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A FILLED FUNCTION METHOD FOR CONSTRAINED NONLINEAR EQUATIONS*

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Dedicated to Professor Liansheng Zhang on the occasion of his 70th birthday.

Abstract: We consider the problem of solving a constrained system of nonlinear equations. After reformulating the system into an equivalent constrained global optimization problems, we construct a filled function based on a special property of the reformulated problem. A filled function method is then proposed to solve the constrained system of nonlinear equations. Some numerical examples are presented to illustrate the usefulness of the present techniques.

Key words: *constrained system of nonlinear equations, global optimization, filled function method*

Mathematics Subject Classification: *65H10, 65H20, 65K05*

1 Introduction

Consider the following constrained system of nonlinear equations:

$$(CSNE) \qquad \begin{aligned} F(x) &= 0 \\ G(x) &\le 0, \end{aligned}$$

where $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable. The system $(CSNE)$ has found myriad applications in various industrial and economic areas.

If $n = m$ and $l = 0$, then the system $(CSNE)$ reduces to an unconstrained square system of nonlinear equations, which is a classical problem in mathematics, and there are many well-known techniques such as Newton-type methods, secant methods and trust-region methods for solving it, see e.g. [1, 2, 3]. The solving of a general form of the system $(CSNE)$, however, has not been intensively investigated, see e.g. [6].

A typical way of solving $(CSNE)$ is to reformulate it into the following constrained optimization problem

$$(COP) \qquad \begin{aligned} \min \quad & \|F(x)\|^2 \\ \text{s.t.} \quad & G(x) \le 0, \end{aligned}$$

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where $\|\cdot\|$ denotes the Euclidean norm, and then solve (COP) using well-developed optimization methods. It is clear that global optimal solutions of problem (COP) with the zero objective function value correspond to solutions of system $(CSNE)$. Therefore, efficient global optimization methods are crucial for successfully solving system $(CSNE)$.

Filled function methods have been initially proposed for solving unconstrained or box-constrained global optimization problems, see e.g. [4, 5]. Recently, a new filled function method has been developed to solve constrained global optimization problems as well, see [7].

In [8], we proposed a filled function method to solve a box-constrained system of nonlinear equations. To the best of our knowledge, it was the first filled function method in the literature proposed especially for solving nonlinear equations. The present paper deals with a more general case of constrained system of nonlinear equations where general nonlinear inequality constraints $G(x) \leq 0$ are addressed by employing the idea of penalty function in constrained optimization. The obtained numerical experiments show that our present method works quite well.

Note that the optimization problems discussed in both [8] and the present paper are special cases of the optimization problems discussed in [7]. In [8] the fact that the optimal objective function value of the reformulated optimization problem is zero if and only if the original system of nonlinear equations has at least one solution has been used in constructing the filled function there, with the property that the objective function value of the reformulated optimization problem can be reduced by half in each iteration of the corresponding filled function algorithm. The same fact is used in the present paper to construct an appropriate filled function with a similar property to that in [8].

The rest of this paper is organized as follows. In Section 2, a filled function is constructed for the reformulated constrained optimization problem (COP). The corresponding filled function algorithm is presented in Section 3. Several numerical examples are reported in Section 4.

2 Filled Function for Problem (COP)

Let

$$\begin{aligned} F(x) &:= (f_1(x), \ldots, f_m(x))^T, \\ G(x) &:= (g_1(x), \ldots, g_l(x))^T, \\ f(x) &:= \frac{1}{2}\sum_{i=1}^{m} f_i^2(x). \end{aligned}$$

Then problem (COP) can be rewritten as

$$\begin{array}{lll} (COP) & \min & f(x) \\ & \text{s.t.} & g_j(x) \leq 0, j = 1, \ldots, l. \end{array}$$

The following assumptions are needed in developing the filled function method to solve $(CSNE)$.

Assumption 2.1. *The constrained system of nonlinear equations $(CSNE)$ has at least one solution.*

Let

$$S = \{x \in \mathbb{R}^n \mid g_j(x) \le 0,\ j = 1, \dots, l\}, \tag{2.1}$$

$$S^\circ = \{x \in \mathbb{R}^n \mid g_j(x) < 0,\ j = 1, \dots, l\}. \tag{2.2}$$

Note that S° is not necessarily identical to the interior of S.

