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ON DUALITY FOR A CLASS OF NONDIFFERENTIABLE PROGRAMMING PROBLEMS *

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Dedicated to Professor Liansheng Zhang on the occasion of his 70th birthday.

Abstract: In this paper, first order and second order dual models for a class of nondifferentiable programming problems in which the objective function contains a support function of a compact convex set are formulated. Weak and converse duality theorems for the two dual models are established by using Fritz John necessary optimality conditions and some suitable conditions.

Key words: *first and second order dual models, duality theorems, nondifferentiable programming problems, generalized convexity*

Mathematics Subject Classification: *90C30, 90C46*

1 Introduction

In [5], Mond first established a first order dual theorem for the following nondifferentiable programming problem

$$(P_1) \qquad \text{minimize} \qquad f(x) + (x^T Bx)^{\frac{1}{2}}$$
$$\text{subject to} \qquad x \in \mathbb{R}^n,\ g(x) \geq 0,$$

where f and g are twice differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively, and B is an $n \times n$ positive semi-definite (symmetric) matrix. Later, many authors gave first-order duality theorems for nondifferentiable programming problem (P_1) using first order optimality conditions.

Second order dual models have also received extensive attentions for (P_1). Mangasarian [2] introduced a second order dual and obtained the duality result under a so-called "inclusion condition". Mond [3] proved duality theorems under the condition which is called "second-order convexity". This condition is much simpler than that used by Mangasarian [2]. Furthermore, Mond and Weir [4] reformulated a new type of second order duals. Later, second order dualities in nonlinear programming were considered by Husain and Rueda and

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Jabeen [1], Yang et. al [8, 9] and Zhang and Mond [11], while Zhang and Mond [11] formulated general first order and second-order dual models for nondifferentiable programming problem (P_1) and established weak, strong and converse duality theorems under certain conditions. On the other hand, second-order dual models for a convex composite optimization problem have also been studied in Yang [10] under a generalized representation condition.

We note that Mond and Schechter [6] studied nondifferentiable symmetric duality, in which the objective function contains a support function. In this paper, based on Mond and Schechter's ideas in [6] and Zhang and Mond's works in [11], we replace the term $(x^TBx)^{\frac{1}{2}}$ in the objective function of (P_1) by a somewhat more general function, namely, the support function of a compact convex set, for which the subdifferential may be simply expressed. That is, we will consider the following nondifferentiable programming problem:

$$\begin{array}{lll} \text{(P)} & \text{minimize} & f(x) + s(x|C) \\ & \text{subject to} & x \in \mathbb{R}^n, g(x) \geq 0, \end{array} \tag{1.1}$$

where f and g are twice differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively, and C is a compact convex set of $\mathbb{R}^n$, the support function $s(x|C)$ of C is defined by

$$s(x|C) := max\{x^Ty, y \in C\}.$$

The support function $s(x|C)$, being convex and everywhere finite, has a subdifferential at every x in the sense of Rockafellar, that is, there exists z such that

$$s(y|C) \geq s(x|C) + z^T(y - x) \quad \text{for all} \quad y \in C.$$

Equivalently,

$$z^Tx = s(x|C).$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) := \{z \in C : z^Tx = s(x|C)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y | y^T(z - x) \leq 0, \quad \text{for all} \quad z \in S\}.$$

It is readily verified that for a compact convex set C, y is in $N_C(x)$ if and only if $s(y|C) = x^Ty$, or equivalently, x is in the subdifferential of s at y.

In this paper, we will construct first order and second order dual models and establish weak and converse duality theorems under suitable generalized convexity conditions.

2 First Order Duality

In this section, we introduce the following first order dual (GD_1) to (P).

$$\begin{array}{lll} (\mathbf{GD_1}) & \text{maximize} & f(u) - \sum_{i \in I_0} y_i g_i(u) + u^Tw, \\ & \text{subject to} & \nabla f(u) + w - \nabla(y^Tg(u)) = 0, \qquad (2.1) \\ & & \sum_{i \in I_\alpha} y_i g_i(u) \leq 0, \alpha = 1, 2, \cdots, r, \qquad (2.2) \\ & & w \in C, \qquad (2.3) \\ & & y \geq 0, \qquad (2.4) \end{array}$$

where $u, w \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $I_\alpha \subset M = \{1, 2, \cdots, m\}$, $\alpha = 0, 1, 2, \cdots, r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \emptyset$ if $\alpha \neq \beta$. We will obtain some weak and converse duality results for (P) under generalized $(F, \rho)-$convexity assumptions.

