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POLYHEDRALLY TIGHT SET FUNCTIONS AND DISCRETE CONVEXITY

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Abstract: This paper studies the class of polyhedrally tight functions in terms of the basic theorems on convex functions over $\Re^n$, such as the Fenchel Duality Theorem, Separation Theorem etc.(Polyhedrally tight functions are those for which the inequalities

$$y^T x \leq f(y), \quad y \in \mathcal{A},\ x \in \mathcal{A}^*,\ \mathcal{A}, \mathcal{A}^* \subseteq \Re^n$$

can be satisfied as equalities for some vector x, not necessarily simultaneously for all y.) It is shown, using results in [9], that the basic theorems hold for polyhedrally tight set functions provided the concerned functions can be extended to convex/concave functionals retaining certain essential features. These essential features carry over only if the functions are compatible in the sense that the normal cone structures of the associated polyhedra are related in a strong way.

Key words: *discrete convexity, polyhedrally tight set functions, discrete separation theorem*

Mathematics Subject Classification: *90C27, 90C46*

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1 Introduction

Combinatorial optimization as a subject has benefited from convexity based methods and a whole subarea, namely polyhedral combinatorics, is concerned with this viewpoint towards combinatorics. In polyhedral combinatorics, structures are studied by first building an appropriate set function (a function which takes real values on subsets), associating a polyhedron with the set function and studying properties of the original structure through the geometrical properties of the polyhedron. A very good example of this approach is the case of the polyhedron associated with a submodular set function. For each subset under consideration one writes the inequality $\chi_X^T x \leq f(X)$ where χ_X is the characteristic vector associated with the set X. The set of all vectors x which satisfy these inequalities is the polyhedron associated with the set function. Now, clearly, the function $f(\cdot)$ can be recovered using the geometrical object, namely the polyhedron, provided for each X, some vector in the polyhedron actually satisfies $\chi_X^T x = f(X)$. We call such functions polyhedrally tight and use them as the basis for our study.

There is a strong case for regarding polyhedrally tight set functions as 'discrete convex' functions because they are precisely the class of functions which can be extended to convex

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functionals (i.e., convex functions $\hat{f}$ which satisfy $\hat{f}(\lambda z) = \lambda \hat{f}(z), \lambda \geq 0$) (see [9]). Submodular set functions possess a number of properties such as the one implied by the Discrete Separation Theorem of Frank [3] which are analogous to properties of convex functions over $\Re^n$ and a Fenchel-Duality analogue of [4]. Quite naturally, they also possess some other special properties. Now, submodular function theory 'rests' on four basic equivalent theorems which may be called Minkowski Sum Theorem, Discrete Separation Theorem, Fenchel Duality Theorem, and the Intersection Theorem due to Edmonds [2]. We show in this paper using results in [9] that three of these theorems go through also for 'compatible' polyhedrally tight set functions. The last does not appear to generalize. Further, in the case of submodular set functions all four results have integrality counterparts which are equivalent. These do not appear to generalize. Our approach downplays integrality aspects although these techniques have been used for obtaining special integrality results ([9]). Other approaches for studying 'discrete convexity' keep integrality ideas in the forefront ([1],[8]).

The primary motivation behind this paper is to understand better certain techniques, which have worked well for submodular functions. Our main tool is that of extending the set function to a suitable convex functional.

The outline of the paper is as follows:

- Section 2 is on preliminaries,
- Section 3 is on the equivalence of the basic 'key fact' convex function theorems at an elementary level,
- Section 4 is on issues related to extending a polyhedrally tight set function to a convex functional,
- Section 5 is on an idea due to Hirai [6] which allows us to study polyhedrally tight set functions in terms of characteristic inequalities (in the manner submodular functions are studied in terms of submodular inequalities),
- Section 6 is on Conclusions.

2 Notation and Preliminaries

Vectors are treated as functions such as $a : E \to \Re$ where E is the underlying finite nonempty set. Set functions $f : 2^E \to \Re$ are treated as functions over collections of characteristic vectors (characteristic vector χ_X of $X \subseteq E$ takes value 1 on $e \in X$ and 0 on $e' \notin X$). This collection is denoted by $\mathcal{A} \subseteq \Re^E$. For $V \subseteq \mathcal{A}, C(V)$ denotes the cone of nonnegative linear combinations of vectors in V. A function $\hat{f} : \Re^E \to \Re$ is said to be a *convex functional* iff $\hat{f}(\sum_{i=1,\cdots,k} \lambda_i y_i) \leq \sum_{i=1,\cdots,k} \lambda_i \hat{f}(y_i)$, whenever $\lambda_i \geq 0$ and $\hat{f}(\lambda y) = \lambda \hat{f}(y)\ \forall \lambda \geq 0$.

The *polyhedron* P_f (P^f) associated with $f : \mathcal{A} \to \Re$ is defined by

$$x \in \Re^E,\ y^T x \leq f(y), \quad y \in \mathcal{A} \qquad (x \in \Re^E,\ y^T x \geq f(y), \quad y \in \mathcal{A}).$$

We say f is *polyhedrally tight* (*pt*) (*dually polyhedrally tight* (*dpt*)) iff each inequality in P_f (P^f) is satisfied as an equality, not necessarily simultaneously. A *face* F of P_f (P^f) is defined by imposing the additional condition that some of these inequalities be satisfied as equalities. We associate the corresponding set of row vectors with F and denote it by V_F. The *normal cone* of P_f (P^f) at a face is the collection of all vectors c such that $\max_{x\in P_f} c^T x$ ($\min_{x\in P^f} c^T x$) is achieved at the face.

