$$\max_{\substack{(x,y,t) \in \mathcal{G}_\epsilon \\ \hat{F}(x,y,t) \leq \hat{F}(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)}} \ \min_{\substack{u \in \mathbb{R}^m \\ g(x,u) \leq 0}} [f(x, u) - t] = 0 \tag{$\mathcal{OC}_\epsilon$}$$

Then, $(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)$ solves the problem $(\hat{S}_\epsilon)$.

Proof. Set $\alpha_\epsilon = \hat{F}(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)$ and let $(\mathcal{P}_0)$ and $(\mathcal{Q}_{\alpha_\epsilon})$, denote respectively the problem $(\hat{S}_\epsilon)$ and the problem

$$\operatorname*{Max}_{\substack{(x,y,t) \in \mathcal{G}_\epsilon \\ \hat{F}(x,y,t) \leq \alpha_\epsilon}} [v(x) - t].$$

The latter can be rewritten as

$$\operatorname*{Max}_{\substack{(x,y,t)\in\mathcal{G}_\epsilon \\ \hat{F}(x,y,t)\le\alpha_\epsilon}} \min_{\substack{u\in\mathbb{R}^m \\ g(x,u)\le 0}} [f(x,u)-t].$$

Then, since $(\hat{S}_\epsilon)$ is stable, it follows from Proposition 4.3, that $\alpha_\epsilon \le \inf \hat{S}_\epsilon = \inf \mathcal{P}_0$. Hence, $(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)$ is a solution of $(\hat{S}_\epsilon)$. □

Now, we can state our main result in this section.

Theorem 4.10. *Let $\epsilon_k \searrow 0^+$. Let assumptions of one of Propositions 4.6 to 4.8 hold. Let $(\hat{x}_k, \hat{y}_k, \hat{t}_k)$ be a feasible point of $(\hat{S}_k)$ that satisfies the condition $(\mathcal{OC}_{\epsilon_k})$. Let $(\hat{x}, \hat{y}, \hat{t})$ be an accumulation point of the sequence $(\hat{x}_k, \hat{y}_k, \hat{t}_k)_k$. Then, $\hat{x}$ solves the original problem (S).*

Proof. Use respectively Theorems 4.9 and 3.2. □

Let us give the following example in the differentiable case where assumptions of Proposition 4.8 are satisfied.

Example 4.11. Let $F(x,y) = |x-y| - 2x,\ G = (G_1, G_2)$,

$$G_1(x) = 1 - x, G_2(x) = x - 5,$$

$$f(x,y) = -x - y,\ g = (g_1, g_2),$$
$$g_1(x,y) = -y - 1,\ g_2(x,y) = -x + y + 1,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$.We have $X = \{x \in \mathbb{R} / 1 \le x \le 5\}$, which is a compact set, and $Y(x) = \{y \in \mathbb{R} / -1 \le y \le x - 1\}$. Let $\mathcal{Z} = [-1, 4]$. Then, $Y(x) \subset \mathcal{Z}$, for all $x \in X$. Hence assumptions (2.1), (2.2), (3.1) are satisfied. Let us verify assumption (4.3). Let $l = 1$, and for $\epsilon > 0$ sufficiently small,

$$\mathcal{D}_{\epsilon,1} = \big\{(x,y,t) \in \mathcal{G}_\epsilon / G_1(x) = 0\big\} \bigcup \bigcup_{j=1,2} \big\{(x,y,t) \in \mathcal{G}_\epsilon / g_j(x,y) = 0\big\}$$
$$\bigcup \big\{(x,y,t) \in \mathcal{G}_\epsilon / f(x,y) - t = \epsilon\big\}.$$

We have

$$f_{\min} = \inf_{\substack{(x,y)\in\mathbb{R}\times\mathbb{R} \\ G(x)\le 0 \\ g(x,y)\le 0}} f(x,y) = -9, \qquad f_{\max} = \sup_{\substack{(x,y)\in\mathbb{R}\times\mathbb{R} \\ G(x)\le 0 \\ g(x,y)\le 0}} f(x,y) = 0.$$

Let $f_1 = 0$, $x(1) = 5 \in X$, $y(1) = 4 - \epsilon/2 \in \mathrm{int}Y(x(1))$, and $t(1) = -9$. Then,

$$f(x(1), y(1)) = -9 + \epsilon/2 < -9 + \epsilon = t(1) + \epsilon,$$

and

$$\hat{F}(x(1), y(1), t(1)) = -9 + \epsilon/2 < \inf_{\substack{(x,y,t)\in\mathcal{D}_{\epsilon,1} \\ t\le f_1}} \hat{F}(x,y,t) = 0.$$

Let $x \in X$ and $y \in \mathbb{R}$, such that $g(x,y) < 0$. We have $I(x) = \{2\}$ [see i) of Remark 4.4], and

$$G_2^{'}(x) = 1, \quad f_x^{'}(x,y) = -1, \quad g_{1,x}^{'}(x,y) = 0, \quad g_{2,x}^{'}(x,y) = -1.$$

Then,

$$G_2^{'}(x) f_x^{'}(x,y) = -1 < 0,\ G_2^{'}(x) g_{j,x}^{'}(x,y) \le 0, j = 1,2, \text{ et } 0 \notin \{G_2^{'}(x)\}.$$

Hence, assumption (4.3) is satisfied.