Assumption 2.2. *$S^\circ \neq \emptyset$ and $\mathrm{cl}S^\circ = S$, where $\mathrm{cl}A$ denotes the closure of set A.*

By Assumption 2.2, we know that for any $x_0 \in S$, there exists a sequence $\{x_k\} \subset S^\circ$, such that $\lim_{k\to\infty} x_k = x_0$.

Throughout the rest of the present paper, we suppose that Assumptions 2.1 and 2.2 hold.

It is easy to see that x^* is a solution of $(CSNE)$ if and only if it is a global minimizer of problem (COP) and satisfies that $f(x^*) = 0$.

For a given point x^* with $f(x^*) > 0$, a filled function at x^* is defined as follows.

Definition 2.3. A continuously differentiable function $p_{x^*}(x)$ is said to be a filled function of problem (COP) at x^* with $f(x^*) > 0$, if

1° x^* is a strict global maximizer of $p_{x^*}(x)$ on $\mathbb{R}^n$;

2° any local minimizer $\bar{x}$ of $p_{x^*}(x)$ on $\mathbb{R}^n$ satisfies

$$f(\bar{x}) < \frac{f(x^*)}{2} \text{ and } \bar{x} \in S^\circ;$$

and

3° there exist infinite points in S° which are local minimizers of $p_{x^*}(x)$ on $\mathbb{R}^n$.

In what follows, we introduce a filled function at x^* with $f(x^*) > 0$, which satisfies Definition 2.3. To begin with, we present a continuously differentiable function $h_{r,a}(t)$ with the following properties: it equals to zero when t is less than a negative number $-r$, and equals to a positive constant a when $t > 0$. More specifically, for any given $r > 0$ and $a > 0$, let

$$h_{r,a}(t) = \begin{cases} a, & t \ge 0 \\ -\dfrac{2a}{r^3}t^3 - \dfrac{3a}{r^2}t^2 + a, & -r < t \le 0 \\ 0, & t \le -r \end{cases}. \tag{2.3}$$

Note that the requirement for continuous differentiability of $h_{r,a}(t)$ justifies the use of cubic polynomial in constructing $h_{r,a}(t)$.

Consequently we have that

$$h'_{r,a}(t) = \begin{cases} 0, & t \ge 0 \\ -\dfrac{6a}{r^3}t^2 - \dfrac{6a}{r^2}t, & -r < t \le 0 \\ 0, & t \le -r \end{cases}. \tag{2.4}$$

Note that $h_{r,a}(t)$ is increasing on $\mathbb{R}$. Given $x^* \in S$ with $f(x^*) > 0$, let

$$p_{q,x^*}(x) = \frac{1}{\|x - x^*\|^2 + 1} h_{\frac{f(x^*)}{4},1}\Bigg(h_{\frac{f(x^*)}{4},f(x^*)}\Big(f(x) - \frac{f(x^*)}{2}\Big) + \sum_{i=1}^{l} h_{\frac{f(x^*)}{q},f(x^*)}\Big(g_i(x)\Big) - \frac{f(x^*)}{2}\Bigg), \tag{2.5}$$

where $q > 0$ is a parameter. It is clear that $p_{q,x^*}(x)$ is continuously differentiable on $\mathbb{R}^n$. Note that $p_{q,x^*}(x)$ includes $\sum_{i=1}^{l} h_{\frac{f(x^*)}{q},f(x^*)}(g_i(x))$ as a penalty term to penalize unfeasible points.

The following theorems show that $p_{q,x^*}(x)$ satisfies Definition 2.3 when q is sufficiently large.