We need the notion of a 'Legal Dual Generator' structure (see [9]) which is a generalization of the structure of generators of normal cones at vertices of a polyhedron.

A *legal dual generator structure* (LDG) $\mathcal{G}$ on E is a collection of sets V of vectors in $\mathcal{A} \subseteq \Re^E$ such that

1. If $c \in \Re^E$ and c belongs to the cone $C(\mathcal{A})$, then there exist $V \in \mathcal{G}$ and $\lambda_i \geq 0$ such that $\sum_i \lambda_i v_i = c$ and $v_i \in V$.
2. (Intersection property) If $V^1, V^2 \in \mathcal{G}$, then $C(V^1 \cap V^2) = C(V^1) \cap C(V^2)$.

As noted before $\mathcal{A}$ would be made up of $0, 1$ vectors. When each $V \in \mathcal{G}$ has linearly independent vectors we say $\mathcal{G}$ is *simplicial*. Given two LDGs $\mathcal{G}_1$ and $\mathcal{G}_2$, we write $\mathcal{G}_1 \geq \mathcal{G}_2$ iff for every $V_2 \in \mathcal{G}_2$, there exists a $V_1 \in \mathcal{G}_1$ s.t. $C(V_2) \subseteq C(V_1)$. The set of all V_F, where F is a vertex of P_f (P^f), is seen to be an LDG structure which is denoted by $\mathcal{G}_f$. We have made use of the following results from [9].

Theorem 2.1. *Let $f : 2^E \to \Re$ and $g : 2^E \to \Re$ be pt and dpt functions, respectively. Let $f \geq g$ and let there be a simplicial LDG structure $\mathcal{G}$ s.t. $\mathcal{G}_f \geq \mathcal{G}$ and $\mathcal{G}_g \geq \mathcal{G}$. Then there exists a modular function h s.t. $f \geq h \geq g$.*

Theorem 2.2. *Let f and g be pt and dpt functions, respectively, on subsets of S. Let $\mathcal{G}_f$ and $\mathcal{G}_g$ be LDGs but let $\mathcal{G}_f \not\geq \mathcal{G}_g$ and $\mathcal{G}_g$ be simplicial. Then there exists a modular function α s.t. $f \geq g + \alpha$ but such that no modular function exists between f and $g + \alpha$.*

The most well known legal dual generator structure is of course that associated with submodular functions. Let $f(\cdot)$ be a submodular function on subsets of E. The most natural LDG $\mathcal{G}^s$ compatible with $f(\cdot)$ (i.e. $\mathcal{G}^s \leq \mathcal{G}_f$) is defined as follows :

Let V_σ, σ a permutation of $\{1, 2, \cdots, |E|\}$, be composed of the row vectors v_j ($j = 1, 2, \cdots, |E|$),

$$v_j(\sigma(i)) = \begin{cases} 1 & (i \leq j), \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{G}^s \equiv \{V_\sigma \mid \sigma \text{ is a permutation of } \{1, 2, \cdots, |E|\}\}$.

$\boxed{3}$ Basic Convexity Theorems

Let $\mathcal{A}$ and $\mathcal{A}^*$ be collections of vectors on $\Re^E$. We assume $\mathcal{A}^*$ to be an abelian group (closed under subtraction). No condition is imposed on $\mathcal{A}$. Let $f : \mathcal{A} \to \Re$. Then $P_f \cap \mathcal{A}^*$ ($P^f \cap \mathcal{A}^*$) is the collection of all vectors x in $\mathcal{A}^*$ which satisfy

$$y^T x \leq f(y), \quad \forall y \in \mathcal{A} \qquad (y^T x \geq f(y), \quad \forall y \in \mathcal{A}).$$

For two such functions f_1 and f_2 we have

$$P_{f_1} + P_{f_2} \equiv \{x \mid x = x_1 + x_2,\ x_1 \in P_{f_1},\ x_2 \in P_{f_2}\}$$

(the Minkowski sum of P_{f_1} and P_{f_2}). We also define

$$f^*(x) \equiv \max_{y \in \mathcal{A}} (x^T y - f(y)), \qquad x \in \mathcal{A}^*$$

$$f_*(x) \equiv \min_{y \in \mathcal{A}} (x^T y - f(y)), \qquad x \in \mathcal{A}^*.$$

We call f^* and f_* convex and concave Fenchel duals, respectively, of f. We allow vectors in $\mathcal{A}^*$ to define functions on $\mathcal{A}$ in the usual way as

$$x(y) \equiv x^T y, \quad x \in \mathcal{A}^*,\ y \in \mathcal{A}.$$

If $f : \mathcal{A} \to \Re$ and $x \in \mathcal{A}^*$, then $x + f$ denotes the function whose value on $y \in \mathcal{A}$ is $x^T y + f(y)$. Let $f_1, f_2, g : \mathcal{A} \to \Re$. We say f_1 and f_2 satisfy *MS* (*Minkowski Sum*) *property* if $P_{f_1+f_2} \cap \mathcal{A}^* = P_{f_1} \cap \mathcal{A}^* + P_{f_2} \cap \mathcal{A}^*$. (Note that $P_{f_1+f_2} \cap \mathcal{A}^* \supseteq P_{f_1} \cap \mathcal{A}^* + P_{f_2} \cap \mathcal{A}^*$ always holds.) We say f, g satisfy *DST* (*Discrete Separation Theorem*) *property* if there exist $h \in \mathcal{A}^*$ and $\delta \in \Re$ s.t.

