Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 4, No. 2, pp. 195 – 212, May 2008

A GENERAL FRAMEWORK OF CONTRACTION METHODS FOR MONOTONE VARIATIONAL INEQUALITIES

Bingsheng He* and Ming-Hua Xu

Abstract: Variational inequality is a uniform approach for some important problems in optimization and equilibrium problems. In this paper, we establish a general framework of contraction methods. Based on the directions offered by the framework, we construct different discrete methods and continuous models for monotone variational inequalities. Besides some existing methods are illustrated to be accordant with the framework, we present a new method for structured variational inequalities.

Key words: *variational inequality, monotone, contraction method, discrete and continuous methods*

Mathematics Subject Classification: *90C25, 90C30*

1 Introduction

Let Ω be a nonempty subset of $\Re^n$, F be a continuous mapping from $\Re^n$ to itself. The variational inequality problem, denoted by $\mathrm{VI}(\Omega, F)$, is to find a vector $u^* \in \Omega$ such that

$$\mathrm{VI}(\Omega, F) \qquad (u - u^*)^T F(u^*) \geq 0, \qquad \forall u \in \Omega. \tag{1.1}$$

Throughout this paper we assume that the solution set of $\mathrm{VI}(\Omega, F)$, denoted by Ω^*, is nonempty. Moreover, it is assumed that F is monotone and Lipschitz continuous, i.e.,

$$(v - u)^T (F(v) - F(u)) \geq 0$$

and there is a constant $L > 0$ such that

$$\|F(v) - F(u)\| \leq L\|v - u\|.$$

Variational inequality problems contain as special cases system of equations (when $\Omega = \Re^n$), complementarity problem (when Ω is the nonnegative orthant of $\Re^n$). $\mathrm{VI}(\Omega, F)$ arising from many practical equilibrium problems which can not be transformed into an equivalent classical minimization problem and development of efficient solution methods remains desirable. In the last several years, a class of iterative projection and contraction methods for solving variational inequality problems was proposed by one of the authors [4, 6, 7] and extended by other researchers [12]. In such methods, although the solution u^* is an unknown vector to be determined, some descent directions of the unknown distance function $\|u-u^*\|^2$

*This author was supported by the NSFC grant 10571083 and MOEC grant 20060284001.

are constructed. The methods (without line search) make one (or two) projection(s) on a closed convex set at each iteration, and the distance of the iterates to the solution set monotonically converges to zero. According to terminology in the literature [2], these methods belong to the class of Fejér contraction methods in Euclidean norm. The main advantages of the contraction methods are their simplicity (almost as simple as the basic projection method) and ability to handle problems under mild conditions.

The concepts of the discrete contraction methods can be straightforwardly extended to models of continuous methods. For given u, let

$$\operatorname{dist}\big(u,\Omega^*\big) = \min\{\|u-u^\star\| \mid u^\star \in \Omega^*\} \tag{1.2}$$

denote the distance of the current point u to the solution set. In the continuous models we take $\operatorname{dist}^2(u,\Omega^*)$ as the *energy function*, and use the similar directions as in the discrete methods as the *motion equation*. The motorial track $u(t)$ of the system of differential equations will converge to a $u^* \in \Omega^*$ as $t \to \infty$.

The purpose of this paper is to give a general framework of contraction methods. Using the convex combination of the directions offered from the framework, we construct generalized discrete and continuous methods for monotone variational inequalities.

The outline of this paper is as follows. Section 2 gives some preliminaries of projection mapping and variational inequalities. In Section 3 we give the general framework which offers us two useful descent directions. Section 4 presents the discrete contraction methods by using the convex combination of the descent directions. The continuous models based on the framework are given in Section 5. In Section 6, we demonstrate some existing methods belong to the presented framework. The last section, Sections 7, gives new applications of the presented framework for structured variational inequalities. Finally, conclusions remarks are addressed.

In the following u^* denotes a solution point. A superscript such as in u^k refers to a specific vector and usually denotes an iteration index. For any real matrix M and vector v, we denote the transpose by M^T and v^T, respectively. The Euclidean norm will be denoted by $\|\cdot\|$.

2 Some Preliminaries

For given $v \in \Re^n$, the solution of problem

$$\min\{\|u-v\| \mid u \in \Omega\}$$

is called the projection of v on Ω, denoted by $P_\Omega(v)$. In other words,

$$P_\Omega(v) = \operatorname{argmin}\{\|u-v\| \mid u \in \Omega\}.$$

Since Ω is convex and closed, the projection on to Ω is unique. It is assumed that the projection on Ω is simple to carry out. The following properties of the projection mapping can be found in textbooks, e. g., [2].

Lemma 2.1. *Let $\Omega \subset \Re^n$ be a convex closed set, then we have*

$$(v - P_\Omega(v))^T(u - P_\Omega(v)) \le 0, \quad \forall\ v \in \Re^n, \forall\ u \in \Omega. \tag{2.1}$$

Consequently, we have

$$\|P_\Omega(u) - P_\Omega(v)\| \le \|u-v\|, \qquad \forall\ u,\ v \in \Re^n \tag{2.2}$$

and

$$\|u - P_\Omega(v)\|^2 \le \|v-u\|^2 - \|v - P_\Omega(v)\|^2, \quad \forall\ v \in \Re^n, \forall\ u \in \Omega. \tag{2.3}$$

2.1 Preliminaries of Variational Inequalities

It is well known (e. g., see [8]) that the variational inequality $\mathrm{VI}(\Omega, F)$ problem is equivalent to the following projection equation

$$u = P_\Omega[u - F(u)].$$

In other words, to solve $\mathrm{VI}(\Omega, F)$ is equivalent to finding a zero point of the continuous residue function

$$e(u) := u - P_\Omega[u - F(u)].$$

Hence,

$$e(u) = 0 \quad \Leftrightarrow \quad u \in \Omega^*.$$

In the literature for variational inequalities, $\|e(u)\|$ is called *error bound* of $\mathrm{VI}(\Omega, F)$. It quantitatively measures how much u fails to be in Ω^*.

Notice that the variational inequality $\mathrm{VI}(\Omega, F)$ is invariant when we multiply F by some positive scalar $\beta > 0$. Let

$$e(u, \beta) := u - P_\Omega[u - \beta F(u)].$$

For $\beta = 1$, sometimes, instead of $e(u, 1)$ we write $e(u)$. The next lemma tells us $\|e(u, \beta)\|$ is a non-decreasing function of $\beta > 0$ for given $u \in \Re^n$.

