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A NEW TUNNEL FUNCTION METHOD FOR GLOBAL OPTIMIZATION*

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Dedicated to Professor Liansheng Zhang on the occasion of his 70th birthday.

Abstract: We introduce a new tunnel function for finding the global minimum of a general coercive C^1 function over its domain. A tunnel function is constructed at a local minimizer of the objective function such that it achieves local maximum at the current solution. Moreover, a local minimizer of the tunnel function leads to a new solution to the original problem with lower objective value. Iteration follows in this manner to reach a global minimizer. Promising computational results are included and discussed.

Key words: *global optimization, nonlinear programming, tunnel function*

Mathematics Subject Classification: *90C30, 65H20, 65K05*

1 Introduction

Optimization theory and methods have been widely used with real life applications for many years. Yet, most classical optimization techniques only find local optimizers. In the past several decades, researchers have turned their attentions to global optimization problems [13]. Numerous meta-heuristics and softcomputing techniques, including genetic algorithm [10, 18], tabu search [7, 8, 9], simulated annealing [14, 19], neural networks [11, 12, 23], MARS [17] and electromagnetism method [2], have been developed for finding a global optimal (or near-optimal) solution to problems with nonlinear, nonconvex and discrete structure. Classical continuous optimization methods have also been refined to invoke certain auxiliary functions to move from one local optimal solution to a better one in search of the optimum. The methods of "filled function" [5, 6, 16, 20, 22] and "tunnel function" [1, 4, 15, 21] belong to the latter category.

In this paper, we propose a new tunnel function method for solving minimization problems with a general objective function over a box-constrained domain. The method iterates from one local minimum to a better one. In each iteration, we construct a tunnel function that attains strict local maximum at the current solution. A local minimizer of the tunnel function then leads to a new solution of reduced objective function value. Some promising computational results are reported by running all testing problems listed in [22].

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We introduce the new tunnel function and study its properties for global minimization problems in Section 2. Then we propose a solution method that uses the tunnel function to find a global minimizer in Section 3. Following that we report computational results on the proposed method in Section 4. Concluding remarks are given in the last section.

2 A New Tunnel Function

Let $f:\mathbb{R}^n \to \mathbb{R}$ be a C^1 function that is coercive, i.e., $\lim_{\|x\|\to+\infty} f(x) = +\infty$. Assume that $f(x)$ has a global minimizer in the interior of a box $B = \{x \in \mathbb{R}^n | a \le x \le b\}$ for some $a, b \in \mathbb{R}^n$. Let $X = int\,(B)$. In this paper, we are interested in finding a global optimizer of the following minimization problem

$$(P) \quad \min_{x\in X} f(x). \tag{2.1}$$

Let us first recall the definition of tunnel function.

Definition 2.1. Given $x^\star \in X$ and $f:\mathbb{R}^n \to \mathbb{R}$, a function $T(x; x^\star, r, q)$ is called a "tunnel function" of f at $x^\star$ if, for any $x^0 \in X$ with $r > 0$ and $q > 0$, $T\left(x^0; x^\star, r, q\right) = 0$ if and only if $f\left(x^0\right) - f(x^\star) + r = 0$.

In this paper, we define a function $T(x; x^\star, r, q)$ of f at a given $x^\star \in X$ with $r > 0$ and $q > 0$ as follows

$$T(x; x^\star, r, q) \triangleq \frac{\ln\left(1 + q\left(f(x) - f(x^\star) + r\right)^2\right)}{1 + \|x - x^\star\|^2}, \forall x \in X. \tag{2.2}$$

It is easy to see that $T(x; x^\star, r, q)$ is a tunnel function. Moreover, this function has the following properties:

Property 2.2.

(1) *For any given $x^\star \in X, r > 0, q > 0$, we have $T(x; x^\star, r, q) \ge 0, \forall x \in X$.*

(2) *For any given $r > 0, q > 0$ and $x^0 \in X$, if $f\left(x^0\right) = f(x^\star) - r$, then x^0 is a minimizer of $T(x; x^\star, r, q)$.*

For any $x \in X$, the gradient of $T(x; x^\star, r, q)$ can be calculated as

$$\begin{aligned}\nabla T(x; x^\star, r, q) = &\frac{\left(1 + \|x - x^\star\|^2\right) \frac{2q(f(x) - f(x^\star) + r)}{1 + q(f(x) - f(x^\star) + r)^2} \nabla f(x)}{\left(1 + \|x - x^\star\|^2\right)^2} \\ &- \frac{\ln\left(1 + q\left(f(x) - f(x^\star) + r\right)^2\right) 2(x - x^\star)}{\left(1 + \|x - x^\star\|^2\right)^2}.\end{aligned} \tag{2.3}$$

Assume that $x^\star$ is a local minimizer of f, since $x^\star$ is an interior point, we have $\nabla f(x^\star) = 0$. Then based on (2.3), it is not difficult to see that $\nabla T(x^\star; x^\star, r, q)$ has the following properties:

Property 2.3. *Let $x^\star$ be a local minimizer of f, then*

(1) $\nabla T\left(x^{\star}; x^{\star}, r, q\right)=0$;

(2) *If there exist* $x^{0} \in X$ *and* $r>0$ *such that* $f\left(x^{0}\right)-f\left(x^{\star}\right)+r=0$, *then* $\nabla T\left(x^{0}; x^{\star}, r, q\right)=0$.