$$f(\hat{y}) - \min_{y \in \mathcal{A}}(f(y) - g(y)) \geq h^T \hat{y} + \delta \geq g(\hat{y}), \quad \hat{y} \in \mathcal{A}.$$

We say f, g satisfy *FDT* (*Fenchel Duality Theorem*) *property* if

$$\min_{y \in \mathcal{A}}(f(y) - g(y)) = \max_{x \in \mathcal{A}^*}(g_*(x) - f^*(x)).$$

Remark 3.1. For DST to be satisfied it is necessary that $\min_{y \in \mathcal{A}}(f(y) - g(y))$ exists. For FDT to be satisfied g_*, f^* must exist and the min and max of the left-hand side and right-hand side must exist. We have deliberately used min and max in place of inf and sup since, for our arguments of equivalence of the theorems, we need the appropriate 'min' and 'max' values to exist.

We show that f, g satisfy DST iff they satisfy FDT. We show a more limited equivalence between the MS property and the DST property.

Theorem 3.2. *Let $f, g : \mathcal{A} \to \Re$. Let $\mathbf{0} \in \mathcal{A}$, $f(\mathbf{0}) = g(\mathbf{0}) = 0$. Then f and $-g$ satisfy MS iff $f, g + x$ ($\forall x \in \mathcal{A}^*$ s.t. $f \geq g + x$) satisfy DST.*

Proof. (DST $\Longrightarrow$ MS) Let $x \in P_{(f-g)} \cap \mathcal{A}^*$. Then

$$y^T x \leq f(y) - g(y), \qquad \forall y \in \mathcal{A}$$

i.e.,

$$x^T y + g(y) \leq f(y), \qquad \forall y \in \mathcal{A}.$$

Hence

$$x + g \leq f.$$

By DST there exists a vector $h \in \mathcal{A}^*$ s.t.

$$f(y) \geq h(y) \geq (x + g)(y)$$

(noting that $f(\mathbf{0}) = (x+g)(\mathbf{0}) = 0$, so that $\min_{y \in \mathcal{A}}(f(y) - g(y) - x^T y) = 0$). Hence, $h \in P_f \cap \mathcal{A}^*$ and $x - h \in P_{-g} \cap \mathcal{A}^*$. So MS is satisfied by f and $-g$.

(MS $\Longrightarrow$ DST) Let $f \geq g + x, x \in \mathcal{A}^*$. Now, $x^T y \leq (f - g)(y)$. So $x \in P_{(f-g)} \cap \mathcal{A}^*$. We have $P_{(f-g)} \cap \mathcal{A}^* = P_f \cap \mathcal{A}^* + P_{-g} \cap \mathcal{A}^*$.

Hence by MS there exists $h \in \mathcal{A}^*$ s.t. $h \in P_f \cap \mathcal{A}^*$ and $x - h \in P_{-g} \cap \mathcal{A}^*$, i.e.,

$$\begin{aligned} h^T y &\leq f(y), && \forall y \in \mathcal{A}, \\ (x - h)^T y &\leq -g(y), && \forall y \in \mathcal{A}. \end{aligned}$$

Hence $f(y) \geq h^T y \geq g(y) + x^T y, \forall y \in \mathcal{A}$, i.e., $f \geq h \geq g + x$. Thus DST is satisfied by f and $g + x$. $\square$

Theorem 3.3. *Let $f, g : \mathcal{A} \to \Re$. Then f, g satisfy DST iff they satisfy FDT.*

Proof. (DST $\Longrightarrow$ FDT) We have

$$f(\hat{y}) - (\min_{y \in \mathcal{A}}(f(y) - g(y)) \geq g(\hat{y}), \quad \forall \hat{y} \in \mathcal{A}.$$

By DST, there exist $h \in \mathcal{A}^*$ and $\delta \in \Re$ s.t.

$$f(\hat{y}) - (\min_{y \in \mathcal{A}}(f(y) - g(y)) \geq h^T\hat{y} + \delta \geq g(\hat{y}), \quad \forall \hat{y} \in \mathcal{A}.$$

We now have

$$\begin{aligned} h^T\hat{y} - f(\hat{y}) &\leq -\delta - (\min_{y \in \mathcal{A}}(f(y) - g(y))), && \forall \hat{y} \in \mathcal{A}, \\ h^T\hat{y} - g(\hat{y}) &\geq -\delta, && \forall \hat{y} \in \mathcal{A}. \end{aligned}$$

Hence

$$\begin{aligned} f^*(h) &\leq -\delta - (\min_{y \in \mathcal{A}}(f(y) - g(y))), \\ g_*(h) &\geq -\delta. \end{aligned}$$

Hence (by adding the inequalities),

$$g_*(h) - f^*(h) \geq \min_{y \in \mathcal{A}}(f(y) - g(y)).$$

However, from the definition of f^* and g_*, it is clear that

$$g_*(x) - f^*(x) \leq f(y) - g(y), \quad \forall x \in \mathcal{A}^*, \quad \forall y \in \mathcal{A}.$$

Hence

$$g_*(h) - f^*(h) = \min_{y \in \mathcal{A}}(f(y) - g(y))$$

and clearly

$$g_*(h) - f^*(h) = \max_{x \in \mathcal{A}^*}(g_*(x) - f^*(x)),$$

which proves that f, g satisfy FDT.