Theorem 2.4. *Let $f(x^*) > 0$. For any $q > 0$, x^* is a strict global maximizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$.*

Proof. For any $q > 0$ and $x \neq x^*$, we have that

$$p_{q,x^*}(x^*) = 1 \text{ and } p_{q,x^*}(x) \leq \frac{1}{\|x - x^*\|^2 + 1} < 1.$$

Thus, x^* is a strict global maximizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$. □

Theorem 2.5. *Let $f(x^*) > 0$. Any local minimizer $\bar{x}$ of $p_{q,x^*}(x)$ on $\mathbb{R}^n$ satisfies that*

$$f(\bar{x}) < \frac{f(x^*)}{2} \text{ and } \bar{x} \in S^\circ.$$

Proof. Let $\bar{x}$ be a local minimizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$, then $\nabla_x p_{q,x^*}(\bar{x}) = 0$ and $\bar{x} \neq x^*$ since x^* is a strict global maximizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$. By contradiction, suppose that

$$f(\bar{x}) < \frac{f(x^*)}{2} \text{ and } \bar{x} \in S^\circ$$

do not hold. Then

$$\nabla_x p_{q,x^*}(\bar{x}) = \frac{-2(\bar{x} - x^*)}{(\|\bar{x} - x^*\|^2 + 1)^2} \neq 0,$$

which is impossible since $\nabla_x p_{q,x^*}(\bar{x}) = 0$. Therefore, any local minimizer $\bar{x}$ of $p_{q,x^*}(x)$ on $\mathbb{R}^n$ satisfies that

$$f(\bar{x}) < \frac{f(x^*)}{2} \text{ and } \bar{x} \in S^\circ.$$

□

Theorem 2.6. *Let $f(x^*) > 0$. For any $\bar{x}$ with $f(\bar{x}) \leq \frac{f(x^*)}{4}$ and $\bar{x} \in S^\circ$, there exists $q_0 > 0$ such that when $q \geq q_0$, $\bar{x}$ is a local minimizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$. Furthermore, the number of point $\tilde{x}$ with $f(\tilde{x}) \leq \frac{f(x^*)}{4}$ and $\tilde{x} \in S^\circ$ is infinite.*

Proof. For any $\bar{x}$ satisfy $f(\bar{x}) \leq \frac{f(x^*)}{4}$ and $\bar{x} \in S^\circ$, we have that

$$f(\bar{x}) - \frac{f(x^*)}{2} \leq -\frac{f(x^*)}{4} \text{ and } g_i(\bar{x}) < 0, i = 1, \ldots, l.$$

Thus, there exists $q_0 > 0$ such that $g_i(\bar{x}) < -\frac{f(x^*)}{q_0}$ for any $i = 1, \ldots, l$. It follows that $p_{q,x^*}(\bar{x}) = 0$ when $q \geq q_0$. Since $p_{q,x^*}(x) \geq 0$ for any $x \in \mathbb{R}^n$, $\bar{x}$ is a global minimizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$. Therefore, x^* is a local minimizer of $p_{q,x^*}(x)$ on $\mathbb{R}^n$.

Let $\hat{x}$ be a solution of $(CSNE)$. Then we have that $\hat{x} \neq x^*$. By $\text{cl}S^\circ = S$, there exists a sequence $\{x_k\} \subset S^\circ$ such that $x_i \neq x_j$ for $i \neq j$ and

$$\lim_{k \to +\infty} x_k = \hat{x}.$$

Hence, there exists a positive integer number k_0 such that when $k \geq k_0$, $f(x_k) \leq \frac{f(x^*)}{4}$. Therefore the number of point $\tilde{x}$ with $f(\tilde{x}) \leq \frac{f(x^*)}{4}$ and $\tilde{x} \in S^\circ$ is infinite. □

Remark 2.7. Note that the present filled function $p_{q,x^*}(x)$ is essentially different from the filled function $p_{r,c,q,x^*}(x)$ proposed in [7] for constrained global optimization. In $p_{r,c,q,x^*}(x)$, x^* is assumed to be a given local minimizer of the original optimization problem, while in $p_{q,x^*}(x)$, x^* is only assumed to be a given point with $f(x^*) > 0$ and it is not necessarily a local minimizer of (COP), even not necessarily a feasible point of (COP). Furthermore, as it is shown in Theorem 2.5, any local minimizer $\bar{x}$ of the function $p_{q,x^*}(x)$ on $\mathbb{R}^n$ satisfies $f(\bar{x}) < \frac{f(x^*)}{2}$, while any local minimizer $\bar{x}$ of the function $p_{r,c,q,x^*}(x)$ on a box X, except for a vertex of X, satisfies only $f(\bar{x}) < f(x^*)$ instead, when the parameters r, c and q are appropriately chosen (see Theorem 2.2 in [7]).