We begin by recalling the following definitions of the generalized (F, ρ)-convexity due to Preda [7].

Definition 2.1. A functional $F : D \times D \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be sublinear if for any $x, u \in D$,

$$F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2), \forall a_1, a_2 \in \mathbb{R}^n \quad \text{and}$$
$$F(x, u; \alpha a) = \alpha F(x, u; a), \forall \alpha \geq 0, \quad \text{and} \quad a \in \mathbb{R}^n.$$

Let F be a sublinear functional, the function $\phi : D \longrightarrow \mathbb{R}$ be differentiable at $u \in D$, $\rho \in \mathbb{R}$, and $d(\cdot, \cdot) : D \times D \longrightarrow \mathbb{R}$.

Definition 2.2. The function ϕ is said to be (F, ρ)-quasiconvex at u, if

$$\phi(x) \leq \phi(u) \Longrightarrow F(x, u; \nabla\phi(u)) \leq -\rho d^2(x, u), \quad \forall x \in D.$$

Definition 2.3. The function ϕ is said to be (F, ρ)-pseudoconvex at u, if

$$F(x, u; \nabla\phi(u)) \geq -\rho d^2(x, u) \Longrightarrow \phi(x) \geq \phi(u), \quad \forall x \in D.$$

Theorem 2.4 (Weak duality). *Let x be feasible for (P) and (u, y, w) be feasible for (GD_1). If for any feasible (x, u, y, w), $f(\cdot) - \sum_{i \in I_0} y_i g_i(\cdot) + (\cdot)^T w$ is (F, ρ_0)-pseudoconvex and $-\sum_{i \in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \cdots, r$ is (F, ρ_α)-quasiconvex, and $\sum_{\alpha=1}^{r} \rho_\alpha + \rho_0 \geq 0$, then*

$$f(x) + s(x|C) \geq f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T w.$$

Proof. As x is feasible for (P) and (u, y, w) is feasible for (GD_1), we have

$$\sum_{i \in I_\alpha} y_i g_i(x) \geq 0 \geq \sum_{i \in I_\alpha} y_i g_i(u), \qquad \alpha = 1, 2, \cdots, r.$$

By the (F, ρ_α)-quasiconvexity of $-\sum_{i \in I_\alpha} y_i g_i(\cdot)$, $\forall \alpha = 1, 2, \cdots, r$, it follows that

$$F(x, u; -\nabla \sum_{i \in I_\alpha} y_i g_i(u)) \leq -\rho_\alpha d^2(x, u), \qquad \alpha = 1, 2, \cdots, r. \tag{2.5}$$

On the other hand, by the sublinearity of F and (2), we have

$$F(x, u; \nabla f(u) - \sum_{i \in I_0} \nabla y_i g_i(u) + w) + \sum_{\alpha=1}^{r} F(x, u; -\nabla \sum_{i \in I_\alpha} y_i g_i(u))$$
$$\geq F(x, u; \nabla f(u) + w - \nabla y^T g(u)) = 0. \tag{2.6}$$

Combining (2.5) and (2.6), as well as $\sum_{\alpha=1}^{r} \rho_\alpha + \rho_0 \geq 0$, we get

$$F(x, u; \nabla f(u) - \sum_{i \in I_0} \nabla y_i g_i(u) + w) \geq \sum_{\alpha=1}^{r} \rho_\alpha d^2(x, u) \geq -\rho_0 d^2(x, u).$$

The (F, ρ_0)-pseudoconvexity of $f(\cdot) - \sum_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T w$ then yields

$$f(x) - \sum_{i\in I_0} y_i g_i(x) + x^T w \geq f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T w.$$

From $y \geq 0$, $g(x) \geq 0$ and $x^T w \leq s(x|C)$, it follows that

$$f(x) + s(x|C) \geq f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T w.$$

□

Theorem 2.5 (Converse duality). *Let* (x^*, y^*, w^*) *be an optimal solution of* (GD_1) *such that*

(A1) *the matrix* $\nabla^2 f(x^*) - \nabla^2 {y^*}^T g(x^*)$ *is positive or negative definite;*

(A2) *the vectors* $\{\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*),\ \alpha = 1, 2, \cdots, r\}$ *are linearly independent.*

If, for all feasible (x, u, y, w), $f(\cdot) - \sum_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T w$ is (F, ρ_0)-pseudoconvex and $-\sum_{i\in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \cdots, r$ is (F, ρ_α)-quasiconvex, and $\sum_{\alpha=1}^r \rho_\alpha + \rho_0 \geq 0$, then x^* is an optimal solution to (P).