(FDT $\Longrightarrow$ DST) We have

$$\min_{y \in \mathcal{A}}(f(y) - g(y)) = \max_{x \in \mathcal{A}^*}(g_*(x) - f^*(x)).$$

Let $g_*(h) - f^*(h)$ correspond to the right-hand side of the above equation. Now, by definition of f^* and g_*,

$$h^T y - f(y) \leq f^*(h), \qquad \forall y \in \mathcal{A}$$

and

$$h^T y - g(y) \geq g_*(h), \qquad \forall y \in \mathcal{A}.$$

Hence,

$$f(y) - (g_*(h) - f^*(h)) \geq h^T y - g_*(h) \geq g(y) \qquad \forall y \in \mathcal{A},$$

i.e.,

$$f(y) - \min_{\hat{y} \in \mathcal{A}} (f(\hat{y}) - g(\hat{y})) \geq h^T y - g_*(h) \geq g(y) \qquad \forall y \in \mathcal{A}.$$

□

Remark 3.4. If $\mathcal{A}$, $\mathcal{A}^*$ are collections of integral vectors, then $g_*(h)$ would be integral provided g is integral. Thus if FDT is satisfied, then DST would be satisfied with δ integral.

At the level of generality that we are working, we also have the following result. The proof is essentially that of Theorem 6.1 of [5].

Theorem 3.5. *If f is pt (dpt) s.t. for each $y \in \mathcal{A}$, there exists $x_y \in P_f \cap \mathcal{A}^*$ ($x_y \in P^f \cap \mathcal{A}^*$) with $x_y^T y - f(y) = 0$, then*

$$f^{**} = f \quad (f_{**} = f).$$

Proof. We consider only the *pt* case. By definition

$$f^*(x) \geq x^T y - f(y), \quad \forall y \in \mathcal{A}.$$

Hence

$$f(y) \geq x^T y - f^*(x), \quad \forall y \in \mathcal{A}.$$

Hence

$$f(y) \geq \max_{x \in \mathcal{A}^*} (x^T y - f^*(x)) = f^{**}(y), \quad \forall y \in \mathcal{A}.$$

We construct a vector $x_y \in \mathcal{A}^*$ s.t.

$$f(y) = x_y^T y - f^*(x_y).$$

By the conditions of the theorem, there exists a vector $x_y \in P_f \cap \mathcal{A}^*$ s.t. $f(y) = x_y^T y$. Now,

$$f^*(x_y) = \max_{y \in \mathcal{A}} (x_y^T y - f(y)) = 0,$$

since $x_y \in P_f$. Hence,

$$f(y) = x_y^T y - f^*(x_y).$$

This proves the result. □

Corollary 3.6. *If $f_1 = f + \delta$ where f is pt (dpt) s.t. for each $y \in \mathcal{A}$, there exists $x_y \in P_f \cap \mathcal{A}^*$ ($x_y \in P^f \cap \mathcal{A}^*$) with $x_y^T y - f(y) = 0$ and $\delta \in \Re$, then*

$$f_1^{**} = f_1 \quad ((f_1)_{**} = f_1).$$

Proof. We note that $(f + \delta)^* = f^* - \delta$ and $(f^* - \delta)^* = f^{**} + \delta$. The result follows. □

4 Studying *pt* Functions through Convex Extensions

The discussion of the previous section indicates that, in order to consider discrete functions to be 'convex' it is desirable that they satisfy one of the basic convexity theorems, say Separation or Fenchel Duality Theorem. For *pt* functions one could also use Minkowski Sum theorem equivalently. We show in this section that *pt* and *dpt* functions do satisfy the basic theorems provided they are 'compatible' (i.e., there exists an LDG structure $\mathcal{G}$ s.t.

$\mathcal{G} \leq \mathcal{G}_f$ and $\mathcal{G} \leq \mathcal{G}_g$). If they are incompatible, they do not satisfy the basic theorems for all practical purposes, due to Theorem 2.2. The theme in this section is that *pt* and *dpt* functions satisfy the basic theorem provided they can be extended to convex and concave functionals respectively, retaining properties essential for the theorem to be true for the concerned functionals.

In combinatorial optimization it is often convenient to permit set functions to take nonzero value on the null set. It is therefore natural to work with functions of the form $f + \delta$ where f is a *pt* function and δ is a constant. The ideas of extension that we use for polyhedrally tight functions carry through in this case by introducing an additional dimension. Here we only sketch the ideas since they have already been elaborated for *pt* functions in [9]. Henceforth, we will invariably work with *pt* functions when we use P_f. We say an LDG structure $\mathcal{G}$ is compatible with f iff $\mathcal{G} \leq \mathcal{G}_f$. We extend f to a convex function $\hat{f}$ over $\Re^E$ by

$$\begin{aligned} \hat{f}(c) &\equiv \max_{y \in P_f} c^T y, \quad c \in C(\mathcal{A}) \\ &\equiv +\infty \quad \text{otherwise.} \end{aligned}$$

It is easily directly verified that $\hat{f}$ is convex. The extension of f when f is *dpt* is similar except that we use 'min' in place of 'max', P^f in place of P_f and $-\infty$ in place of $+\infty$. The extension would of course be concave in that case. LDGs enter into the picture here. The value of $\max c^T x$ for $c \in C(\mathcal{A})$ is attained at a vertex v of P_f; equivalently, c belongs to the normal cone of P_f at v which is generated by vectors y_i, where $y_i^T v = f(y_i)$. These are precisely the vectors in $V_v \in \mathcal{G}_f$. Thus $\hat{f}(c)$ may be computed by first expressing c as $\sum \lambda_i y_i, \lambda_i \geq 0$ where y_i are some of the vectors in V_v and taking $\hat{f}(c) = \sum \lambda_i f(y_i)$. Even if c is expressed in a different way in terms of vectors of V_v, the computed value of $\hat{f}(c)$ would be the same.