Lemma 2.2. *For all $u \in \Re^n$ and $\tilde{\beta} \geq \beta > 0$, it holds that*

$$\|e(u, \tilde{\beta})\| \geq \|e(u, \beta)\|$$

and

$$\frac{\|e(u, \tilde{\beta})\|}{\tilde{\beta}} \leq \frac{\|e(u, \beta)\|}{\beta}.$$

Proof. A proof can be found in [14] and here is omitted. □

2.2 A Framework of Continuous Models for Optimization Problems

As Tank and Hopfield [13], and Kennedy and Chua [11], we use the concepts of *energy function* and *motion equation* in analysis of our continuous models for $\mathrm{VI}(\Omega, F)$. The minimizer of the energy function should be the solution of the problem. Let $E(u(t))$ be a given energy function, according to Tank and Hopfield [13], constructing a continuous model is to give a motion equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = m(u(t)), \tag{2.4}$$

which satisfies

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(\frac{\partial E}{\partial u}\right)^T \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) \leq 0 \tag{2.5}$$

for all t, and the equality holds only at the equilibrium (the point with $e(u) = 0$). Therefore, to construct a continuous model, mathematically, we need only to focus our attention on constructing the motion equations (system of ordinary differential equations) which satisfy (2.5) for proper defined energy function $E(u)$. The continuous model or dynamic solver will become a powerful tool with the developments and advances of neural computers.

2.3 The Discrete Methods and Continuous Models

In general, the discrete methods generate a sequence $\{u^k\}$ by

$$u^{k+1} = u^k + \alpha_k d(u^k), \tag{2.6}$$

where $d(u^k)$ is a descent direction of a certain function and $\alpha_k > 0$ is a step size. To ensure the convergence, the computational load for choosing step size is sometimes costly. With $\alpha_k \equiv h > 0$, the discrete method (2.6) can be viewed as the implementation of Euler method for differential equation

$$\dot{u} = d(u).$$

The continuous methods are advantageous only if the computational load of $d(u)$ in (2.6) is much less than the one for the decision of the step size.

3 General Framework of the Contraction Methods

Definition 3.1. For given $u \in \Omega$, $\tilde{u} \in \Omega$ is said to be a test vector of u if $\tilde{u}$ is generated from u by some well-defined rule such that

$$\|u - \tilde{u}\| \geq c_0 \|e(u)\| \qquad \text{or} \qquad \|u - \tilde{u}\| \geq c_0 \|e(\tilde{u})\|, \tag{3.1}$$

for a constant $c_0 > 0$ and

$$u = \tilde{u} \quad \Leftrightarrow \quad u \in \Omega^*.$$

For given $u \in \Omega$, there are different ways to get $\tilde{u}$ which is a test vector of u. For example, set $\tilde{u} = P_\Omega[u - \beta F(u)]$, then according to Lemma 2.2 we have

$$\|u - \tilde{u}\| \geq \min\{\beta, 1\} \|e(u)\|$$

and thus $\tilde{u}$ can be viewed as a test vector of u. Now, we describe the general framework.

The General Framework

For given $u \in \Omega$, let $\tilde{u} \in \Omega$ be a test vector of u. For this pair of u and $\tilde{u}$, we find $d_1(u, \tilde{u})$, $d_2(u, \tilde{u}) \in \Re^n$ and $\varphi(u, \tilde{u}) \in \Re$ which satisfy following conditions:

1. it holds the projection equation

$$\tilde{u} = P_\Omega\{\tilde{u} - [d_2(u, \tilde{u}) - d_1(u, \tilde{u})]\}, \tag{3.2a}$$

2. there is a constant $K > 0$ such that

$$\|d_1(u, \tilde{u})\| \leq K \|u - \tilde{u}\|, \tag{3.2b}$$

3. for any $u^* \in \Omega^*$,

$$(\tilde{u} - u^*)^T d_2(u, \tilde{u}) \geq \varphi(u, \tilde{u}) - (u - \tilde{u})^T d_1(u, \tilde{u}), \tag{3.2c}$$

4. $\varphi(u, \tilde{u})$ is an *error measure function* of VI(Ω, F), i.e., there is a constant $\tau > 0$, such that

$$\varphi(u, \tilde{u}) \geq \tau \|u - \tilde{u}\|^2 \qquad \& \qquad \varphi(u, \tilde{u}) = 0 \;\Leftrightarrow\; u = \tilde{u}. \tag{3.2d}$$

All the discrete methods and the continuous models are based on the above framework. The following properties play important role in the contraction methods.

Lemma 3.2. *Let the conditions* (3.2) *in the general framework be satisfied, then we have*

$$(u - u^*)^T d_1(u, \tilde{u}) \geq \varphi(u, \tilde{u}), \quad \forall u \in \Omega, u^* \in \Omega^*. \tag{3.3}$$

Proof. Let $v = \tilde{u} - (d_2(u, \tilde{u}) - d_1(u, \tilde{u}))$, it follows from (3.2a) that $P_\Omega(v) = \tilde{u}$. Setting them in (2.1), we get

$$\{[\tilde{u} - (d_2(u, \tilde{u}) - d_1(u, \tilde{u}))] - \tilde{u}\}^T (u' - \tilde{u}) \leq 0, \quad \forall\, u' \in \Omega \tag{3.4}$$

and thus (because $u^* \in \Omega$) we have

$$(\tilde{u} - u^*)^T d_1(u, \tilde{u}) \geq (\tilde{u} - u^*)^T d_2(u, \tilde{u}).$$

It follows from the above inequality and (3.2c) that

$$(\tilde{u} - u^*)^T d_1(u, \tilde{u}) \geq \varphi(u, \tilde{u}) - (u - \tilde{u})^T d_1(u, \tilde{u})$$

and thus

$$(u - u^*)^T d_1(u, \tilde{u}) \geq \varphi(u, \tilde{u}). \tag{3.5}$$

The lemma is proved. □

Lemma 3.3. *Let the conditions* (3.2) *in the general framework be satisfied, then we have*

$$(u - u^*)^T d_2(u, \tilde{u}) \geq \varphi(u, \tilde{u}), \quad \forall u \in \Omega, u^* \in \Omega^*. \tag{3.6}$$

Proof. Since $u \in \Omega$, set $u' = u$ in (3.4) we get

$$(u - \tilde{u})^T d_2(u, \tilde{u}) \geq (u - \tilde{u})^T d_1(u, \tilde{u}). \tag{3.7}$$

Adding (3.7) and (3.2c) we obtain

$$(u - u^*)^T d_2(u, \tilde{u}) \geq \varphi(u, \tilde{u})$$

and the assertion of this lemma is proved. □

For any $u^* \in \Omega^*$, $(u - u^*)$ is the gradient of the unknown distance function $\frac{1}{2}\|u - u^*\|^2$ at point u. A direction d is called a descent direction of $\frac{1}{2}\|u - u^*\|^2$ if and only if the inner-product $\langle u - u^*, d \rangle < 0$. Therefore, Lemmas 3.2 and 3.3 tell us that the directions $-d_1(u, \tilde{u})$ and $-d_2(u, \tilde{u})$ in the general framework are descent directions of the unknown distance function $\|u - u^*\|^2$ when u is not a solution point.

Note that Condition (3.2b) in the general framework means that $\|d_1(u, \tilde{u})\| \to 0$ as $\|u - \tilde{u}\| \to 0$. However, the framework does not claim the same request for $\|d_2(u, \tilde{u})\|$.