Proof. Since (x^*, y^*, w^*) is an optimal solution of (GD_1), by the generalized Fritz John necessary conditions [6], there exist $\tau_0 \in \mathbb{R}$, $v \in \mathbb{R}^n$, $\tau_\alpha \in \mathbb{R}, \alpha = 1, 2, \cdots, r$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^m$, such that

$$\tau_0\{-\nabla f(x^*) + \sum_{i\in I_0} \nabla {y^*}_i g_i(x^*) - w\} + \{\nabla^2 f(x^*) - \nabla^2 {y^*}^T g(x^*)\}^T v + \sum_{\alpha=1}^r \tau_\alpha\{\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)\} = 0, \quad (2.7)$$

$$\tau_0 g_i(x^*) - v^T g_i(x^*) - \gamma_i = 0, i \in I_0, \quad (2.8)$$

$$\tau_\alpha g_i(x^*) - v^T \nabla g_i(x^*) - \gamma_i = 0, i \in I_\alpha, \alpha = 1, 2, \cdots, r, \quad (2.9)$$

$$\tau_0 x^* - v = \beta \in N_C(w^*), \quad (2.10)$$

$$\tau_\alpha \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) = 0, \alpha = 1, 2, \cdots, r, \quad (2.11)$$

$$\gamma^T y^* = 0, \quad (2.12)$$

$$(\tau_0, \tau_1, \tau_2, \cdots, \tau_r, \beta, \gamma) \geq 0, \quad (2.13)$$

$$(\tau_0, \tau_1, \tau_2, \cdots, \tau_r, \beta, \gamma, v) \neq 0. \quad (2.14)$$

Right multiplying (2.9) by ${y^*}_i, i \in I_\alpha, \alpha = 1, 2, \cdots, r$ and using (2.11), we have

$$\tau_\alpha {y^*}_i g_i(x^*) - v^T \nabla {y^*}_i g(x^*) = 0, i \in I_\alpha, \alpha = 1, 2, \cdots, r,$$

thus

$$\tau_\alpha \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) - v^T \sum_{i\in I_\alpha} \nabla {y^*}_i g(x^*) = 0, \alpha = 1, 2, \cdots, r.$$

From (2.11), it follows that

$$v^T \sum_{i\in I_\alpha} \nabla {y^*}_i g(x^*) = 0, \alpha = 1, 2, \cdots, r. \tag{2.15}$$

Using (2.1) in (2.7), we have

$$\sum_{\alpha=1}^{r} (\tau_\alpha - \tau_0) \nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) + [\nabla^2 f(x^*) - \nabla^2 y^T g(x^*)]^T v = 0. \tag{2.16}$$

Left multiplying (2.16) by v and using (2.15), we have

$$v^T [\nabla^2 f(x^*) - \nabla^2 {y^*}^T g(x^*)] v = 0. \tag{2.17}$$

By the assumption that $\nabla^2 f(x^*) - \nabla^2 {y^*}^T g(x^*)$ is positive or negative definite at (x^*, y^*, w^*), it follows that

$$v = 0.$$

Then (2.16) gives

$$\sum_{\alpha=1}^{r} (\tau_\alpha - \tau_0) \nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) = 0. \tag{2.18}$$

Since the vectors $\{\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*), \alpha = 1, 2, \cdots, r\}$ are linearly independent, (2.18) yields

$$\tau_\alpha = \tau_0, \quad \alpha = 1, 2, \cdots, r. \tag{2.19}$$

If $\tau_0 = 0$, then $\tau_\alpha = 0, \alpha = 1, 2, \cdots, r$ from (2.19), $\gamma = 0$ from (2.8), (2.9) and $v = 0$, and $\beta = 0$ from (2.10), i.e., $(\tau_0, \tau_1, \tau_2, \cdots, \tau_r, \beta, \gamma, v) = 0$, contradicts (2.14). So, $\tau_0 > 0$. This gives $\tau_\alpha > 0, \alpha = 1, 2, \cdots, r$. It follows from (2.8) and (2.9) that

$$\tau_0 g_i(x^*) - \gamma_i = 0, i \in I_0, \tag{2.20}$$

$$\tau_\alpha g_i(x^*) - \gamma_i = 0, i \in I_\alpha, \alpha = 1, 2, \cdots, r. \tag{2.21}$$

Therefore $g(x^*) \geq 0$ since $\gamma \geq 0$ and $\tau_\alpha > 0, \alpha = 0, 1, 2, \cdots, r$. Thus, x^* is feasible for (P), and the objective functions of (P) and (GD_1) are equal.