Let $\mathcal{G} \leq \mathcal{G}_f$ and let c belong to some $V \in \mathcal{G}$ s.t. $C(V) \subseteq C(V_f)$, $V_f \in \mathcal{G}_f$. The value of $\hat{f}(c)$ computed as $\sum \lambda_i f(y'_i)$, where $c = \sum \lambda_i y'_i, \lambda_i \geq 0$ and $y'_i \in V$, would be the same as earlier when the computation was in terms of $\mathcal{G}_f$ (which can be seen by using the above argument using vertices). Now suppose g is *dpt* and $\mathcal{G}$ is compatible with both $\mathcal{G}_f$ and $\mathcal{G}_g$ with $\mathcal{G} \leq \mathcal{G}_f$ and $\mathcal{G} \leq \mathcal{G}_g$. We can again use $V \in \mathcal{G}$ for computing $\hat{g}(c)$. If $g \leq f$, then we can proceed as follows: we write c as $\sum \lambda_i y'_i$, $y'_i \in V$, and $\lambda_i \geq 0$. Then,

$$\hat{f}(c) = \sum \lambda_i f(y'_i), \qquad \hat{g}(c) = \sum \lambda_i g(y'_i).$$

But then $\hat{g}(c) \leq \hat{f}(c)$. Since $\hat{f}$ and $\hat{g}$ are convex and concave, respectively, and $\hat{f}(\mathbf{0}) = \hat{g}(\mathbf{0}) = 0$, there exists a vector $h \in \Re^E$ such that

$$\hat{f}(c) \geq h^T c \geq \hat{g}(c), \quad \forall c \in C(\mathcal{A}).$$

Now, $\hat{f}(y) = f(y)$ ($\forall y \in \mathcal{A}$) since f is *pt*. Hence we have $\hat{f}(y) = f(y) \geq h^T y \geq \hat{g}(y) = g(y), \quad \forall y \in \mathcal{A}$. Thus we have that f and g satisfy DST.

Let us next consider a function of the type $f_1 = f + \delta$ where f is *pt*.

A natural attempt to extend f_1 to a convex function is to introduce an additional dimension. We enlarge E to $E \cup \{e_0\}$ with a new element e_0. Each $y \in \mathcal{A}$ is now changed to $y^0 \in \mathcal{A}^0$ where

$$\begin{aligned} &y^0(e) = y(e), \quad \forall e \in E, \\ &y^0(e_0) = 1. \end{aligned}$$

We can therefore denote y^0 by $(y, 1)$. The function f_1 is replaced by f_{10} where

$$f_{10}(y^0) \equiv f_1(y) = f(y) + \delta,$$
$$f_{10}(e_0) \equiv \delta.$$

$P_{f_{10}}$ would be the polyhedron

$$(y^0)^T x^0 \leq f_{10}(y^0), \quad y^0 \in \mathcal{A}^0.$$

$$x^0(e_0) \leq \delta.$$

The function f_{10} is clearly *pt* when f is *pt*. Similarly, if g is *dpt* and $g_1 = g + \theta$, we can define g_{10} suitably so that it is *dpt*. If $f(y) \geq g(y)$ and $\delta \geq \theta$ we will have

$$f_{10}(y, 1) = (f + \delta)(y) \geq (g + \theta)(y) = g_{10}(y, 1).$$

We next examine the LDG structure $\mathcal{G}_{f_{10}}$ associated with f_{10}.

We claim that a set V_{10} of vectors of the form $(y, 1)$ belongs to $\mathcal{G}_{f_{10}}$ under the following conditions:
(a) $(y, 1) \in V_{10}, y \neq \mathbf{0}$ iff $y \in V$
(b) $(\mathbf{0}, 1) \in V_{10}$
Proof of Claim:- Every vector ${y_1}^0 \in \mathcal{A}^0$ has ${y_1}^0(e_0) = 1$. Hence the inequalities of P_f permit $x^0 \in P_f$ to have $x^0(e_0)$ less than any negative number and therefore $\max_{x^0 \in P_{f_{10}}} {c_1}^T x^0 = \infty$, if $c_1(e_0) < 0$.

Next let $c_1(e_0) = \alpha \geq 0$. By LP duality, if the primal optimum exists,

$$\begin{aligned}
&\max_{x^0 \in P_{f_{10}}} c_1^T x^0 \\
&= \min\{\sum_i \lambda_i f(y_{1i}^0) \mid \lambda_i \geq 0, y_{1i}^0 = (y_i, 1) \in \mathcal{A}^0, \sum_i \lambda_i [{y_i}^T, 1] = [c^T, \alpha]\} \\
&= \min\{\sum_i \lambda_i \hat{f}(y_i) + (\sum_i \lambda_i)\delta \mid \sum_i \lambda_i {y_i}^T = c^T, \lambda_i \geq 0, \sum_i \lambda_i = \alpha, y_i \in \mathcal{A}\} \\
&\text{(noting that } \hat{f}(y_i) = f(y_i), y_i \in \mathcal{A}, \hat{f}(\mathbf{0}) = 0) \\
&= \hat{f}(c) + \alpha\delta
\end{aligned}$$

Thus if the primal optimum exists, it is clear that (c, α), $\alpha \geq 0$ lies in the cone generated by $(\mathbf{0}, 1)$ and $(y_i, 1)$, $y_i \in V$ if c lies in the cone generated by y_i, $y_i \in V$.