4 The Discrete Methods Based on the General Framework

Based on the descent directions offered by the general framework, we can construct the discrete contraction methods. Unlike the existing discrete methods we use the direction

$$d(u, \tilde{u}) = (1 - t) d_1(u, \tilde{u}) + t d_2(u, \tilde{u}), \quad t \in [0, 1], \tag{4.1}$$

the convex combination of $d_1(u,\tilde{u})$ and $d_2(u,\tilde{u})$ as the search direction, and let

$$u(\alpha) = P_\Omega[u - \alpha d(u,\tilde{u})], \tag{4.2}$$

be the step size dependent vector. For any solution point $u^* \in \Omega^*$, we define

$$\theta(\alpha) := \|u - u^*\|^2 - \|u(\alpha) - u^*\|^2 \tag{4.3}$$

as the profit function of the general algorithm. The following theorem introduces a lower bound of $\theta(\alpha)$, namely $q(\alpha)$, which does not include the unknown solution u^*.

Theorem 4.1. *For any $u^* \in \Omega^*$ and $\alpha \ge 0$, we have*

$$\theta(\alpha) \ge q(\alpha), \tag{4.4}$$

where

$$q(\alpha) = 2\alpha\varphi(u,\tilde{u}) - \alpha^2\|d_1(u,\tilde{u})\|^2. \tag{4.5}$$

Proof. First, since $u(\alpha) = P_\Omega[u - \alpha d(u,\tilde{u})]$ and $u^* \in \Omega$, it follows from (2.3) that

$$\|u(\alpha) - u^*\|^2 \le \|u - \alpha d(u,\tilde{u}) - u^*\|^2 - \|u - \alpha d(u,\tilde{u}) - u(\alpha)\|^2. \tag{4.6}$$

Consequently, using the definition of $\theta(\alpha)$, we get

$$\begin{aligned} \theta(\alpha) &\ge \|u - u^*\|^2 - \|u - \alpha d(u,\tilde{u}) - u^*\|^2 + \|u - \alpha d(u,\tilde{u}) - u(\alpha)\|^2 \\ &= \|u - u(\alpha)\|^2 + 2\alpha(u - u^*)^T d(u,\tilde{u}) + 2\alpha(u(\alpha) - u)^T d(u,\tilde{u}) \\ &= \|u - u(\alpha)\|^2 + 2\alpha(u(\alpha) - u^*)^T d(u,\tilde{u}). \end{aligned} \tag{4.7}$$

By using (4.1), we have

$$\theta(\alpha) \ge (1-t)\theta_1(\alpha) + t\theta_2(\alpha), \tag{4.8}$$

where

$$\theta_1(\alpha) = \|u - u(\alpha)\|^2 + 2\alpha(u(\alpha) - u^*)^T d_1(u,\tilde{u}) \tag{4.9}$$

and

$$\theta_2(\alpha) = \|u - u(\alpha)\|^2 + 2\alpha(u(\alpha) - u^*)^T d_2(u,\tilde{u}). \tag{4.10}$$

In the following we show that both $\theta_1(\alpha)$ and $\theta_2(\alpha)$ have lower bound $q(\alpha)$. From (3.3) and (4.5) we have

$$\begin{aligned} \theta_1(\alpha) &= \|u - u(\alpha)\|^2 + 2\alpha(u(\alpha) - u^*)^T d_1(u,\tilde{u}) \\ &= \|u - u(\alpha)\|^2 + 2\alpha(u - u^*)^T d_1(u,\tilde{u}) + 2\alpha(u(\alpha) - u)^T d_1(u,\tilde{u}) \\ \text{(use (3.3))} \quad &\ge \|u - u(\alpha)\|^2 + 2\alpha\varphi(u,\tilde{u}) + 2\alpha(u(\alpha) - u)^T d_1(u,\tilde{u}) \\ &= \|u - u(\alpha) - \alpha d_1(u,\tilde{u})\|^2 + 2\alpha\varphi(u,\tilde{u}) - \alpha^2\|d_1(u,\tilde{u})\|^2 \\ \text{(use (4.5))} \quad &= q(\alpha) + \|u - u(\alpha) - \alpha d_1(u,\tilde{u})\|^2. \end{aligned} \tag{4.11}$$

Now we turn to prove $\theta_2(\alpha) \ge q(\alpha)$. From

$$\theta_2(\alpha) = \|u - u(\alpha)\|^2 + 2\alpha(\tilde{u} - u^*)^T d_2(u,\tilde{u}) + 2\alpha(u(\alpha) - \tilde{u})^T d_2(u,\tilde{u}) \tag{4.12}$$

and (3.2c) we have

$$\theta_2(\alpha) \ge \|u - u(\alpha)\|^2 + 2\alpha\varphi(u,\tilde{u}) - 2\alpha(u - \tilde{u})^T d_1(u,\tilde{u}) + 2\alpha(u(\alpha) - \tilde{u})^T d_2(u,\tilde{u}) \tag{4.13}$$

Since $u(\alpha) \in \Omega$, it follows from (3.4) that

$$(u(\alpha) - \tilde{u})^T d_2(u, \tilde{u}) \geq (u(\alpha) - \tilde{u})^T d_1(u, \tilde{u}). \tag{4.14}$$

Substituting (4.14) in the right hand side of (4.13), we get

$$\begin{aligned} \theta_2(\alpha) &\geq \|u - u(\alpha)\|^2 + 2\alpha\varphi(u, \tilde{u}) - 2\alpha(u - \tilde{u})^T d_1(u, \tilde{u}) + 2\alpha(u(\alpha) - \tilde{u})^T d_1(u, \tilde{u}) \\ &= \|u - u(\alpha)\|^2 + 2\alpha\varphi(u, \tilde{u}) + 2\alpha(u(\alpha) - u)^T d_1(u, \tilde{u}) \\ &= \|u - u(\alpha) - \alpha d_1(u, \tilde{u})\|^2 + 2\alpha\varphi(u, \tilde{u}) - \alpha^2\|d_1(u, \tilde{u})\|^2 \\ &= q(\alpha) + \|u - u(\alpha) - \alpha d_1(u, \tilde{u})\|^2. \end{aligned} \tag{4.15}$$

The proof is complete. □

Note that $q(\alpha)$ is a quadratic function of α, it reaches its maximum at

$$\alpha^* = \frac{\varphi(u, \tilde{u})}{\|d_1(u, \tilde{u})\|^2}.$$

In the discrete methods, by setting $u^k = u$ and $\tilde{u}^k = \tilde{u}$, the new iterate is updated by

$$u^{k+1} = P_\Omega[u^k - \gamma\alpha_k^* d(u^k, \tilde{u}^k)], \tag{4.16}$$

where $\gamma \in [1, 2)$ is a relaxation factor. Two special cases are taking $t = 0$ and $t = 1$ in (4.1), and the update formula are

$$u^{k+1} = P_\Omega[u^k - \gamma\alpha_k^* d_1(u^k, \tilde{u}^k)], \tag{4.17a}$$

and

$$u^{k+1} = P_\Omega[u^k - \gamma\alpha_k^* d_2(u^k, \tilde{u}^k)], \tag{4.17b}$$

respectively. Using Theorem 4.1 and Conditions (3.2), by a manipulation we get

$$\begin{aligned} \|u^{k+1} - u^*\|^2 &\leq \|u^k - u^*\|^2 - q(\gamma\alpha_k^*) \\ &= \|u^k - u^*\|^2 - 2\gamma\alpha_k^*\varphi(u^k, \tilde{u}^k) + \gamma^2(\alpha_k^*)^2\|d_1(u^k, \tilde{u}^k)\|^2 \\ &= \|u^k - u^*\|^2 - \gamma(2 - \gamma)\alpha_k^*\varphi(u^k, \tilde{u}^k) \\ &\leq \|u^k - u^*\|^2 - \gamma(2 - \gamma)\frac{\tau^2}{K^2}\|u^k - \tilde{u}^k\|^2. \end{aligned} \tag{4.18}$$