$\boxed{5}$ The Third Approach

In this section, we will improve the approach given in Section 4. In fact, when the functions f and g are respectively polyhedral and linear, we can reduce the problem (S) to a maximization problem of a polyhedral convex function over a compact convex set (similar results are given in [5]). Let f and g be the functions defined by

$$f(x,y) = \max_{i=1,\ldots,r} \left[\langle A_i, x\rangle + \langle B_i, y\rangle - c_i\right], \quad g(x,y) = Cx + Dy - e,$$

where

$$A_i = (a_{i1}, ..., a_{in})^T, B_i = (b_{i1}, ..., b_{im})^T, c_i \in \mathbb{R}, i = 1, 2, ..., r,$$

$$C \in \mathbb{R}^{q\times n}, D \in \mathbb{R}^{q\times m}, e \in \mathbb{R}^q.$$

Then, the lower level problem $P(x)$ is equivalent to the following linear programming problem [7]

$$\hat{P}(x) \qquad \operatorname*{Min}_{\substack{y\in\mathbb{R}^m, z\in\mathbb{R} \\ \langle A_i,x\rangle+\langle B_i,y\rangle-c_i\leq z\ \forall i \\ Cx+Dy\leq e}} \quad z$$

in the sense that if $(\hat{y}, \hat{z})$ solves $\hat{P}(x)$, then, $\hat{y}$ solves $P(x)$ and $\hat{z} = v(x)$. Let A and B denote the matrices with the ith rows being equal to A_i^T and $(B_i^T, -1)$ respectively, and set $c = (c_1, ..., c_r)^T$. Then, the problem $\hat{P}(x)$ can be rewritten as follows

$$\operatorname*{Min}_{\substack{t\in\mathbb{R}^{m+1} \\ \hat{B}t\leq d-\hat{A}x}} \langle \hat{c}, t\rangle$$

where $\hat{A} = \begin{pmatrix} A \\ C \end{pmatrix}$, $\hat{B} = \begin{pmatrix} B \\ (D,0) \end{pmatrix}$, $\hat{c} = (\underbrace{0,...,0}_{m}, 1)^T$, $d = \begin{pmatrix} c \\ e \end{pmatrix}$ and $t = \begin{pmatrix} y \\ z \end{pmatrix}$.

Consequently, by duality, we have

$$v(x) = \max_{\substack{u\in(\mathbb{R}_+)^{r+q} \\ \hat{B}^T u=-\hat{c}}} \langle \hat{A}x - d, u\rangle = \max_{j=1,\ldots,s} [\langle \hat{A}^T u_j, x\rangle - \langle d, u_j\rangle],$$

with $u_1, ..., u_s$, the vertices of the polyhedral set $\left\{u \in (\mathbb{R}_+)^{r+q} \setminus \{0\} / \hat{B}^T u = -\hat{c}\right\}$.

Then, Theorem 4.9 is improved to the following:

Theorem 5.1. *Let $\epsilon > 0$. Let assumptions of one of Propositions 4.6 to 4.8 hold. Let $(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)$ be a feasible point of $(\hat{S}_\epsilon)$ that satisfies the condition*

$$\max_{\substack{(x,y,t)\in\mathcal{G}_\epsilon \\ \hat{F}(x,y,t)\leq\hat{F}(\hat{x}_\epsilon,\hat{y}_\epsilon,\hat{t}_\epsilon)}} \left[\max_{j=1,\ldots,s} [\langle \hat{A}^T u_j, x\rangle - \langle d, u_j\rangle] - t\right] = 0 \qquad (\widehat{\mathcal{OC}}_\epsilon)$$

Then, $(\hat{x}_\epsilon, \hat{y}_\epsilon, \hat{t}_\epsilon)$ solves the problem $(\hat{S}_\epsilon)$.