For a given function $f \in C^{1}(X)$, we say its gradient function ∇f satisfies the Lipschitz condition on X, if there exists an $L>0$ such that $\left\|\nabla f\left(x_{1}\right)-\nabla f\left(x_{2}\right)\right\| \leq L\left\|x_{1}-x_{2}\right\|, \forall x_{1}, x_{2} \in X$. The following proposition shows an important property of the tunnel function $T\left(x^{0}; x^{\star}, r, q\right)$.

Proposition 2.4. *For Problem* (P), *assume that (i)* ∇f *satisfies the Lipschitz condition, and (ii)* $x^{\star} \in X$ *is a local minimizer of* f *over a neighborhood* $Nbd\left(x^{\star}, \delta\right)$. *If* $x^{\star}$ *is not the global minimizer of* f *and* r *is sufficiently small such that* $f\left(x^{0}\right)=f\left(x^{\star}\right)-r$ *holds for some* $x^{0} \in X$, *then* $T\left(x^{\star}; x^{\star}, r, q\right)>T\left(x; x^{\star}, r, q\right), \forall x \in Nbd\left(x^{\star}, \delta\right) \cap X-\left\{x^{\star}\right\}$ *with* q *being sufficiently large.*

Proof. Notice that $\nabla f\left(x^{\star}\right)=0$ and

$$\nabla T\left(x; x^{\star}, r, q\right)=\frac{\left(1+\left\|x-x^{\star}\right\|^{2}\right) \frac{2 q\left(f(x)-f\left(x^{\star}\right)+r\right)}{1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}} \nabla f(x)}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}} -\frac{\ln \left(1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}\right) 2\left(x-x^{\star}\right)}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}}.$$

Now for each $x \in Nbd\left(x^{\star}, \delta\right) \cap X$, we have $f(x) \geq f\left(x^{\star}\right)$, and hence $f(x)-f\left(x^{\star}\right)+r \geq r>0$. Let $x=x^{\star}+\lambda d, \lambda \geq 0$, then

$$\begin{aligned}
d^{T} \nabla T\left(x; x^{\star}, r, q\right)= & \frac{\left(1+\left\|x-x^{\star}\right\|^{2}\right) \frac{2 q\left(f(x)-f\left(x^{\star}\right)+r\right)}{1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}} d^{T} \nabla f(x)}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}} \\
& -\frac{\ln \left(1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}\right) 2 \lambda\|d\|^{2}}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}} \\
\leq & \frac{\left(1+\left\|x-x^{\star}\right\|^{2}\right) \frac{2 q\left(f(x)-f\left(x^{\star}\right)+r\right)}{1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}}\|d\|\|\nabla f(x)\|}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}} \\
& -\frac{\ln \left(1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}\right) 2 \lambda\|d\|^{2}}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}} \\
= & \frac{\left(1+\left\|x-x^{\star}\right\|^{2}\right) \frac{2 q\left(f(x)-f\left(x^{\star}\right)+r\right)}{1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}}\|d\|\left\|\nabla f(x)-\nabla f\left(x^{\star}\right)\right\|}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}} \\
& -\frac{\ln \left(1+q\left(f(x)-f\left(x^{\star}\right)+r\right)^{2}\right) 2 \lambda\|d\|^{2}}{\left(1+\left\|x-x^{\star}\right\|^{2}\right)^{2}}
\end{aligned}$$

$$\leq \frac{\left(1+\|x-x^\star\|^2\right)\frac{2q(f(x)-f(x^\star)+r)}{1+q(f(x)-f(x^\star)+r)^2}L\lambda\|d\|^2}{\left(1+\|x-x^\star\|^2\right)^2}$$

$$-\frac{\ln\left(1+q\left(f\left(x\right)-f\left(x^\star\right)+r\right)^2\right)2\lambda\|d\|^2}{\left(1+\|x-x^\star\|^2\right)^2}.$$