3 Filled Function Algorithm

In this section, we present a global optimization method for solving problem (COP), which leads to a solution or an approximate solution to $(CSNE)$.

The general idea of the global optimization method is as follows. Consider the following unconstrained optimization problem:

$$(UOP) \qquad \min_{x\in\mathbb{R}^n} \quad p_{q,x^*}(x),$$

where $p_{q,x^*}(x)$ is given in (2.5). Let $\bar{x}_1$ be a local minimizer of problem (UOP) on $\mathbb{R}^n$, then we have $f(\bar{x}_1) < \frac{f(x^*)}{2}$ and $\bar{x}_1 \in S^\circ$. By locally solving the problem (COP) starting from the point $\bar{x}_1$, we are able to obtain a local minimizer x_1^* of problem (COP), which also satisfies that $f(x_1^*) < \frac{f(x^*)}{2}$. If $f(x_1^*) = 0$, then x_1^* is the solution of system $(CSNE)$; otherwise locally solve problem (UOP) with x^* replaced by x_1^*. Let $\bar{x}_2$ be the obtained local minimizer, then we have that $f(\bar{x}_2) < \frac{f(x_1^*)}{2}$ and $\bar{x}_2 \in S^\circ$. Repeat this process, we can finally obtain a solution of system $(CSNE)$ or a sequence $\{x_k^*\}$ with $f(x_k^*) < \frac{f(x_1^*)}{2^{k-1}}, k = 1, 2, \ldots$. For such a sequence $\{x_k\}, k = 1, 2, \ldots$, when k is sufficiently large, x_k^* can be regarded as an approximate solution of system $(CSNE)$.

Let $x^* \in \mathbb{R}^n$, $\mu > 0$. x^* is said to be a μ-approximate solution of system $(CSNE)$ if $x^* \in S$ and $f(x^*) \le \mu$. The corresponding filled function algorithm for the global optimization problem (COP) is described below.

Algorithm 3.1 (Filled Function Algorithm for (COP)).

Step 0. Choose a small positive number μ and an initial value q_1 for the parameter q. Choose an initial point $x_0^* \in \mathbb{R}^n$ with $f(x_0^*) > 0$ (if $f(x_0^*) = 0$ and $x_0^* \in S$, then stop, and x_0^* is already a solution of nonlinear system $(CSNE)$). Let $k := 0$, and go to *Step 1.*

Step 1. Let

$$\begin{aligned} p_{q_k,x_k^*}(x) &= \frac{1}{\|x - x_k^*\|^2 + 1} h_{\frac{f(x_k^*)}{4},1}\Bigg(h_{\frac{f(x_k^*)}{4},f(x_k^*)}\Big(f(x) - \frac{f(x_k^*)}{2}\Big) \\ &\quad + \sum_{i=1}^{m} h_{\frac{f(x_k^*)}{q_k},f(x_k^*)}\Big(g_i(x)\Big) - \frac{f(x_k^*)}{2}\Bigg), \end{aligned} \tag{3.1}$$

where $h_{r,c}(t)$ is defined by (2.3). Choose a point y_k^* with $y_k^* \neq x_k^*$(in the numerical examples of Section 4, y_k^* is taken from the proximity of x_k^*). Solve the following problem:

$$\min_{x\in\mathbb{R}^n} \quad p_{q_k,x_k^*}(x) \tag{3.2}$$

by local search method starting from y_k^* (in the numerical examples of Section 4, the conjugate gradient method is used to search for a local minimizer of problem (3.2)). Let $\bar{x}_k$ be the obtained local minimizer. Go to *Step 2*.

Step 2. Solve problem (COP) by local search method starting from $\bar{x}_k$ (in the numerical examples of Section 4, the SQP method is used to search for a local minimizer of problem (COP)). Let x_{k+1}^* be the obtained local minimizer. If $f(x_{k+1}^*) \leq \mu$, then stop, and x_{k+1}^* is a solution or a μ-approximate solution of the constrained nonlinear system $(CSNE)$; otherwise, let $k := k+1$ and go to *Step 1*.

From Theorems 2.4-2.6, it is clear that **Algorithm 3.1** will terminate within finite steps.