Multiplying (2.20) by ${y^*}_i, i \in I_0$ and using (2.12), we have

$$\tau_0 {y^*}_i g_i(x^*) = 0, i \in I_0.$$

By $\tau_0 > 0$, it follows that

$$ {y^*}_i g_i(x^*) = 0, i \in I_0. \tag{2.22}$$

Also, $v = 0$, $\tau_0 > 0$ and (9) give

$$x^* \in N_C(w^*).$$

Hence

$$s(x^*|C) = x^{*T} w^*. \tag{2.23}$$

Therefore, from (2.22) and (2.23), we have

$$f(x^*) + s(x^*|C) = f(x^*) - \sum_{i \in I_0} y^*{}_i g_i(x^*) + u^{*T} w^*.$$

Thus, if for any feasible (x, u, y, w), $f(\cdot) - \sum_{i \in I_0} y_i g_i(\cdot) + (\cdot)^T w$ is (F, ρ_0)-pseudoconvex and $-\sum_{i \in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \cdots, r$ is (F, ρ_α)-quasiconvex, and $\sum_{\alpha=1}^r \rho_\alpha + \rho_0 \geq 0$, by Theorem 2.4, then x^* is an optimal solution to (P). □

3 Second Order Duality

In this section, following Mond and Weir [4], we propose a second-order dual model for nondifferentiable programming problem (P).

$$(\mathbf{GD_2}) \text{ maximize } \quad f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T w - \frac{1}{2} p^T [\nabla^2 f(u) - \nabla^2 \sum_{i \in I_0} y_i g_i(u)] p,$$

$$\text{subject to } \quad \nabla f(u) + w - \nabla(y^T g(u)) + \nabla^2 f(u) p - \nabla^2 y^T g(u) p = 0, \tag{3.1}$$

$$\sum_{i \in I_\alpha} y_i g_i(u) - \frac{1}{2} p^T \nabla^2 \sum_{i \in I_\alpha} y_i g_i(u) p \leq 0, \alpha = 1, 2, \cdots, r, \tag{3.2}$$

$$w \in C, \tag{3.3}$$

$$y \geq 0, \tag{3.4}$$

where $u, w, p \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $I_\alpha \subset M = \{1, 2, \cdots, m\}$, $\alpha = 0, 1, 2, \cdots, r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \emptyset$ if $\alpha \neq \beta$. This model is a generalization of the one in Zhang and Mond [11].

Before giving weak and converse duality theorems, we introduce the following second order (F, ρ)-convex definitions.

Let F be a sublinear functional, the function $\phi : D \longrightarrow \mathbb{R}$ be twice differentiable at $u \in D$, $\rho \in \mathbb{R}$, and $d(\cdot, \cdot) : D \times D \longrightarrow \mathbb{R}$ be a distance function.

Definition 3.1. The function ϕ is said to be second order (F, ρ)-quasiconvex at u, if for all $p \in \mathbb{R}^n$,

$$\phi(x) \leq \phi(u) - \frac{1}{2} p^T \nabla^2 \phi(u) p \Longrightarrow F(x, u; \nabla \phi(u) + \nabla^2 \phi(u)) \leq -\rho d^2(x, u), \quad \forall x \in D.$$

Definition 3.2. The function ϕ is said to be second order (F, ρ)-pseudoconvex at u, if for all $p \in \mathbb{R}^n$,

$$F(x, u; \nabla \phi(u) + \nabla^2 \phi(u)) \geq -\rho d^2(x, u) \Longrightarrow \phi(x) \geq \phi(u) - \frac{1}{2} p^T \nabla^2 \phi(u) p, \quad \forall x \in D.$$