This proves the claim.

(The above discussion also shows that if (c, α) is such that $c \in C(V)$, but whenever $c^T = \sum_i \lambda_i y_i^T, \lambda_i \geq 0$, we have $\sum_i \lambda_i > \alpha$, then the primal optimum will not exist.)

Since f_{10} and g_{10} are *pt* and *dpt*, respectively, it follows that the extensions $\hat{f}_{10}$ and $\hat{g}_{10}$ are convex and concave, respectively. If $f_1 \geq g_1$, we have $f_{10} \geq g_{10}$. If an LDG structure $\mathcal{G}$ exists such that $\mathcal{G} \leq \mathcal{G}_f$ as well as $\mathcal{G} \leq \mathcal{G}_g$, we can construct an LDG structure $\mathcal{G}^0$ from $\mathcal{G}$ in the way $\mathcal{G}_{f_{10}}$ was built from $\mathcal{G}_f$ and it would follow that $\mathcal{G}^0 \leq \mathcal{G}_{f_{10}}$ and $\mathcal{G}^0 \leq \mathcal{G}_{g_{10}}$, and therefore $\hat{f}_{10} \geq \hat{g}_{10}$. Hence there would be a vector h^0 in $\Re^{E \cup \{e_0\}}$ s.t. $\hat{f}_{10}(c^0) \geq (h^0)^T c^0 \geq \hat{g}_{10}(c^0)$ for every $c^0 \in C(\mathcal{A}^0)$. Hence $f_{10}(y^0) = \hat{f}_{10}(y^0) \geq (h^0)^T y^0 \geq \hat{g}_{10}(y^0)$ or equivalently $f_1(y) \geq h^T y + h(e_0) \geq g_1(y)$, where $(h^0)^T = (h^T, h(e_0))$. If the LDG $\mathcal{G}^0$ permits an integral h^0, we would have the advantage that $h(e_0)$ is an integer.

From the preceding discussion it is clear that if f and g are *pt* and *dpt*, respectively, then $f + \delta$ and $g + \theta$ satisfy the discrete separation theorem provided f and g are *compatible*, i.e., if there exists an LDG structure $\mathcal{G}$ s.t. $\mathcal{G}_f \geq \mathcal{G}$ and $\mathcal{G}_g \geq \mathcal{G}$. By Theorem 2.2, we know that if either $\mathcal{G}_f$ or $\mathcal{G}_g$ is simplicial (i.e., every member V having $|E|$ linearly independent vectors) and $f \geq g$, unless $\mathcal{G}_f$ and $\mathcal{G}_g$ are compatible they can not always satisfy the Discrete Separation Theorem. From the results of Section 3, we know that for *pt* and *dpt* functions, DST, MS and FDT hold together or not at all. Essentially, therefore, the situation is as follows. For convex and concave functionals on $\Re^E$ all three results — Separation Theorem, the Minkowski sum theorem ($P_{f_1+f_2} = P_{f_1} + P_{f_2}$) and Fenchel Duality Theorem are always true. But things go wrong when we extend *pt* and *dpt* functions to convex functionals unless the functions are compatible. Thus if $f \geq g$ but f and g are not compatible, it would not be true that $\hat{f} \geq \hat{g}$. If *pt* function f and *dpt* function g are not compatible, $\min_{y\in\mathcal{A}}(f(y) - g(y)) \neq \min_{y\in C(\mathcal{A})}(\hat{f}(y) - \hat{g}(y))$. Similarly, if f_1 and f_2 are not compatible, the extension of $f_1 + f_2$ would not be the sum of the extensions of f_1 and f_2. What if we extend incompatible f and g using the same LDG $\mathcal{G}$? In this case $f \geq g$ will clearly lead to $\hat{f} \geq \hat{g}$. Unfortunately we lose convexity during extension so that no separation theorem is guaranteed. The following result is due to Sohoni [10]. We give a different proof consistent with the approach in this paper.

Theorem 4.1. *Let f be pt and let $\mathcal{G}_f \not\geq \mathcal{G}$. Let $\hat{f}(c) \equiv \sum \lambda_i f(y_i), \lambda_i \geq 0, y_i \in V \in \mathcal{G}$ s.t. $\sum \lambda_i y_i = c$. Then, the function $\hat{f}$ is not convex.*

Proof. Since $\mathcal{G}_f \not\geq \mathcal{G}$, there exists a $V \in \mathcal{G}$ s.t. $C(V)$ is not contained in any $C(V_f)$, where $V_f \in \mathcal{G}_f$. Let $V_f \in \mathcal{G}_f$ be such that $C(V) \cap C(V_f)$ has nonzero volume. Let $c \in$ Interior$(C(V) \cap C(V_f))$. Let $c = \sum \lambda_i y_i, \lambda_i \geq 0$ when expressed in terms of vectors in V_f and equal to $\sum \sigma_j y'_j, \sigma_j \geq 0$ when expressed in terms of vectors of V. Observe that at least one of the y'_j, say y'_k, would not be in $C(V_f)$.