4 Numerical Examples

Example 4.1 (See Problem 32 in the "Polynomially Constrained Problems" section on the website "http://icwww.epfl.ch/~sam/Coconut-benchs/". We incorporate $g_i(x), i = 1,2$ into the original problem.).

$$\begin{aligned} f_1(x) &= 4x_1^3 - 3x_1 - x_2 = 0 \\ f_2(x) &= x_1^2 - x_2 = 0 \\ g_1(x) &= x_1 + x_2^2 - 1 \leq 0 \\ g_2(x) &= x_1x_2 - x_1 - 2 \leq 0. \end{aligned}$$

The solution provided in the above source is $(0,0)^T$.

Table 1 records the numerical results of solving Example 4.1 by **Algorithm 3.1**. The initial point x_0^* in the table is obtained by locally solving the corresponding constrained optimization problem (COP) starting from

$$(0.5000000, -0.5000000)^T.$$

The algorithm terminated after x_4^* had been obtained. Note that the obtained approximate solution x_4^* is very close to the solution provided in the source of the example.

Table 1: Numerical results for Example 4.1

k	x_k^*	$f(x_k^*)$	$\bar{x}_k$	$f(\bar{x}_k)$
0	$\begin{pmatrix} -6.5798857E-5 \\ 3.4764635E-6 \end{pmatrix}$	3.7617063E−8	$\begin{pmatrix} -1.4918665E-5 \\ 5.4908469E-5 \end{pmatrix}$	3.1179883E−9
1	$\begin{pmatrix} -1.4918665E-5 \\ 5.4908469E-5 \end{pmatrix}$	3.1179883E−9	$\begin{pmatrix} 3.9115548E-7 \\ 1.5321442E-5 \end{pmatrix}$	5.0682858E−10
2	$\begin{pmatrix} 3.9115548E-7 \\ 1.5321442E-5 \end{pmatrix}$	5.0682858E−10	$\begin{pmatrix} 2.4704084E-6 \\ -3.3367971E-7 \end{pmatrix}$	5.0202995E−11
3	$\begin{pmatrix} 2.4704084E-6 \\ -3.3367971E-7 \end{pmatrix}$	5.0202995E−11	$\begin{pmatrix} 5.5533303E-7 \\ -3.0155272E-6 \end{pmatrix}$	1.0914633E−11
4	$\begin{pmatrix} 5.5533303E-7 \\ -3.0155272E-6 \end{pmatrix}$	1.0914633E−11	−−	−−

Example 4.2 (See Problem 93 in the same source as Example 4.1. Here $g_i(x), i = 1, 2$ have been incorporated into the original problem.).

$$\begin{aligned}
f_1(x) &= x_1x_2^2 + x_1x_3^2 + x_1x_4^2 - 1.1x_1 + 1 = 0\\
f_2(x) &= x_2x_1^2 + x_2x_3^2 + x_2x_4^2 - 1.1x_2 + 1 = 0\\
f_3(x) &= x_3x_1^2 + x_3x_2^2 + x_3x_4^2 - 1.1x_3 + 1 = 0\\
f_4(x) &= x_4x_1^2 + x_4x_2^2 + x_4x_3^2 - 1.1x_4 + 1 = 0\\
g_1(x) &= x_1^2 - 1 \leq 0\\
g_2(x) &= x_1x_2 + x_3x_4^2 - 2 \leq 0.
\end{aligned}$$

The solution provided in the source of the example is

$$\begin{pmatrix} -0.8667443047168157 \\ -0.8667443047168157 \\ -0.8667443047168157 \\ -0.8667443047168158 \end{pmatrix}.$$

Example 4.3 (See Problem 128 in the same source as Example 4.1. We incorporate $g_i(x), i = 1, 2$ into the original problem.).

$$\begin{aligned}
f_1(x) &= 200x_1^3 - 200x_1x_2 + x_1 - 1 = 0\\
f_2(x) &= -100x_1^2 + 110.1x_2 + 9.9x_4 - 20 = 0\\
f_3(x) &= 180x_3^3 - 180x_3x_4 + x_3 - 1 = 0\\
f_4(x) &= -90x_3^2 + 9.9x_2 + 100.1x_4 - 20 = 0\\
g_1(x) &= x_1^2x_3 - x_4 - 1 \leq 0\\
g_2(x) &= x_3x_4^2 - x_1x_2 - 2 \leq 0.
\end{aligned}$$

The solution provided in the source of the example is $(1, 1, 1, 1)$.