The term of $\left(1+\|x-x^\star\|^2\right)\frac{2q(f(x)-f(x^\star)+r)}{1+q(f(x)-f(x^\star)+r)^2}L$ is bounded by $\left(1+\|b-a\|^2\right)\frac{2}{r}L$. But when q becomes sufficiently large, $2\ln\left(1+q\left(f\left(x\right)-f\left(x^\star\right)+r\right)^2\right)$ goes to infinity. Therefore, if q is sufficiently large such that $2\ln\left(1+qr^2\right)>\left(1+\|b-a\|^2\right)\frac{2}{r}L$, then

$$\left(1+\|x-x^\star\|^2\right)\frac{2q\left(f\left(x\right)-f\left(x^\star\right)+r\right)}{1+q\left(f\left(x\right)-f\left(x^\star\right)+r\right)^2}L$$

$$-2\ln\left(1+q\left(f\left(x\right)-f\left(x^\star\right)+r\right)^2\right)<0, \qquad \forall x\in Nbd\left(x^\star,\delta\right)\cap X.$$

Thus $d^T\nabla T\left(x;x^\star,r,q\right)<0$. Since d is chosen arbitrarily, and q, which is chosen to satisfy $2\ln\left(1+qr^2\right)>\left(1+\|b-a\|^2\right)\frac{2}{r}L$, is independent of x, we conclude that starting from $x^\star$, any direction is a descent direction of $T\left(\cdot\right)$.

Without loss of generality, we can modify the set $Nbd\left(x^\star,\delta\right)\cap X$ such that $\forall\bar{x}\in Nbd\left(x^\star,\delta\right)\cap X, x=x^\star+\lambda\left(\bar{x}-x^\star\right)\in Nbd\left(x^\star,\delta\right)\cap X$, for $0\leq\lambda\leq1$. Suppose $\exists\bar{x}\in Nbd\left(x^\star,\delta\right)\cap X$ such that $T\left(x^\star;x^\star,r,q\right)\leq T\left(\bar{x};x^\star,r,q\right)$, let us consider the direction vector $d=\bar{x}-x^\star$ over a straight line. There must exist $\tilde{x}=x^\star+\lambda d$ with $0<\lambda<1$ such that the directional derivative $T'\left(\tilde{x};x^\star,r,q;d\right)=T\left(\bar{x};x^\star,r,q\right)-T\left(x^\star;x^\star,r,q\right)\geq0$. On the other hand, $T'\left(\tilde{x};x^\star,r,q;d\right)=d^T\nabla T\left(\tilde{x},x^\star,r,q\right)<0$, which causes a contradiction. Therefore, $x^\star$ is a strict local maximizer of $T\left(\cdot\right)$ on $Nbd\left(x^\star,\delta\right)\cap X$. □

From the proof of the above proposition, it is straightforward to have the following proposition:

Proposition 2.5. *The $x^\star$ in Proposition 2.4 is a strict local maximizer of $T\left(x;x^\star,r,q\right)$. Moreover, starting from $x^\star$, any direction is a descent direction and $T\left(x;x^\star,r,q\right)$ is monotone along this direction within $Nbd\left(x^\star,\delta\right)\cap X$.*

It is interesting to note that a function satisfying the properties stated in the above proposition is known as a "filled function" [5]. Therefore, our new tunnel function $T\left(\cdot\right)$ is also a "filled function" of $f\left(\cdot\right)$.

For the problem (P), we can show that the function $T\left(x;x^\star,r,q\right)$ holds the property of Proposition 2.5 not only on $Nbd\left(x^\star,\delta\right)\cap X$, but also on a larger area for some proper q. Denote $\overline{Nbd}\left(x^\star,\epsilon\right)$ with $r>\epsilon>0$, the level set such that $\forall x\in\overline{Nbd}\left(x^\star,\epsilon\right)$, $f\left(x\right)-f\left(x^\star\right)+r\geq\epsilon$. Then the following proposition holds:

Proposition 2.6. *For Problem (P) with f, T as defined before, we have $T\left(x^\star;x^\star,r,q\right)>T\left(x;x^\star,r,q\right),\forall x\in\overline{Nbd}\left(x^\star,\epsilon\right)\cap X$ with q being sufficiently large. Moreover, $T\left(\cdot\right)$ is monotone along any direction starting from $x^\star$ within the area of $\overline{Nbd}\left(x^\star,\epsilon\right)\cap X$.*

Proof. This proof is similar to that of Proposition 2.4. Here we only need to use a sufficiently large q such that $2\ln\left(1+q\epsilon^2\right)>\left(1+\|b-a\|^2\right)\frac{2}{\epsilon}L$. □

From Propositions 2.5 and 2.6, one can see that by selecting a proper q, the function $T(x;x^\star,r,q)$ has no local minimizer within the range that $x^\star$ is a local minimizer of $f(x)$, say, $Nbd(x^\star,\delta)\cap X$. Furthermore, by using a proper q, $T(x;x^\star,r,q)$ has no local minimizer within a larger range $\overline{Nbd}(x^\star,\epsilon)\cap X$.