Convergence follows from (4.18) and (3.1) directly. Note that the step size α^* is dependent on $\varphi(u, \tilde{u})$ and $d_1(u, \tilde{u})$, even if the search direction $d(u, \tilde{u})$ is any convex combinations of $d_1(u, \tilde{u})$ and $d_2(u, \tilde{u})$. In other words, the methods use different search directions but the same step length sizes !

5 The Continuous Models Based on the General Framework

The continuous models in this paper are based on the consideration there are efficient solvers for system of differential equations. The solver is a 'black box', or an 'oracle'. Therefore, our task is only to convert the VI(Ω, F) to an equivalent system of ODE. For this purpose, we use

$$E(u) = \text{dist}^2\big(u, \Omega^*\big), \tag{5.1}$$

where $\text{dist}\big(u, \Omega^*\big)$ is defined in (1.2). $E(u)$ is a differential function due to the following lemma.

Lemma 5.1. *Let $h(u,v) : \Re^{n\times n} \to \Re$ be a continuous differentiable function and S be a closed convex set in $\Re^n$. If $v_u = \arg\min\{h(u,v) \,|\, v \in S\}$ is unique and*

$$f(u) = \min\{h(u,v) \,|\, v \in S\},$$

then f is differentiable and

$$\nabla_u f(u) = \nabla_u h(u, v_u). \tag{5.2}$$

Proof. The assertion is used popularly in the literature [3]. A proof can be found in [1]. □

Since the solution set of VI(Ω, F) is convex and closed. Therefore, it follows from Lemma 5.1 that

$$\frac{\partial E}{\partial u} = 2(u - u^*), \tag{5.3}$$

where $u^* = \arg\min\{\|u - u^\star\| \,|\, u^\star \in \Omega^*\}$.

5.1 Allowance the Motorial Track Outside the Constraint Set

If we allow the motorial track outsides Ω, we take

$$\frac{\mathrm{d}u}{\mathrm{d}t} := -d_1(u, \tilde{u}) \tag{5.4}$$

as the motion equation.

Theorem 5.2. *Let the conditions* (3.2a)-(3.2d) *in the general framework be satisfied. If we take $E(u)$ defined in* (5.1) *as the energy function and* (5.4) *as the motion equation, then the motorial track will converge to the solution set.*

Proof. Clearly, $E(u)$ is a proper energy function because its minimizer coincides with the solution of the VI problems. It follows from (5.3), (5.4), Lemma 3.2 and (3.2d) that

$$\begin{aligned} \frac{\mathrm{d}E(u)}{\mathrm{d}t} &= \left(\frac{\partial E}{\partial u}\right)^T \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) \\ &= 2(u - u^*)^T(-d_1(u, \tilde{u})) \\ &\le -2\tau\|u - \tilde{u}\|^2. \end{aligned} \tag{5.5}$$

According to the framework in Section 2.2, this is a continuous model the motion equation will leads the motorial track to the solution set. □

Since $\|d_1(u, \tilde{u})\| \le K\|u - \tilde{u}\|$ (see (3.2b)), it follows from (5.5) that

$$\left(\frac{\mathrm{d}E(u)}{\mathrm{d}t}\right) \Big/ \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\| \le -\frac{2\tau}{K}\|\tilde{u} - u\|.$$

In comparison with (2.5), this is a sharper property than the general model introduced in Sec. 2.2 .

Note that Inequality (3.3) is hold for all $u \in \Re^n$ while Inequality (3.6) is true only for $u \in \Omega$. Therefore, we do not use $-d_2(u, \tilde{u})$ to construct a motion equation in the continuous model because it can not ensure the motorial track in Ω.

5.2 Ensuring the Motorial Track in the Constraint Set

In general, we can not guarantee the motorial track followed from motion equation (5.4) contained in Ω even if with a slight movement. In order to overcome this shortcoming, we consider another motion equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} := u(\alpha) - u. \tag{5.6}$$

where $u(\alpha)$ is defined in (4.2). Note that for any constant α

$$\begin{aligned} q(\alpha) &= 2\alpha\varphi(u,\tilde{u}) - \alpha^2\|d_1(u,\tilde{u})\|^2 \\ &\geq 2\alpha\tau\|u-\tilde{u}\|^2 - \alpha^2K^2\|u-\tilde{u}\|^2 \\ &= \alpha(2\tau - \alpha K^2)\|u-\tilde{u}\|^2. \end{aligned} \tag{5.7}$$

Therefore, for any fixed $\alpha \in (0, 2\tau/K^2)$, there is a constant $\sigma > 0$, such that

$$\mathrm{dist}^2\big(u(\alpha),\Omega^*\big) \leq \mathrm{dist}^2\big(u,\Omega^*\big) - \sigma\|u-\tilde{u}\|^2. \tag{5.8}$$

For $u \in \Omega\backslash\Omega^*$,

$$\mathrm{dist}\big(u(\alpha),\Omega^*\big) - \mathrm{dist}\big(u,\Omega^*\big) \leq -\frac{\sigma\|u-\tilde{u}\|^2}{2\mathrm{dist}\big(u,\Omega^*\big)}. \tag{5.9}$$

Since $E(u) = \mathrm{dist}^2(u,\Omega^*)$ is the energy function and $\dfrac{\mathrm{d}u}{\mathrm{d}t} = (u(\alpha)-u)$ is the motion equation, we have

$$\begin{aligned} \frac{\mathrm{d}E(u)}{\mathrm{d}t} &= \lim_{t\to 0^+} \frac{\mathrm{dist}^2\big((u+t(u(\alpha)-u)),\Omega^*\big) - \mathrm{dist}^2\big(u,\Omega^*\big)}{t} \\ &= 2\mathrm{dist}\big(u,\Omega^*\big)\cdot \lim_{t\to 0^+} \frac{\mathrm{dist}\big((1-t)u + tu(\alpha),\Omega^*\big) - \mathrm{dist}\big(u,\Omega^*\big)}{t}. \end{aligned} \tag{5.10}$$

Because the solution set Ω^* is closed and convex, the distance function $\mathrm{dist}(u,\Omega^*)$ is convex, i.e., for all $t \in [0,1]$, we have

$$\begin{aligned} &\mathrm{dist}\big((1-t)u + tu(\alpha),\Omega^*\big) \\ &\leq (1-t)\mathrm{dist}\big(u,\Omega^*\big) + t\mathrm{dist}\big(u(\alpha),\Omega^*\big) \\ &= t\big(\mathrm{dist}\big(u(\alpha),\Omega^*\big) - \mathrm{dist}\big(u,\Omega^*\big)\big) + \mathrm{dist}\big(u,\Omega^*\big). \end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11) we have