Table 3 records the numerical results of solving Example 4.3 by **Algorithm 3.1**. The initial point x_0^* is obtained by locally solving the corresponding constrained optimization problem (COP) starting from

$$(2.000000, 2.000000, 2.000000, 2.000000)^T.$$

The algorithm terminated after x_8^* had been obtained. The obtained approximate solution x_8^* provides a new approximate solution to the example.

Table 2 records the numerical results of solving Example 4.2 by **Algorithm 3.1**. The initial point x_0^* in the table is obtained by locally solving the corresponding constrained optimization problem (COP) starting from

$$(2.000000, -2.000000, 2.000000, -2.000000)^T.$$

The algorithm terminated after x_{11}^* had been obtained. Note that the obtained approximate solution x_{11}^* is very close to the solution provided in the source of the example.

Table 2: Numerical results for Example 4.2

k	x_k^*	$f(x_k^*)$	$\bar{x}_k$	$f(\bar{x}_k)$
0	$\begin{pmatrix} 0.3500557 \\ 0.3532820 \\ 0.3410559 \\ 0.3532815 \end{pmatrix}$	2.212058	$\begin{pmatrix} -0.9473325 \\ -0.4817399 \\ -0.6425902 \\ -0.8257586 \end{pmatrix}$	1.638936
1	$\begin{pmatrix} -0.8868951 \\ -0.8275127 \\ -0.8812694 \\ -0.8678638 \end{pmatrix}$	2.6905691E−04	$\begin{pmatrix} -0.8751874 \\ -0.8494877 \\ -0.8726978 \\ -0.8684734 \end{pmatrix}$	5.1920211E−5
2	$\begin{pmatrix} -0.8753983 \\ -0.8497847 \\ -0.8729223 \\ -0.8687181 \end{pmatrix}$	4.2430343E−5	$\begin{pmatrix} -0.8733780 \\ -0.8684126 \\ -0.8626447 \\ -0.8627962 \end{pmatrix}$	1.0453566E−5
3	$\begin{pmatrix} -0.8733780 \\ -0.8684126 \\ -0.8626447 \\ -0.8627962 \end{pmatrix}$	1.0453566E−5	$\begin{pmatrix} -0.8694260 \\ -0.8684902 \\ -0.8638253 \\ -0.8652345 \end{pmatrix}$	2.5523368E−6
4	$\begin{pmatrix} -0.8694260 \\ -0.8684902 \\ -0.8638253 \\ -0.8652345 \end{pmatrix}$	2.5523368E−6	$\begin{pmatrix} -0.8674767 \\ -0.8672043 \\ -0.8653255 \\ -0.8669978 \end{pmatrix}$	3.4384695E−7
5	$\begin{pmatrix} -0.8674767 \\ -0.8672043 \\ -0.8653255 \\ -0.8669978 \end{pmatrix}$	3.4384695E−7	$\begin{pmatrix} -0.8668536 \\ -0.8667719 \\ -0.8663250 \\ -0.8669671 \end{pmatrix}$	6.0641263E−8
6	$\begin{pmatrix} -0.8668536 \\ -0.8667719 \\ -0.8663250 \\ -0.8669671 \end{pmatrix}$	6.0641263E−8	$\begin{pmatrix} -0.8668450 \\ -0.8668247 \\ -0.8666589 \\ -0.8666413 \end{pmatrix}$	5.1467350E−9
7	$\begin{pmatrix} -0.8668450 \\ -0.8668247 \\ -0.8666589 \\ -0.8666413 \end{pmatrix}$	5.1467350E−9	$\begin{pmatrix} -0.8667801 \\ -0.8667427 \\ -0.8667115 \\ -0.8667474 \end{pmatrix}$	2.9517841E−10
8	$\begin{pmatrix} -0.8667801 \\ -0.8667427 \\ -0.8667115 \\ -0.8667474 \end{pmatrix}$	2.9517841E−10	$\begin{pmatrix} -0.8667552 \\ -0.8667444 \\ -0.8667303 \\ -0.8667516 \end{pmatrix}$	4.9410889E−11
9	$\begin{pmatrix} -0.8667552 \\ -0.8667444 \\ -0.8667303 \\ -0.8667516 \end{pmatrix}$	4.9410889E−11	$\begin{pmatrix} -0.8667511 \\ -0.8667451 \\ -0.8667399 \\ -0.8667451 \end{pmatrix}$	1.0076683E−11
10	$\begin{pmatrix} -0.8667511 \\ -0.8667451 \\ -0.8667399 \\ -0.8667451 \end{pmatrix}$	1.0076683E−11	$\begin{pmatrix} -0.8667474 \\ -0.8667444 \\ -0.8667455 \\ -0.8667434 \end{pmatrix}$	1.0448101E−12
11	$\begin{pmatrix} -0.8667474 \\ -0.8667444 \\ -0.8667455 \\ -0.8667434 \end{pmatrix}$	1.0448101E−12	−−	−−