Proposition 2.6 also indicates that if q is sufficiently large such that $2\ln\left(1+q\epsilon^2\right) > \left(1+\|b-a\|^2\right)\frac{2}{\epsilon}L$, then starting from $x^\star$ and moving along any direction d, we have $d^T\nabla T(x;x^\star,r,q) < 0, \forall x = x^\star+\lambda d \in \overline{Nbd}(x^\star,\epsilon)\cap X, \lambda > 0$. Moreover, $T(x;x^\star,r,q)$ is monotone along d within $\overline{Nbd}(x^\star,\epsilon)\cap X$.

Proposition 2.7. *If there exists an $\tilde{x}\in\overline{Nbd}(x^\star,\epsilon)\cap X$ such that $\tilde{x}\neq x^\star$ and $\nabla T(\tilde{x};x^\star,r,q) = 0$, then $f(\tilde{x}) < f(x^\star)-r+\epsilon$ with $0<\epsilon<r$, and hence $f(\tilde{x}) < f(x^\star)$.*

Proof. Suppose the statement is not true, then we have $f(\tilde{x}) \geq f(x^\star)-r+\epsilon$ with $0<\epsilon<r$. From the proof of Proposition 2.6 we know that in this case, $d^T\nabla T(\tilde{x};x^\star,r,q) < 0$ where $d=\tilde{x}-x^\star$. This is a contradiction. Hence $f(\tilde{x}) < f(x^\star)-r+\epsilon < f(x^\star)$. Notice that such $\tilde{x}$ does exist, since for any x^0 such that $f\left(x^0\right) = f(x^\star)-r$, we have $\nabla T\left(x^0;x^\star,r,q\right)=0$ and hence $d^T\nabla T\left(x^0;x^\star,r,q\right)=0$. □

Proposition 2.8. *Given any $r>0, r_1>0, r_2>0, q>0$, for any $x_1,x_2\in X$, if $\|x_1-x^\star\| > \|x_2-x^\star\| > 0$ with $f(x_1)=f(x^\star)-r_1$ and $f(x_2)=f(x^\star)-r_2$ where $r\geq r_1>r_2>0$, then $T(x_1;x^\star,r,q) < T(x_2;x^\star,r,q)$.*

Proof. For $x_1,x_2\in X$ where $\|x_1-x^\star\| > \|x_2-x^\star\| > 0$, we have $\frac{1}{1+\|x_1-x^\star\|^2} < \frac{1}{1+\|x_2-x^\star\|^2}$. Since $0 \leq f(x_1)-f(x^\star)+r = r-r_1 < f(x_2)-f(x^\star)+r = r-r_2$, we have $\ln\Big(1+q\left(f(x_1)-f(x^\star)+r\right)^2\Big) < \ln\Big(1+q\left(f(x_2)-f(x^\star)+r\right)^2\Big)$. Consequently, $T(x_1;x^\star,r,q) < T(x_2;x^\star,r,q)$. □

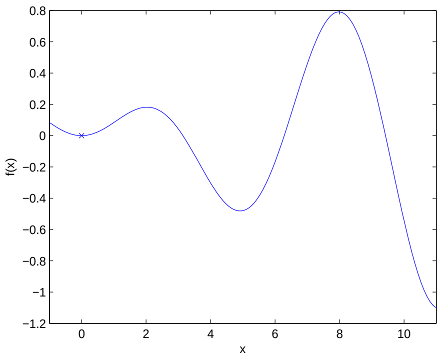
To illustrate the properties of the proposed tunnel function, we consider an example of $f(x)=\frac{1}{10}x\sin(x)$. This function has a local minimum at $x^\star=0$ with $f(x^\star)=0$. For $r=0.1$ and $q=10^3$, the corresponding $T(x;x^\star,r,q)$ monotonically decreases to a minimum at $\tilde{x}=3.4368$ with $\nabla T(\tilde{x};x^\star,r,q)=0$. The figures of $f(x)$ and $T(x;x^\star,r,q)$ are plotted in Figure 1(a) and Figure 1(b), respectively. Notice that at $x=\tilde{x}$, $f(\tilde{x})=-0.1$, which equals $f(x^\star)-r$ as stated in Proposition 2.7. The tunnel function T of $f(x)$ with $r=1$, $q=10^3$ and $r=0.1$, $q=10^8$ are plotted in Figure 1(c) and Figure 1(d), respectively. Comparing these figures with Figure 1(b), one can see that a larger q results in a tunnel function with larger amplitude and a larger r leads to farther local minimizers of the tunnel function from $x^\star$.

3 Solution Method

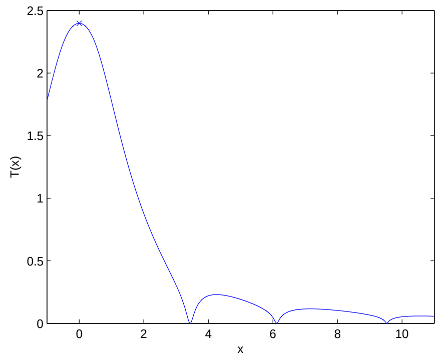
Assume $x^\star\in X$ is a local minimizer of $f(x)$. Based on the properties of the tunnel function proposed in the previous section, if one can find a local minimizer of $T(x;x^\star,r,q)$ in an appropriate neighborhood of $x^\star$ for some r and q, then a solution with lower objective value than $f(x^\star)$ can be obtained. If no such r and q exist, the current local minimizer is the global solution of Problem (P).