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t} \leq 2\mathrm{dist}\big(u,\Omega^*\big)\cdot\Big(\mathrm{dist}\big(u(\alpha),\Omega^*\big) - \mathrm{dist}\big(u,\Omega^*\big)\Big). \tag{5.12}$$

Substituting (5.9) in (5.12), we get

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t} \leq -\sigma\|u-\tilde{u}\|^2, \tag{5.13}$$

i.e.,

$$\left(\frac{\partial E}{\partial u}\right)^T\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) \leq -\sigma\|u-\tilde{u}\|^2.$$

Since $\|u(\alpha)-u\| \leq 2\mathrm{dist}(u,\Omega^*)$, it follows from (5.6) and (5.13) that

$$\left(\frac{\mathrm{d}E(u)}{\mathrm{d}t}\right)\Bigg/\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\| \leq -\sigma\frac{\|u-\tilde{u}\|^2}{2\mathrm{dist}(u,\Omega^*)}.$$

This is a sharper property than the general model introduced in Sec. 2.2.

6 Relations to Some Existing Methods

In order to use the methods in Sections 4 and 5, for given u, we should find $\tilde{u}$, $d_1(u,\tilde{u})$, $d_2(u,\tilde{u})$ and $\varphi(u,\tilde{u})$ which satisfy the conditions (3.2) described in the general framework. Especially, the continuous models are meaningful only when the motion equation is easy to construct. This section illustrates that some existing projection type methods are accordant with this framework.

6.1 Example for Linear Variational Inequalities

Consider the monotone linear variational inequality

$$u^* \in \Omega, \qquad (u' - u^*)^T (Mu^* + q) \geq 0, \qquad \forall\, u' \in \Omega,$$

where M is positive definite but not necessary symmetric. For current point u, we let

$$\tilde{u} = P_\Omega[u - (Mu + q)]. \tag{6.1}$$

Clearly, $\tilde{u}$ is a test vector. By denoting

$$d_1(u,\tilde{u}) = (M^T + I)(u - \tilde{u}), \tag{6.2}$$

$$d_2(u,\tilde{u}) = M^T(u - \tilde{u}) + (Mu + q) \tag{6.3}$$

and

$$\varphi(u,\tilde{u}) = \|u - \tilde{u}\|^2, \tag{6.4}$$

we show that Conditions (3.2) in the general framework are satisfied. It follows from (6.2) and (6.3) that

$$\begin{aligned} P_\Omega\{\tilde{u} - [d_2(u,\tilde{u}) - d_1(u,\tilde{u})]\} &= P_\Omega\{\tilde{u} - [(Mu + q) - (u - \tilde{u})]\} \\ &= P_\Omega\{u - (Mu + q)\}. \end{aligned}$$

The right hand side is $\tilde{u}$ (see (6.1)) and we get Condition (3.2a). Set $K = \|M^T + I\|$, Condition (3.2b) follows from the definition of $d_1(u,\tilde{u})$ directly. Now we turn to check Condition (3.2c). Since $\tilde{u} \in \Omega$, we have

$$(\tilde{u} - u^*)^T (Mu^* + q) \geq 0, \qquad \forall\, u^* \in \Omega^*$$

and it can be rewritten as

$$\{(u - u^*) - (u - \tilde{u})\}^T \{(Mu + q) - M(u - u^*)\} \geq 0, \qquad \forall\, u^* \in \Omega^*.$$

Consequently

$$(u - u^*)^T \{M^T(u - \tilde{u}) + (Mu + q)\} \geq (u - \tilde{u})^T (Mu + q), \qquad \forall\, u^* \in \Omega^*,$$

Using the notations of $\tilde{u}$ and $d_2(u,\tilde{u})$, it can be rewritten as

$$(u - u^*)^T d_2(u,\tilde{u}) \geq (u - \tilde{u})^T (Mu + q), \qquad \forall\, u^* \in \Omega^*$$

and thus

$$(\tilde{u} - u^*)^T d_2(u,\tilde{u}) \geq (u - \tilde{u})^T \{(Mu + q) - d_2(u,\tilde{u})\}, \qquad \forall\, u^* \in \Omega^*. \tag{6.5}$$

By using

$$(Mu + q) - d_2(u, \tilde{u}) = (u - \tilde{u}) - d_1(u, \tilde{u})$$

and

$$\|u - \tilde{u}\|^2 = \varphi(u, \tilde{u}),$$

it follows from (6.5) that

$$\begin{aligned} (\tilde{u} - u^*)^T d_2(u, \tilde{u}) &\geq (u - \tilde{u})^T \{(u - \tilde{u}) - d_1(u, \tilde{u})\} \\ &= \varphi(u, \tilde{u}) - (u - \tilde{u})^T d_1(u, \tilde{u}) \end{aligned} \tag{6.6}$$

and thus Condition (3.2c) is satisfied. Set $\tau = 1$, Condition (3.2d) follows from the definition of $\varphi(u, \tilde{u})$ immediately. The directions $d_1(u, \tilde{u})$ (resp. $d_2(u, \tilde{u})$) defined in (6.2) (resp. (6.3)) were used in He [5] and Solodov and Tseng [12] for constructing discrete methods.

6.2 Example for Nonlinear Variational Inequalities

Consider the monotone nonlinear variational inequality

$$u^* \in \Omega, \qquad (u' - u^*)^T F(u^*) \geq 0, \qquad \forall\, u' \in \Omega,$$

for a current point u, we let

$$\tilde{u} = P_\Omega[u - \beta F(u)], \tag{6.7}$$

where $\beta > 0$ is selected (under the condition that F is Lipschitz continuous) to satisfy

$$\beta \|F(u) - F(\tilde{u})\| \leq \nu \|u - \tilde{u}\|, \quad \nu \in (0, 1). \tag{6.8}$$

By setting

$$d_1(u, \tilde{u}) = (u - \tilde{u}) - \beta(F(u) - F(\tilde{u})) \tag{6.9}$$

and

$$d_2(u, \tilde{u}) = \beta F(\tilde{u}), \tag{6.10}$$

Equation (6.7) can be written as

$$\tilde{u} = P_\Omega\{\tilde{u} - [d_2(u, \tilde{u}) - d_1(u, \tilde{u})]\}$$

which is the condition (3.2a). It follows from (6.8) and (6.9) that

$$\|d_1(u, \tilde{u})\| \leq (1 + \nu)\|u - \tilde{u}\|$$

and thus (3.2b) is satisfied. Since F is monotone, we have

$$(\tilde{u} - u^*)^T \beta F(\tilde{u}) \geq (\tilde{u} - u^*)^T \beta F(u^*) \geq 0$$

and consequently (due to $d_2(u, \tilde{u}) = \beta F(\tilde{u})$)

$$(\tilde{u} - u^*)^T d_2(u, \tilde{u}) \geq 0. \tag{6.11}$$

Setting

$$\varphi(u, \tilde{u}) = (u - \tilde{u})^T d_1(u, \tilde{u}), \tag{6.12}$$

and using Cauchy-Schwarz Inequality, it follows from (6.8) that

$$\begin{aligned}\varphi(u,\tilde{u}) &= (u-\tilde{u})^T d_1(u,\tilde{u}) \\ &= \|u-\tilde{u}\|^2 - (u-\tilde{u})^T \beta\big(F(u)-F(\tilde{u})\big) \\ &\geq (1-\nu)\|u-\tilde{u}\|^2. \end{aligned} \tag{6.13}$$

From (6.11)-(6.13), it yields Conditions (3.2c) and (3.2d). The directions $d_1(u,\tilde{u})$ (resp. $d_2(u,\tilde{u})$) defined in (6.9) (resp. (6.10)) were used in He [7, 9] and Solodov and Tseng [12] for constructing discrete methods.