Now $\hat{f}(c) = \sum \sigma_j f(y'_j)$. Let $\max\limits_{x\in P_f} c^T x$ be achieved at a vertex say v_c of P_f whose normal cone is $C(V_f)$. Now $y'_k \notin C(V_f)$. So $(y'_k)^T v_c < f(y'_k)$. Hence

$$c^T v_c = (\sum \sigma_j y'_j)^T v_c = \sum \sigma_j ({y'_j}^T v_c) < \sum \sigma_j f(y'_j),$$

where note that $(y'_j)^T v_c \leq f(y'_j)$ since $v_c \in P_f$.

On the other hand

$$c^T v_c = (\sum \lambda_i y_i)^T v_c = \sum \lambda_i (y_i^T v_c) = \sum \lambda_i f(y_i).$$

Now $\hat{f}(y) = f(y), y \in \mathcal{A}$.

Hence we have

$$\hat{f}(c) = \sum \sigma_j f(y'_j) > \sum \lambda_i f(y_i) = \sum \lambda_i \hat{f}(y_i).$$

Thus $\hat{f}(\sum \lambda_i y_i) = \hat{f}(c) > \sum \lambda_i \hat{f}(y_i)$ ($\lambda_i \geq 0$), which contradicts the fact that $\hat{f}$ is a convex functional. □

5 Natural Inequalities for Polyhedrally Tight Functions

In the case of submodular functions, the subject was largely developed in terms of the defining inequalities. The use of the natural LDG for this class arose during the convex extension carried out in [7]. For polyhedrally tight functions our approach has been entirely in terms of LDGs. It is natural to ask whether there are inequalities in this case analogous to the case of submodular functions. In [6] such inequalities are defined and exploited. Here we cast some of these results in our language.

Let $\mathcal{F}$ be any family of subsets of $\mathcal{A}$ (the collection of characteristic vectors of subsets of E) with the property that if $V_i, V_j \in \mathcal{F}$ and $i \neq j$, then $V_i \nsubseteq V_j$. Let $\mathcal{F}_*$ be the family of minimal subsets of $\mathcal{A}$ not contained in any element of $\mathcal{F}$ and let $\mathcal{F}^*$ be the family of maximal subsets of $\mathcal{A}$ not containing any element of $\mathcal{F}$. It is clear that

$$(\mathcal{F}^*)_* = (\mathcal{F}_*)^* = \mathcal{F}.$$

Let $T \subseteq \mathcal{A}$. Given a function $f : \mathcal{A} \to \Re$, a T_f *inequality* (which may or may not be valid) for a function $g : \mathcal{A} \to \Re$ is generated as follows.

> Let $c = \sum_{y_i \in T} y_i$. Now $c \in C(V)$ for some $V \in \mathcal{G}_f$. Let $c = \sum_{y'_i \in V} \lambda_i y'_i$ with $\lambda_i \geq 0$. Then
>
> $$\sum_{y_i \in T} g(y_i) \geq \sum_{y'_i \in V} \lambda_i g(y'_i)$$
>
> is a T_f inequality for g.

If the inequality is strict, it is a *strict* T_f *inequality* for g.

Remark 5.1. If the vectors in V are not linearly independent, there would be many T_f inequalities for g corresponding to a single subset T.

The collection of all such T_f inequalities for g, $T \in (\mathcal{G}_f)_*$ would be the $(\mathcal{G}_f)_*$ inequalities for g. We would say 'strict' $(\mathcal{G}_f)_*$ inequalities if we make every inequality concerned strict. The following lemma summarizes essential ideas concerning T_f inequalities.

Lemma 5.2. *Let $f, g : \mathcal{A} \to \Re$ be pt.*

(a) *Let T be any set. Then f satisfies T_f inequalities.*

(b) *Let T be a set not contained in any member of $\mathcal{G}_f$. Then f satisfies T_f inequalities strictly.*

(c) *Let T be a set contained in a member of $\mathcal{G}_g$ and not contained in any member of $\mathcal{G}_f$. Then g satisfies every T_f inequality for g but violates a strict T_f inequality for g.*

Proof. (a) Let $c = \sum_{y_i \in T} y_i$. Now $c \in C(V)$ for some $V \in \mathcal{G}_f$. Let $c = \sum_{y'_i \in V} \lambda_i y'_i$ with $\lambda_i \geq 0$. Let $\max_{x \in P_f} c^T x$ be achieved at a vertex, say v_c of P_f, whose normal cone is $C(V)$. We have

$$c^T v_c = (\sum_{y_j \in T} y_j)^T v_c = \sum_{y_j \in T} (y_j^T v_c) \leq \sum_{y_j \in T} f(y_j).$$

But $c^T v_c = \sum_{y'_i \in V} \lambda_i f(y'_i)$. Hence, $\sum_{y_j \in T} f(y_j) \geq \sum_{y'_i \in V} \lambda_i f(y'_i)$.