For $f(x)$ and $T(x;x^\star,r,q)$ as defined in (2.1) and (2.2), we propose the following algorithm to find a global minimizer of Problem (P):

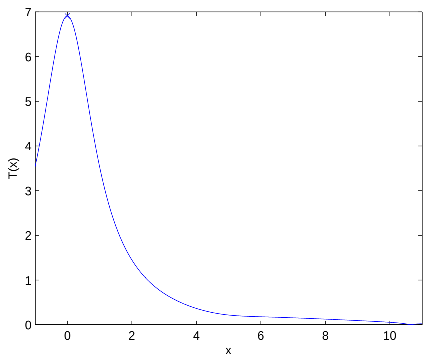
Algorithm 3.1. *Tunnel function method.*



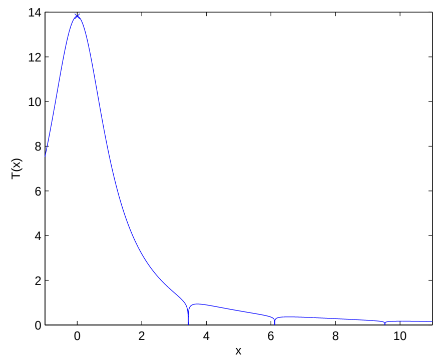
(a) $f(x) = \frac{1}{10} x \sin(x)$



(b) $T\left(x; x^0 = 0, r = 0.1, q = 10^3\right)$



(c) $T\left(x; x^0 = 0, r = 1, q = 10^3\right)$



(d) $T\left(x; x^0 = 0, r = 0.1, q = 10^8\right)$

Figure 1: $f(x)$ and $T(x; x^\star, r, q)$ with $x^0 = 0$ and different r's, q's

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Step 0 : Choose $\epsilon^0 > 0, r > \epsilon^0, \epsilon = \frac{r}{2}$ and an integer $K > 0$.

Step 1 : Pick any initial point $x^0 \in X$ and use a direction search algorithm to find a local minimizer $x^\star$ of $f(x)$.

Step 2 : Choose a sufficiently large $q > 0$ such that $2 \ln\left(1 + q\epsilon^2\right) > \left(1 + \|b - a\|^2\right) \frac{2}{\epsilon} L$ and construct the function $T(x; x^\star, r, q)$ at $x^\star$. Pick K different directions $d_k, k = 1, \ldots, K$.

Step 3 : For $k = 1$ to K do:
Starting from $x^\star$ and moving along the direction d_k, use a one dimensional search algorithm to find the first local minimizer $\tilde{x}$ of $T(x; x^\star, r, q)$ or $f(\tilde{x}) < f(x^\star)$. If $d_k^T \nabla T(\tilde{x}; x^\star, r, q) = 0$ or $f(\tilde{x}) < f(x^\star)$, then go to Step 4. Otherwise go to Step 5.

Step 4 : Starting from $\tilde{x}$, use a direction search algorithm to find a local minimizer $x^\star$ of $f(x)$. Go to Step 6.

Step 5 : Update $r \leftarrow \frac{r}{2}, \epsilon \leftarrow \frac{\epsilon}{2}$.

Step 6 : If $\epsilon > \epsilon^0$, then go to Step 2. Otherwise, terminate algorithm with an ϵ^0 optimal solution.

Remarks:

1) When there is no *a priori* knowledge on the location of the next better local minimizer, a common way to generate the K directions in Step 2 is to use the K different directions that evenly partition the solution space.

2) The proposed algorithm faces with a point $x^\star$ and K directions $d_k, k = 1, \ldots, K$. Starting from $x^\star$ and for all $x = x^\star + \lambda d_k, x \in X, \lambda > 0$, we have $f(x) > f(x^\star) - \epsilon^0$. Thus, when $\epsilon^0 \to 0$ and K is sufficiently large, we have $f(x) \geq f(x^\star), \forall x \in X$.

4 Computational Results

In this section, we run the proposed algorithm on eight test problems used in [22]. The algorithm is implemented using Matlab. The Matlab function “fmincon” is used in the algorithm to find local minimizers of the objective function. All the experiments are conducted on a computer with a Pentium M 1.60GHz processor and 512MB memory. In the implementation, we select K directions so that they evenly partition the solution space by using the following n-dimensional spherical coordinates [3],

$$d = \Bigg(\cos(\phi_1), \cos(\phi_2)\sin(\phi_1), \cdots, \cos(\phi_j)\prod_{l=1}^{j-1}\sin(\phi_l), \cdots, \\ \cos(\phi_{n-2})\prod_{l=1}^{n-3}\sin(\phi_l), \sin(\theta)\prod_{l=1}^{n-2}\sin(\phi_l), \cos(\theta)\prod_{l=1}^{n-2}\sin(\phi_l) \Bigg), \tag{4.1}$$

where $\phi_l \in \{\frac{i\pi}{\kappa} : i = 0, \cdots, \kappa\}$, $l = 1, \cdots, n-2$; $\theta \in \{\frac{2\pi}{\kappa} : i = 0, \cdots, \kappa\}$ and $K = \kappa^{n-1}$.