Indeed, in the above examples, $d_2(u,\tilde{u}) \to \beta F(u^*)$ as $u \to u^*$ and thus usually $\|d_2(u,\tilde{u})\| \gg \|u-\tilde{u}\|$ as $u \to u^*$ when $F(u^*) \neq 0$. However, since the directions in the above mentioned methods are accordant with the the framework in this paper, the methods can be generalized by using the convex combination of $d_1(\tilde{u},u)$ and $d_2(\tilde{u},u)$ as search direction without changing the step sizes.

7 Applications for Structured Variational Inequalities

In this section, we give applications of the proposed framework for structured variational inequalities. Consider the VI problem with the following structure:

$$(x^*,y^*)\in\mathcal{D}, \quad \begin{cases} (x-x^*)^T f(x^*) \geq 0, \\ (y-y^*)^T g(y^*) \geq 0, \end{cases} \quad \forall\,(x,y)\in\mathcal{D}, \tag{7.1}$$

where

$$\mathcal{D} = \{(x,y) | x\in\mathcal{X},\ y\in\mathcal{Y},\ Ax+By=b\}, \tag{7.2}$$

$\mathcal{X}$ and $\mathcal{Y}$ are given nonempty closed convex subsets of $\Re^{n_1}$ and $\Re^{n_2}$, respectively, $A\in\Re^{m\times n_1}$ and $B\in\Re^{m\times n_2}$ are given matrices, $b\in\Re^m$ is a given vector, $f:\mathcal{X}\to\Re^{n_1}$ and $g:\mathcal{Y}\to\Re^{n_2}$ are monotone operators. Additionally, we assume that f and g are Lipschitz continuous with Lipschitz constants L_f and L_g, respectively.

By attaching a Lagrange multiplier vector $\lambda\in\Re^m$ to the linear constraint $Ax+By=b$, the VI problem (7.1)-(7.2) is converted into the following equivalent form:

$$(x^*,y^*,\lambda^*)\in\Omega, \quad \begin{cases} (x-x^*)^T (f(x^*)-A^T\lambda^*) \geq 0, \\ (y-y^*)^T (g(y^*)-B^T\lambda^*) \geq 0, \\ (\lambda-\lambda^*)^T (Ax^*+By^*-b) \geq 0, \end{cases} \quad \forall\,(x,y,\lambda)\in\Omega \tag{7.3}$$

where

$$\Omega = \mathcal{X}\times\mathcal{Y}\times\Re^m. \tag{7.4}$$

We denote VI problem (7.3)-(7.4) by VI(Ω,F), where

$$F(u) = F(x,y,\lambda) = \begin{pmatrix} f(x)-A^T\lambda \\ g(y)-B^T\lambda \\ Ax+By-b \end{pmatrix}. \tag{7.5}$$

Let $\nu\in(0,1)$, r and s be given constants such that

$$r \geq \frac{L_f + \|A^TA\|}{\nu}, \tag{7.6}$$

and

$$s \geq \frac{L_g + \|B^T B\|}{\nu}. \tag{7.7}$$

Given a triplet $(x, y, \lambda) \in \mathcal{W}$, we can get a test vector $(\tilde{x}, \tilde{y}, \tilde{\lambda})$ as in [10] by the following procedure:

First, set

$$\tilde{x} = P_{\mathcal{X}}\Big\{x - \frac{1}{r}\Big(f(x) - A^T[\lambda - (Ax + By - b)]\Big)\Big\}, \tag{7.8a}$$

then let

$$\tilde{y} = P_{\mathcal{Y}}\Big\{y - \frac{1}{s}\Big(g(y) - B^T[\lambda - (A\tilde{x} + By - b)]\Big)\Big\} \tag{7.8b}$$

finally, update $\tilde{\lambda}$ via

$$\tilde{\lambda} = \lambda - (A\tilde{x} + B\tilde{y} - b). \tag{7.8c}$$

Note that $\tilde{u} = (\tilde{x}, \tilde{y}, \tilde{\lambda}) \in \Omega$ is produced in the following order,

- obtain $\tilde{x} \in \mathcal{X}$ from given $(x, y, \lambda) \in \Omega$;
- obtain $\tilde{y} \in \mathcal{Y}$ from given $(\tilde{x}, y, \lambda) \in \Omega$;
- update $\tilde{\lambda} \in \Re^m$ from given $(\tilde{x}, \tilde{y}, \lambda) \in \Omega$,

this procedure adopts the new information whenever possible and only requires the function values $f(x)$ and $g(y)$. For given $u = (x, y, \lambda)$, similar procedure was used in the alternating projection method [10] for producing a test vector $\tilde{u} = (\tilde{x}, \tilde{y}, \tilde{\lambda})$.

For the $\tilde{u}$ obtained from (7.8), we will prove that it is a test vector and find proper $d_1(u, \tilde{u})$, $d_2(u, \tilde{u})$ and $\varphi(u, \tilde{u})$ which satisfy the conditions (3.2) in the general framework in Section 3. To simplify our following analysis, we denote

$$R = rI_{n_1}, \qquad S = sI_{n_2} \qquad \text{and} \qquad M = S + B^T B, \tag{7.9}$$

$$\xi_x = f(x) - f(\tilde{x}) + A^T A(x - \tilde{x}) \tag{7.10}$$

and

$$\xi_y = g(y) - g(\tilde{y}) + B^T B(y - \tilde{y}). \tag{7.11}$$

With proper large scalar r defined in (7.6), we obtain

$$\|\xi_x\| \overset{(7.10)}{\leq} (L_f + \|A^T A\|)\|x - \tilde{x}\| \overset{(7.6)}{\leq} \nu r\|x - \tilde{x}\|. \tag{7.12}$$

Similarly, we have

$$\|\xi_y\| \overset{(7.11)}{\leq} (L_g + \|B^T B\|)\|y - \tilde{y}\| \overset{(7.7)}{\leq} \nu s\|y - \tilde{y}\|. \tag{7.13}$$