Table 3: Numerical results for Example 4.3

k	x_k^*	$f(x_k^*)$	$\bar{x}_k$	$f(\bar{x}_k)$
0	$\begin{pmatrix} -0.2469030 \\ 0.1978096 \\ 0.4532202 \\ 0.3017455 \end{pmatrix}$	1.4278629E−2	$\begin{pmatrix} 5.3684991E-2 \\ 0.1683867 \\ -0.1114347 \\ 0.2020276 \end{pmatrix}$	1.5322113E−3
1	$\begin{pmatrix} 5.3684991E-2 \\ 0.1683867 \\ -0.1114347 \\ 0.2020276 \end{pmatrix}$	1.5322113E−3	$\begin{pmatrix} -4.8455458E-2 \\ 0.1678947 \\ -1.8444225E-3 \\ 0.1784243 \end{pmatrix}$	1.4261539E−4
2	$\begin{pmatrix} -4.8455458E-2 \\ 0.1678947 \\ -1.8444225E-3 \\ 0.1784243 \end{pmatrix}$	1.4261539E−4	$\begin{pmatrix} -3.8237344E-2 \\ 0.1656266 \\ -1.8925449E-2 \\ 0.1842932 \end{pmatrix}$	2.1158787E−5
3	$\begin{pmatrix} -3.8237344E-2 \\ 0.1656266 \\ -1.8925449E-2 \\ 0.1842932 \end{pmatrix}$	2.1158787E−5	$\begin{pmatrix} -3.5062015E-2 \\ 0.1660308 \\ -2.5907500E-2 \\ 0.1838242 \end{pmatrix}$	4.4843623E−6
4	$\begin{pmatrix} -3.5062015E-2 \\ 0.1660308 \\ -2.5907500E-2 \\ 0.1838242 \end{pmatrix}$	4.4843623E−6	$\begin{pmatrix} -3.3213530E-2 \\ 0.1661449 \\ -2.9969044E-2 \\ 0.1842321 \end{pmatrix}$	5.7323098E−7
5	$\begin{pmatrix} -3.3213530E-2 \\ 0.1661449 \\ -2.9969044E-2 \\ 0.1842321 \end{pmatrix}$	5.7323098E−7	$\begin{pmatrix} -3.1922456E-2 \\ 0.1659903 \\ -3.0865408E-2 \\ 0.1841901 \end{pmatrix}$	6.5348772E−8
6	$\begin{pmatrix} -3.1922456E-2 \\ 0.1659903 \\ -3.0865408E-2 \\ 0.1841901 \end{pmatrix}$	6.5348772E−8	$\begin{pmatrix} -3.1538315E-2 \\ 0.1660062 \\ -3.1218626E-2 \\ 0.1842820 \end{pmatrix}$	9.8290069E−9
7	$\begin{pmatrix} -3.1538315E-2 \\ 0.1660062 \\ -3.1218626E-2 \\ 0.1842820 \end{pmatrix}$	9.8290069E−9	$\begin{pmatrix} -3.1393509E-2 \\ 0.1659961 \\ -3.1268597E-2 \\ 0.1842589 \end{pmatrix}$	2.5058471E−9
8	$\begin{pmatrix} -3.1393509E-2 \\ 0.1659961 \\ -3.1268597E-2 \\ 0.1842589 \end{pmatrix}$	2.5058471E−9	−−	−−

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