Problem 4.1 (Two dimensional function [24]).

$$\begin{array}{ll} \min & f(x) = [1 - 2x_2 + c\sin(4\pi x_2) - x_1]^2 + [x_2 - 0.5\sin(2\pi x_1)]^2 \\ \text{s.t.} & 0 \leq x_1 \leq 10,\ -10 \leq x_2 \leq 0, \end{array}$$

with $c = 0.2, 0.5, 0.05$. The proposed method successfully finds the global minimum solutions with $f(x^\star) = 0$ for all c. The computational results are reported in Tables 1, 2 and 3, respectively.

Problem 4.2 (Three-hump back camel function [5]).

$$\begin{array}{ll} \min & f(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2 \\ \text{s.t.} & -3 \leq x_1 \leq 3,\ -3 \leq x_2 \leq 3. \end{array}$$

Starting from two initial points $x^0 = (-2, -1)$ and $(2, 1)$, the proposed method successfully finds the global minimum solution $x^\star = (0, 0)$ with $f(x^\star) = 0$. The computational results are reported in Tables 4 and 5, respectively.

Problem 4.3 (Six-hump back camel function [5]).

$$\begin{array}{ll} \min & f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4 \\ \text{s.t.} & -3 \le x_1 \le 3,\ -3 \le x_2 \le 3. \end{array}$$

Starting from three initial points $x^0 = (-2, 1)$, $(2, -1)$, and $(-2, -1)$, the proposed method successfully finds the global minimum solutions $x^\star = (0.0898, 0.7127)$ or $(-0.0898, -0.7127)$ with $f(x^\star) = -1.0316$. The computational results are reported in Tables 6, 7 and 8, respectively.

Problem 4.4 (Treccani function [5]).

$$\begin{array}{ll} \min & f(x) = x_1^4 + 4x_1^3 + 4x_1^2 + x_2^2 \\ \text{s.t.} & -3 \le x_1 \le 3,\ -3 \le x_2 \le 3. \end{array}$$

The proposed method successfully finds the global minimum solutions $x^\star = (0, 0)$ with $f(x^\star) = 0$. The computational results are reported in Table 9.

Problem 4.5 (Goldstein and Price function [5]).

$$\begin{array}{ll} \min & f(x) = g(x)h(x) \\ \text{s.t.} & -3 \le x_1 \le 3,\ -3 \le x_2 \le 3. \end{array}$$

where

$$\begin{aligned} g(x) &= 1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2), \\ h(x) &= 30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2). \end{aligned}$$

The proposed method successfully finds the global minimum solutions $x^\star = (0, 0)$ with $f(x^\star) = 3$. The computational results are reported in Table 10.

Problem 4.6 (Two-dimensional Shubert function [5]).

$$\begin{array}{ll} \min & f(x) = \left\{\sum_{i=1}^5 i\cos\left[(i+1)x_1 + i\right]\right\}\left\{\sum_{i=1}^5 i\cos\left[(i+1)x_2 + i\right]\right\} \\ \text{s.t.} & 0 \le x_1 \le 10,\ 0 \le x_2 \le 10. \end{array}$$

The proposed method successfully finds the global minimum solution $x^\star = (5.4829, 4.8581)$ or $(4.8581, 5.4829)$ with $f(x^\star) = -186.7309$. The computational results are reported in Table 11.

Problem 4.7 (Shekel's function [17]).

$$\begin{array}{ll} \min & f(x) = -\sum_{i=1}^5 \left[\sum_{j=1}^4 (x_j - a_{i,j})^2 + c_i\right]^{-1} \\ \text{s.t.} & 0 \le x_j \le 10,\ j = 1, 2, 3, 4, \end{array}$$

where the coefficients $a_{i,j}$, c_i, $i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, 4$ are given in the following:

i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	c_i
1	4.0	4.0	4.0	4.0	0.1
2	1.0	1.0	1.0	1.0	0.2
3	8.0	8.0	8.0	8.0	0.3
4	6.0	6.0	6.0	6.0	0.4
5	3.0	7.0	3.0	7.0	0.5

Starting from two initial points $x = (1, 1, 1, 1)$ and $(6, 6, 6, 6)$, the proposed method successfully finds the global minimum solution $x^\star = (4.0000, 4.0001, 4.0000, 4.0001)$ with $f(x^\star) = -10.153$. The computational results are reported in Tables 12 and 13, respectively.