Note that $\tilde{x}$ and $\tilde{y}$ can be rewritten as

$$\begin{aligned} \tilde{x} \;&\overset{(7.8a)}{=}\; P_{\mathcal{X}}\big\{x - R^{-1}\big(f(x) - A^T[\lambda - (Ax + By - b)]\big)\big\} \\ &\overset{(7.10)}{=}\; P_{\mathcal{X}}\{\tilde{x} - [f(\tilde{x}) - A^T\tilde{\lambda} + A^T B(y - \tilde{y}) + R(\tilde{x} - x) + \xi_x]\} \end{aligned} \tag{7.14}$$

and

$$\begin{aligned}\tilde{y} &\stackrel{(7.8b)}{=} P_{\mathcal{Y}}\{y - S^{-1}[g(y) - B^T[\lambda - (A\tilde{x} + By - b)]]\} \\ &\stackrel{(7.11)}{=} P_{\mathcal{Y}}\{\tilde{y} - [g(\tilde{y}) - B^T\tilde{\lambda} + S(\tilde{y} - y) + \xi_y]\} \\ &\stackrel{(7.9)}{=} P_{\mathcal{Y}}\{\tilde{y} - [g(\tilde{y}) - B^T\tilde{\lambda} + B^TB(y - \tilde{y}) + M(\tilde{y} - y) + \xi_y]\}, \end{aligned} \tag{7.15}$$

respectively. For adopting a compact form for $\tilde{u}$, we denote

$$G = \begin{pmatrix} R & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I_m \end{pmatrix}, \qquad \text{and} \qquad \xi = \begin{pmatrix} \xi_x \\ \xi_y \\ 0 \end{pmatrix}. \tag{7.16}$$

Using the notation of $F(u)$ (see (7.5)), it follows from (7.14), (7.15) and (7.8c) that

$$\tilde{u} = P_\Omega\{\tilde{u} - [F(\tilde{u}) + (A, B, 0)^T B(y - \tilde{y}) + G(\tilde{u} - u) + \xi]\}. \tag{7.17}$$

If $u = \tilde{u}$, it follows from (7.10), (7.11) and the above inequality that $\tilde{u} = P_\Omega\{\tilde{u} - F(\tilde{u})\}$ and thus $u = \tilde{u}$ is a solution of the original VI(Ω, F). Moreover, since

$$\begin{aligned}\|e(\tilde{u})\| &= \|P_\Omega\{\tilde{u} - F(\tilde{u})\} - \tilde{u}\| \\ &\stackrel{(7.17)}{=} \|P_\Omega\{\tilde{u} - F(\tilde{u})\} - P_\Omega\{\tilde{u} - [F(\tilde{u}) + (A, B, 0)^T B(y - \tilde{y}) + G(\tilde{u} - u) + \xi]\}\| \\ &\stackrel{(2.2)}{\le} \|(A, B, 0)^T B(y - \tilde{y}) + G(\tilde{u} - u) + \xi\|, \end{aligned}$$

we can find a $c_0 > 0$ such that $\|u - \tilde{u}\| > c_0\|e(\tilde{u})\|$ and consequently $\tilde{u}$ is a test vector.

Based on the obtained test vector $\tilde{u}$, we define

$$d_1(u, \tilde{u}) = G(u - \tilde{u}) - \xi, \tag{7.18}$$

$$d_2(u, \tilde{u}) = F(\tilde{u}) + (A, B, 0)^T B(y - \tilde{y}), \tag{7.19}$$

and

$$\varphi(u, \tilde{u}) = (\lambda - \tilde{\lambda})^T(By - B\tilde{y}) + (u - \tilde{u})^T d_1(u, \tilde{u}). \tag{7.20}$$

In the following we prove that the conditions (3.2) are satisfied for the above defined $d_1(u, \tilde{u})$, $d_2(u, \tilde{u})$ and $\varphi(u, \tilde{u})$.

It follows from (7.18) and (7.19) that equation (7.17) can be written as

$$\tilde{u} = P_\Omega\{\tilde{u} - [d_2(u, \tilde{u}) - d_1(u, \tilde{u})]\}, \tag{7.21}$$

and thus **Condition (3.2a)** is satisfied. In order to check the satisfactory of other conditions, we prove the following lemmas.

Lemma 7.1. *For $d_1(u, \tilde{u})$ defined in* (7.18)*, we have*

$$\|G^{-1}d_1(u, \tilde{u})\|_G \le (1 + \nu)\|u - \tilde{u}\|_G \tag{7.22}$$

and thus **Condition (3.2b)** *is satisfied.*

Proof. We notice that (since $(M-S)$ is positive semi-definite)

$$(\xi_y)^T S^{-1}\xi_y \ge (\xi_y)^T M^{-1}\xi_y$$

and thus

$$\|S^{-1}\xi_y\|_S^2 \ge \|M^{-1}\xi_y\|_M^2. \tag{7.23}$$

Notice that under the conditions (7.12) and (7.13)

$$\begin{aligned} \|G^{-1}\xi\|_G^2 &\overset{\text{def}}{=} \|R^{-1}\xi_x\|_R^2 + \|M^{-1}\xi_y\|_M^2 \\ &\overset{(7.23)}{\le} \|R^{-1}\xi_x\|_R^2 + \|S^{-1}\xi_y\|_S^2 \\ &\overset{(7.12,7.13)}{\le} \nu^2\big(\|x-\tilde{x}\|_R^2 + \|y-\tilde{y}\|_S^2\big) \\ &\le \nu^2\big(\|x-\tilde{x}\|_R^2 + \|y-\tilde{y}\|_{(S+B^TB)}^2\big) \\ &\overset{\text{def}}{\le} \nu^2\|u-\tilde{u}\|_G^2. \end{aligned} \tag{7.24}$$

Therefore

$$\begin{aligned} \|G^{-1}d_1(u,\tilde{u})\|_G &\overset{(7.18)}{=} \|(u-\tilde{u}) - G^{-1}\xi\|_G \\ &\le \|u-\tilde{u}\|_G + \|G^{-1}\xi\|_G \\ &\overset{(7.24)}{\le} (1+\nu)\|u-\tilde{u}\|_G \end{aligned}$$

and the lemma is proved. □

Lemma 7.2. *For $d_2(u,\tilde{u})$ defined in* (7.19)*, we have*

$$(\tilde{u}-u^*)^T d_2(u,\tilde{u}) \ge (\lambda-\tilde{\lambda})^T B(y-\tilde{y}) \tag{7.25}$$

and thus **Condition (3.2c)** *satisfied.*

Proof. Since $\tilde{u}\in\Omega$ and $u^*\in\Omega^*$ is a solution of VI(Ω,F), we have

$$(\tilde{u}-u^*)^T F(u^*) \ge 0.$$

Using the monotonicity of F it follows that

$$(\tilde{u}-u^*)^T F(\tilde{u}) \ge (\tilde{u}-u^*)^T F(u^*) \ge 0.$$

Because $F(\tilde{u}) = d_2(u,\tilde{u}) - [A,B,0]^T B(y-\tilde{y})$ (see (7.19)), from the above inequality we obtain

$$(\tilde{u}-u^*)^T d_2(u,\tilde{u}) \ge (\tilde{u}-u^*)^T [A,B,0]^T B(y-\tilde{y}). \tag{7.26}$$