Proof. The result follows from Theorem 4.9. □

Theorem 5.2. *Let $\epsilon_k \searrow 0^+$. Let assumptions of one of Propositions 4.6 to 4.8 hold. Let $(\hat{x}_k, \hat{y}_k, \hat{t}_k)$ be a feasible point of $(\hat{S}_k)$ that satisfies the condition $(\widehat{\mathcal{OC}}_{\epsilon_k})$. Let $(\hat{x}, \hat{y}, \hat{t})$ be an accumulation point of the sequence $(\hat{x}_k, \hat{y}_k, \hat{t}_k)_k$. Then, $\hat{x}$ solves the original problem (S).*

Proof. It suffices to use respectively Theorems 5.1 and 3.2. □

6 Conclusion

As it is well-known, the numerical resolution of bilevel programming problems is a difficult task, especially when the solution set of the lower level is not always a singleton. So, in order to get round this difficulty and give other prospects to this situation, under certain assumptions, we have given three approaches of the strong bilevel programming problem (S) by some well-known global optimization problems. Precisely, these three approaches use reverse convex and min-max problems, and problems of maximization of a polyhedral convex function over a compact convex set. In spite of that the involved global optimization problems are also known as difficult problems, such theoretical approaches open other ways which can lead to a possible numerical resolution of the considered class of bilevel programming problems. Then, this connection established between such class of bilevel programming problems and the other global optimization problems gives a contribution in the investigation of bilevel programming problems.

Acknowledgements

The authors would like to thank two anonymous referees for their valuable comments and suggestions which improved the first version of the paper.

References

[1] A. Aboussoror, H. Babahadda and A. Mansouri, Bilevel programs with extremal value function: global optimality, *International Journal of Mathematics and Mathematical Sciences* 2005 (2005) 419–435.

[2] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, New York, 1984.

[3] J.F. Bard, An efficient point algorithm for a linear two-stage optimization problem, *Operations Research* 31 (1983) 670–684.

[4] S. Dempe, A simple algorithm for the linear bilevel programming problem, *Optimization* 18 (1987) 373–385.

[5] S. Dempe, *Foundations of Bilevel Programming*, Kluwer Academic Publishers, Dordrecht, 2002.

[6] R.J. Hillestad, and S.E. Jacobsen, Reverse convex programming, *Journal of Applied Mathematics and Optimization* 6 (1980) 63–78.

[7] J.-B. Hiriart-Urruty, *Optimisation et Analyse Convexe*, Presses Universitaires de France, Paris, 1998.

[8] R. Horst and H. Tuy, *Global Optimization, Deterministic Approaches*, Springer Verlag, New York, 1992.

[9] Y. Ishizuka, Farkas' Theorem of nonconvex type and its application to min-max problem, *J. Optim. Theory Appl.* 57 (1988) 341–354.

[10] D.C. Jiang, K.L. Teo and W.Y. Yan, A new computational method for the functional inequality constrained minimax optimization problem, *Computers Math. Appl.* 33 (1997) 53–63.

[11] M.B. Lignola and J. Morgan, *Regularized bilevel programming problem*, Preprint 22 (1993), Dipartimento di Matematica a Applicazioni, Università degli studi di Napoli, "Federico 2".

[12] N.D. Nghia and N.D. Hieu, A method for solving reverse convex programming problems, *Acta Mathematica Vietnamica* 11 (1986) 241–252.

[13] J.V. Outrata, On the numerical solution of a class of Stackelberg problems, *Zeit. Oper. Res.* 34 (1990) 255–277.

[14] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, N.J., 1970.

[15] K. Shimizu and E. Aiyoshi, A new computational method for Stackelberg and min-max problems by use of a penalty method, *IEEE Transastions on Automatic Control* AC-26 (1981) 460–466.

[16] K. Shimizu, Y. Ishizuka and J. F. Bard, *Nondifferentiable and Two-level Mathematical Programming*, Kluwer Academic Publishers, Boston, 1997.

[17] K. Shimizu and E. Aiyoshi, Necessary conditions for min-max problems and algorithms by a relaxation procedure, *IEEE Transactions on Automatic Control* AC-25 (1980) 62–66.

[18] T. Tanino and T. Ogawa, An algorithm for solving two-level convex optimization problems, *Int. J. Systems Sci.* 15 (1984) 163–174.

[19] H. Tuy, Convex programs with an additional reverse convex constraint, *J. Optim. Theory Appl.* 52 (1987) 463–486.

[20] H. Tuy and Ng. V. Thuong, On the global minimization of a convex function under general nonconvex constraints, *Appl. Math. Optim.* 18 (1988) 119–142.

[21] S. Yamada, T. Tanino and M. Inuiguchi, Inner approximation method for a reverse convex programming problem, *J. Optim. Theory Appl.* 107 (2000) 355–389.

Manuscript received 26 August 2007
revised 5 January 2008
accepted for publication 3 March 2008

Abdelmalek Aboussoror
Université Cadi Ayyad, Faculté Polydisciplinaire de Safi
Département de Mathématiques et Informatique
B.P. 4162, Sidi Bouzid, Safi, Morocco
E-mail address: aboussororabdel@hotmail.com

Zakia Ankhili
Université Cadi Ayyad, Faculté des Sciences Semlalia
Département de Mathématiques, B.P. 2390, Marrakech, Morocco
E-mail address: z.ankhili@ucam.ac.ma

Abdelatif Mansouri
Université Cadi Ayyad, Faculté des Sciences Semlalia
Département de Mathématiques, B.P. 2390, Marrakech, Morocco
E-mail address: mansouri@ucam.ac.ma