Problem 4.8 (n-dimensional function [5]).

$$\begin{array}{ll} \min & f(x) = \frac{\pi}{n}\left[10\sin^2 \pi x_1 + g(x) + (x_n - 1)^2\right] \\ \text{s.t.} & -10 \le x_i \le 10,\ i = 1, 2, \ldots, n, \end{array}$$

where

$$g(x) = \sum_{i=1}^{n-1} \left[(x_i - 1)^2(1 + 10\sin^2 \pi x_{i+1})\right].$$

For $n = 2, 3, 5, 7, 10$, the proposed method successfully finds the global minimum solution $x^\star = (1, 1, \ldots, 1)$ with $f(x^\star) = 0$ for all n. The computational results are reported in Tables 14 to 18, respectively.

5 Concluding Remarks

In this paper, we have proposed a new tunnel function method for solving a class of global minimization problems. In our computational experiments, the proposed method successfully finds the optimal solution in an effective manner for all testing problems listed in [22]. However, some implementation issues deserve extra attentions to guarantee the performance of Algorithm 3.1.

(1) At a current solution $x^\star$, although it is a strict local maximizer of the tunnel function $T(x; x^\star, r, q)$ and any direction is a descent direction, but not every direction can lead to a point $\tilde{x}$ such that $f(\tilde{x}) < f(x^\star)$. More sophisticated methods than searching K different directions may greatly improve the efficiency of the proposed method.

(2) In Step 4, the accuracy of finding an exact local minimizer $x^\star$ of $f(x)$ may influence the construction of the corresponding tunnel function $T(x; x^\star, r, q)$ at $x^\star$ and consequently affects the performance of the proposed method.

(3) When we hit a global minimizer $x^\star$ of $f(\cdot)$, then in theory for any q and r, we cannot find $x \in X$ such that $x \neq x^\star$, $\nabla T(x; x^\star, r, q) = 0$. Therefore, designing a quick stopping criterion becomes interesting.

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$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(6,-2)$	$1.0e17$	0.01	$(1.4513, 0)$	0.2264
2	$(1.1593, 0)$	$1.0e17$	0.01	$(0.9932, -0.0205)$	$1.0669e-5$
3	$(0.9999, 0)$	$1.0e17$	0.001	$(1, 0)$	0

Table 1: Computational results of Problem 4.1 with $c = 0.2$ and $K = 40$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(0, 0)$	$1.0e17$	0.01	$(0.5524, -0.1037)$	0.0332
2	$(1.7936, -0.5069)$	$1.0e17$	0.001	$(1.7681, -0.5558)$	0.0039
3	$(1.0117, -0.0062)$	$1.0e17$	0.0001	$(1, 0)$	0

Table 2: Computational results of Problem 4.1 with $c = 0.5$ and $K = 40$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(10, -10)$	$1.0e17$	0.01	$(1.8513, -0.4021)$	0

Table 3: Computational results of Problem 4.1 with $c = 0.05$ and $K = 40$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-2, -1)$	$1.0e17$	0.01	$(-1.7475, -0.8737)$	0.2986
2	$(-0.4373, -0.2186)$	$1.0e17$	0.01	$(0, 0)$	0

Table 4: Computational results of Problem 4.2 with initial point $x^0 = (-2, -1)$ and $K = 400$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(2, 1)$	$1.0e17$	0.01	$(1.7475, 0.8737)$	0.2986
2	$(0.2305, -0.3399)$	$1.0e17$	0.01	$(0, 0)$	0

Table 5: Computational results of Problem 4.2 with initial point $x^0 = (2, 1)$ and $K = 400$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-2, 1)$	$1.0e17$	0.01	$(-0.0898, -0.7127)$	-1.0316

Table 6: Computational results of Problem 4.3 with initial point $x^0 = (-2, 1)$ and $K = 20$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(2, -1)$	$1.0e17$	0.01	$(0.0898, 0.7127)$	-1.0316

Table 7: Computational results of Problem 4.3 with initial point $x^0 = (2, -1)$ and $K = 20$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-2, -1)$	$1.0e17$	0.01	$(0.0898, 0.7127)$	-1.0316

Table 8: Computational results of Problem 4.3 with initial point $x^0 = (-2, -1)$ and $K = 20$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-1, 0)$	$1.0e17$	0.01	$(-1, 0)$	1
2	$(-0.9770, 0)$	$1.0e17$	0.01	$(0, 0)$	0

Table 9: Computational results of Problem 4.4 and $K = 20$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-1, -1)$	$1.0e17$	0.01	$(0, -1)$	3