(b) Let T be a set not contained in any member of $\mathcal{G}_f$. Let $c = \sum_{y_i \in T} y_i$. As before $c \in C(V)$ for some $V \in \mathcal{G}_f$. Let $c = \sum_{y'_i \in V} \lambda_i y'_i$ with $\lambda_i \geq 0$. At least one of the y_i in T, say y_k, does not belong to V and therefore, by the definition of $\mathcal{G}_f$, does not also belong to $C(V)$. Let $\max_{x \in P_f} c^T x$ be achieved at a vertex, say v_c of P_f, whose normal cone is $C(V)$. Now $y_k \notin C(V)$. So

$$y_k^T v_c < f(y_k).$$

Hence,

$$c^T v_c = (\sum_{y_j \in T} y_j)^T v_c = \sum_{y_j \in T} (y_j^T v_c) < \sum_{y_j \in T} f(y_j).$$

But $c^T v_c = \sum_{y_i' \in V} \lambda_i f(y_i')$. Hence, $\sum_{y_j \in T} f(y_j) > \sum_{y_i' \in V} \lambda_i f(y_i')$.

(c) Let T be a set contained in a member of $\mathcal{G}_g$ and not contained in any member of $\mathcal{G}_f$. Let $c = \sum_{y_j \in T} y_j = \sum_{y_i' \in V} \lambda_i y_i'$ with $\lambda_i \geq 0$, and $V \in \mathcal{G}_f$. We then have

$$\begin{aligned}\max\{c^T x \mid x \in P_g\} &= \sum_{y_j \in T} g(y_j) \\ &\leq \sum_{y_i' \in V} \lambda_i g(y_i'),\end{aligned}$$

where $\lambda_i \geq 0$ and $V \in \mathcal{G}_f$. This is a violation of a strict T_f inequality for g. □

When we have $\mathcal{G}_f \geq \mathcal{G}_g$, it is clear that $f(\cdot)$ belongs to the class of all functions which satisfy $(\mathcal{G}_g)_*$ inequalities (not necessarily strictly). When $f(\cdot)$ is submodular, we know that $\mathcal{G}_f \geq \mathcal{G}^s$ ($\mathcal{G}^s$ as defined in Section 2), and therefore satisfies the $(\mathcal{G}^s)_*$ inequalities. $(\mathcal{G}^s)_*$ can be seen to be composed of sets of characteristic vectors, each set with two characteristic vectors corresponding to subsets of E which are not contained in each other. This would give us the usual inequalities which define submodular functions. In general however $(\mathcal{G}_f)_*$ inequalities would not have such a neat form symmetric with respect to all elements.

Theorem 5.3. *Let $f, g : \mathcal{A} \to \Re$, be pt functions. $\mathcal{G}_f \geq \mathcal{G}_g$ iff g satisfies $(\mathcal{G}_f)_*$ inequalities strictly.*

Proof. (Only if) Let $\mathcal{G}_f \geq \mathcal{G}_g$. Consider the strict T_f inequality for g with $T \in (\mathcal{G}_f)_*$,

$$\sum_{y_j \in T} g(y_j) > \sum_{y_i' \in V_f} \lambda_i g(y_i'), \qquad \lambda_i \geq 0,$$

where $\sum_{y_i' \in V_f} \lambda_i y_i' = \sum_{y_j \in T} y_j$, and $T \nsubseteq V_f, V_f \in \mathcal{G}_f$. But $\sum_{y_i \in T} y_i \in C(V_g) \subseteq C(V_f)$, for some $V_g \in \mathcal{G}_g$. Hence

$$\sum_{y_i' \in V_f} \lambda_i g(y_i') = \sum_{y_i'' \in V_g} \sigma_i g(y_i''), \qquad \sigma_i \geq 0,$$

where

$$\sum_{y_i' \in V_f} \lambda_i y_i' = \sum_{y_i'' \in V_g} \sigma_i y_i'' = \sum_{y_i \in T} y_i.$$

Hence the strict T_f inequality is implied by the strict T_g inequality

$$\sum_{y_i \in T} g(y_i) > \sum_{y_i'' \in V_g} \sigma_i g(y_i''), \qquad \sigma_i \geq 0$$

which by Lemma 5.2, is satisfied by g.

(if) Let $\mathcal{G}_f \ngeq \mathcal{G}_g$. Then there exists $V \in \mathcal{G}_g$ that is not in any member of $\mathcal{G}_f$ and therefore contains some $T \in (\mathcal{G}_f)_*$. Consider the T_f inequality for g.

$$\sum_{y_i \in T} g(y_i) \geq \sum_{y_i' \in V_f} \lambda_i g(y_i'), \qquad \lambda_i \geq 0$$

where

$$\sum_{y_i \in T} y_i = \sum_{y_i' \in V_f} \lambda_i y_i', \qquad \lambda_i \geq 0.$$

Now, $\sum_{y_i \in T} y_i \in C(V)$ for $V \in \mathcal{G}_g$. Hence the T_f inequality cannot be satisfied by g strictly. □

6 Conclusion

In this paper we have studied basic properties of polyhedrally tight set functions which are analogous to those of convex functionals. In particular, it is shown that at a very elementary level Fenchel Duality Theorem and the Separation Theorem are equivalent, as a consequence of which integrality versions of the theorems can be seen to be equivalent. For polyhedrally tight set functions it is shown that these are equivalent to the result which could be called Minkowski Sum Theorem which says that the sum of the polyhedra associated with a pair of convex 'support' functions is the polyhedron associated with the sum of the functions. By using convex extension ideas it is indicated using results from [9] that these theorems hold provided the set functions are compatible, in particular, when the functions have the same normal cone structure (Legal Dual Generator structure) associated with the vertices of the associated polyhedra. We have also made a primitive attempt to study polyhedrally tight set functions in terms of inequalities associated with them.

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