Using $A(\tilde{x}-x^*) + B(\tilde{y}-y^*) = A\tilde{x}+B\tilde{y}-b = (\lambda-\tilde{\lambda})$, the right hand side of (7.26) is $(\lambda-\tilde{\lambda})^T B(y-\tilde{y})$. Since (see (7.20))

$$(\lambda-\tilde{\lambda})^T B(y-\tilde{y}) = \varphi(u,\tilde{u}) - (u-\tilde{u})^T d_1(u,\tilde{u}),$$

Condition (3.2c) follows from (7.25) and the above inequality immediately. □

Lemma 7.3. *For $\varphi(u,\tilde{u})$ defined in* (7.20)*, we have*

$$\varphi(u,\tilde{u}) > \min\left\{\frac{1}{2},(1-\nu)\right\}\|u-\tilde{u}\|_G^2 \tag{7.27}$$

and thus **Condition (3.2d)** *satisfied.*

Proof. First, we have

$$\varphi(u,\tilde{u}) \;=\; (u-\tilde{u})^T G(u-\tilde{u}) + (\lambda-\tilde{\lambda})^T(By-B\tilde{y}) - (u-\tilde{u})^T\xi.$$

By using Cauchy-Schwarz Inequality, we have

$$\begin{aligned}(\lambda-\tilde{\lambda})^T(By-B\tilde{y}) &\geq -\frac{1}{2}\Big(\|y-\tilde{y}\|^2_{B^TB}+\|\lambda-\tilde{\lambda}\|^2\Big)\\ &= -\frac{1}{2}\begin{pmatrix} y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}^T\begin{pmatrix} B^TB & \\ & I_m\end{pmatrix}\begin{pmatrix} y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}\end{aligned}$$

and consequently (see the notation of G (7.16))

$$\begin{aligned}\varphi(u,\tilde{u}) \;\geq\; & \begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}^T\begin{pmatrix} R & 0 & 0\\ 0 & S+\frac{1}{2}B^TB & \\ 0 & & \frac{1}{2}I_m\end{pmatrix}\begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}\\ & -(u-\tilde{u})^T\xi.\end{aligned} \tag{7.28}$$

Using (7.12), (7.13) and Cauchy-Schwarz inequality,

$$-(u-\tilde{u})^T\xi \geq -\nu\begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\end{pmatrix}^T\begin{pmatrix} R & \\ & S\end{pmatrix}\begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\end{pmatrix}.$$

Substituting the above inequality into (7.28), we get

$$\begin{aligned}\varphi(u,\tilde{u}) &\geq \begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}^T\begin{pmatrix} (1-\nu)R & 0 & 0\\ 0 & (1-\nu)S+\frac{1}{2}B^TB & \\ 0 & & \frac{1}{2}I_m\end{pmatrix}\begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}\\ &\geq \min\left\{\frac{1}{2},(1-\nu)\right\}\begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}^T\begin{pmatrix} R & 0 & 0\\ 0 & S+B^TB & \\ 0 & & I_m\end{pmatrix}\begin{pmatrix} x-\tilde{x}\\ y-\tilde{y}\\ \lambda-\tilde{\lambda}\end{pmatrix}\\ &= \min\left\{\frac{1}{2},(1-\nu)\right\}\|u-\tilde{u}\|_G^2\end{aligned}$$

and thus the assertion is proved. □

Now, we have proved that the $\tilde{u}$ obtained from (7.8), with $d_1(u,\tilde{u})$, $d_2(u,\tilde{u})$ and $\varphi(u,\tilde{u})$ defined in (7.18)-(7.20), satisfy the conditions (3.2) in the general framework. Therefore, based on such $\tilde{u}$, $d_1(u,\tilde{u})$, $d_2(u,\tilde{u})$ and $\varphi(u,\tilde{u})$, we can construct both discrete and continuous methods for the structured variational inequalities.

8 Conclusion

In this paper, we presented a general framework of contraction methods for variational inequalities. Based on the convex combination of the two directions offered by the framework, we can construct various discrete and continuous methods. The major generalization of the discrete methods is that the step size remains unchanged even if the search directions are different combinations. The applications for various typical problems indicate that it is easy to construct the directions which satisfy the conditions in the general framework. In the continuous models, we take $E(u) = \mathrm{dist}^2(u, \Omega^*)$ as the energy function. It is easy to get the motion equations based on the general framework, and the motorial track is stable and converges to the solution set.

References

[1] A. Auslender, *Optimization méthodes numŕiques*, Mason, Paris, 1976.

[2] E. Blum and W. Oettli, *Mathematische Optimierung. Grundlagen und Verfahren, Ökonometrie und Unternehmensforschung*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[3] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, *Math. Program.* 53 (1992) 99–110.

[4] B.S. He, A projection and contraction method for a class of linear complementarity problems and its application in convex quadratic programming, *Applied Mathematics and Optimization* 25 (1992) 247–262.

[5] B.S. He, A new method for a class of linear variational inequalities, *Math. Program.* 66 (1994) 137–144.

[6] B.S. He, Solving a class of linear projection equations, *Numerische Mathematik* 68 (1994) 71–80.

[7] B.S. He, A class of projection and contraction methods for monotone variational inequalities, *Applied Mathematics and optimization* 35 (1997) 69–76.

[8] B.S. He, Inexact implicit methods for monotone general variational inequalities, *Math. Program. Series A* 86 (1999) 199–217.

[9] B.S. He and L.Z. Liao, Improvements of some projection methods for monotone nonlinear variational inequalities, *J. Optim. Theory and Appl.* 112 (2002) 111–128.

[10] B.S. He, L.Z. Liao and M.J. Qian, Alternating projection based prediction-correction methods for structured variational inequalities, *J. Comp. Math.* 24 (2006) 693–710.

[11] M.P. Kennedy and L.O. Chua, Neural networks for nonlinear programming, *IEEE Trans. Circ. Syst.* 35 (1988) 554–562.

[12] M.V. Solodov and P. Tseng, Modified projection-type methods for monotone variational inequalities, *SIAM J. Control and Optim.* 34 (1996) 1814–1830.

[13] D.W. Tank and J.J. Hopfied, Simple 'neural' optimization network: An A/D converter, signal decision circuit, and a linear programming circuit, *IEEE Trans. Circ. Syst.* CAS-33 (1986) 533–541.

[14] T. Zhu and Z.G. Yu, A simple proof for some important properties of the projection mapping, *Mathematical Inequalities* & *Applications* 7 (2004) 453-456.

Manuscript received 30 August 2007
revised 29 October 2007
accepted for publication 7 November 2007

Bingsheng He
Department of Mathematics, Nanjing University
Nanjing, 210093, P. R. China
E-mail address: hebma@nju.edu.cn

Ming-Hua Xu
Department of Information Science, Jiangsu Polytechnic University
Changzhou, Jiangsu Province, 213164, P. R. China
E-mail address: xuminghua@jpu.edu.cn