Table 10: Computational results of Problem 4.5 and $K = 400$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(1, 1)$	$1.0e17$	0.01	$(0, 7.6032)$	-13.0520
2	$(4.6060, 7.6032)$	$1.0e17$	0.01	$(6.0878, 6.6174)$	-32.7709
3	$(5.4567, 9.9258)$	$1.0e17$	0.01	$(5.4829, 10.0000)$	-48.5068
4	$(4.6663, 5.7192)$	$1.0e17$	0.01	$(5.4829, 4.8581)$	-186.7309

Table 11: Computational results of Problem 4.6 and $K = 100$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(1, 1, 1, 1)$	$1.0e17$	0.1	$(1.0001, 1.0002, 1.0001, 1.0002)$	-5.0552
2	$(3.8421, 3.8422, 3.8421, 3.8422)$	$1.0e17$	0.1	$(4.0000, 4.0001, 4.0000, 4.0001)$	-10.153

Table 12: Computational results of Problem 4.7 with initial point $x^0 = (1, 1, 1, 1)$ and $K = 64$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(6, 6, 6, 6)$	$1.0e17$	0.1	$(5.9987, 6.0002, 5.9987, 6.0002)$	-2.6822
2	$(4.2657, 4.2672, 4.2657, 4.2672)$	$1.0e17$	0.1	$(4.0000, 4.0001, 4.0000, 4.0001)$	-10.153

Table 13: Computational results of Problem 4.7 with initial point $x^0 = (6, 6, 6, 6)$ and $K = 64$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-4, -4)$	$1.0e17$	0.1	$(1, 1)$	0

Table 14: Computational results of Problem 4.8 with initial point $x^0 = (-4, -4)$ and $K = 9$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-3,-3,-3)$	$1.0e17$	0.1	$(1.0000, 1.0001, 1.0000)$	$3.6388e-9$
2	$(1.0000, 1.0000, 1.0000)$	$1.0e17$	0.1	$(1,1,1)$	0

Table 15: Computational results of Problem 4.8 with initial point $x^0 = (-3,-3,-3)$ and $K = 16$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(-1,-1,-1,-1,-1)$	$1.0e17$	0.1	$(0.0101, 0.0104, 0.0104, 0.0104, 0.0103)$	3.1096
2	$(-0.9479, 0.9684, 0.9684, 0.9684, 0.9683)$	$1.0e17$	0.1	$(-0.9799, 1.0000, 1.0000, 1.0001, 1.0000)$	2.4880
3	$(-0.0349, 0.0550, 0.0550, 1.0001, 1.0000)$	$1.0e17$	0.1	$(0.0101, 0.0103, 0.0102, 1.0000, 1.0000)$	1.8658
4	$(0.0101, 0.0103, 0.9152, 1.0000, 1.0000)$	$1.0e17$	0.1	$(0.0101, 0.0102, 1.0000, 1.0000, 1.0000)$	1.2439
5	$(0.0101, 0.9092, 1.0000, 1.0000, 1.0000)$	$1.0e17$	0.1	$(0.0100, 1.0000, 1.0000, 1.0000, 1.0000)$	0.6220
6	$(0.9040, 1.0000, 1.0000, 1.0000, 1.0000)$	$1.0e7$	0.1	$(1,1,1,1,1)$	0

Table 16: Computational results of Problem 4.8 with initial point $x^0 = (-1,-1,-1,-1,-1)$ and $K = 256$

$iter$	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(2,2,2,2,2,2,2)$	$1.0e17$	0.1	$(1.0000, 1.0001, 1.0000, 1.0000, 1.0000, 1.0000, 0.9996)$	$7.1771e-8$
2	$(1.0000, 1.0001, 1.0000, 1.0000, 1.0000, 1.0000, 1.0001)$	$1.0e17$	0.1	$(1,1,1,1,1)$	0

Table 17: Computational results of Problem 4.8 with initial point $x^0 = (2,2,2,2,2,2,2)$ and $K = 4096$

iter	x^0	p	r	$x^\star$	$f(x^\star)$
1	$(6, 6, 6, 6, 6, 6, 6, 6, 6, 6)$	$1.0e17$	0.1	$(2.9798, 2.9948, 2.9949, 2.9949, 2.9949, 2.9949, 2.9949, 2.9949, 2.9949, 2.9949)$	12.5250
2	$(2.1038, 2.1188, 2.1189, 2.1189, 2.1189, 2.1189, 2.1189, 2.1189, 2.1189, 3.8709)$	$1.0e17$	0.1	$(0.0100, 1.0000, 1.000 2, 1.0000, 1.0004, 1.0002, 1.0001, 0.9996, 0.9999, 0.9996)$	0.3110
3	$(1.0100, 1.0000, 1.0002, 1.0000, 1.0004, 1.0002, 1.0001, 0.9996, 0.9999, 0.9996)$	$1.0e17$	0.1	$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$	0

Table 18: Computational results of Problem 4.8 with initial point $x^0 = (6, 6, 6, 6, 6, 6, 6, 6, 6, 6)$ and $K = 19683$

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