Theorem 3.3 (Weak duality). *Let x be feasible for (P) and (u, y, w, p) be feasible for (GD_2). If, for any feasible (x, u, y, w, p), $f(\cdot) - \sum_{i \in I_0} y_i g_i(\cdot) + (\cdot)^T w$ is second order (F, ρ_0)-pseudoconvex and $-\sum_{i \in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \cdots, r$ is second order (F, ρ_α)-quasiconvex, and $\sum_{\alpha=1}^r \rho_\alpha + \rho_0 \geq 0$, then*

$$f(x) + s(x|C) \geq f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T w T w - \frac{1}{2} p^T [\nabla^2 f(u) - \nabla^2 \sum_{i \in I_0} y i g_i(u)] p.$$

Proof. As x is feasible for (P) and (u, y, w) is feasible for (GD_1), we have

$$\sum_{i\in I_\alpha} y_i g_i(x) \geq 0 \geq \sum_{i\in I_\alpha} y_i g_i(u) - \frac{1}{2}p^T\nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)p, \qquad \alpha = 1, 2, \cdots, r.$$

By the second order (F, ρ_α)-quasiconvexity of $-\sum_{i\in I_\alpha} y_i g_i(\cdot)$, $\forall\ \ \alpha = 1, 2, \cdots, r$, it follows that

$$F(x, u; -\nabla \sum_{i\in I_\alpha} y_i g_i(u) - \nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)) \leq -\rho_\alpha d^2(x, u), \qquad \alpha = 1, 2, \cdots, r. \tag{3.5}$$

On the other hand, by (3.1) and the sublinearity of F, we have

$$\begin{aligned}F(x, u; &\nabla f(u) + \nabla^2 f(u)p + w - \sum_{i\in I_0} \nabla y_i g_i(u) - \sum_{i\in I_0} \nabla^2 y_i g_i(u)p) \\ &+ \sum_{\alpha=1}^{r} F(x, u; -\nabla \sum_{i\in I_\alpha} y_i g_i(u) - \nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)p) \\ &\geq F(x, u; \nabla f(u) + \nabla^2 f(u)p + w - \nabla y^T g(u) - \nabla y^T g(u)p) = 0. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), as well as $\sum_{\alpha=1}^{r} \rho_\alpha + \rho_0 \geq 0$, we get

$$\begin{aligned}F(x, u; \nabla f(u) + \nabla^2 f(u)p + w - \sum_{i\in I_0} \nabla y_i g_i(u) - \sum_{i\in I_0} \nabla^2 y_i g_i(u)p) \\ \geq \sum_{\alpha=1}^{r} \rho_\alpha d^2(x, u) \geq -\rho_0 d^2(x, u).\end{aligned}$$

The second order (F, ρ_0)-pseudoconvexity of $f(\cdot) - \sum_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T w$ then yields

$$f(x) - \sum_{i\in I_0} y_i g_i(x) + x^T w \geq f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T w - \frac{1}{2}p^T\nabla^2[f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T w]p.$$

From $y \geq 0$,$g(x) \geq 0$ and $x^T w \leq s(x|C)$, it follows that

$$f(x) + s(x|C) \geq f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T wTw - \frac{1}{2}p^T[\nabla^2 f(u) - \nabla^2 \sum_{i\in I_0} y_i g_i(u)]p.$$

□

Theorem 3.4 (Converse duality). *Let (x^*, y^*, w^*, p^*) be an optimal solution of (GD_2) such that*

(B1) *for all $\alpha = 1, 2, \cdots, r$, either (a) the $n \times n$ Hessian matrix $\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)$ is positive definite and ${p^*}^T\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) \geq 0$ or (b) the $n \times n$ Hessian matrix $\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)$ is negative definite and ${p^*}^T\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) \leq 0$,*

(B2) *the vectors $\{[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} {y^*}_i g_i(x^*)]_j, [\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)]_j, \alpha = 1, 2, \cdots, r, j = 1, 2, \cdots, n\}$ are linearly independent, where $[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} {y^*}_i g_i(x^*)]_j$ is the j^{th} row of $[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} {y^*}_i g_i(x^*)]$ and $[\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)]_j$ is the j^{th} row of $[\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)]$,* (B3) *the vectors* $\{\nabla \sum_{i\in I_\alpha} y^*{}_i g_i(x^*),\ \alpha = 1,2,\cdots,r\}$ *are linearly independent.*

If, for all feasible (x,u,y,w,p), $f(\cdot) - \sum_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T w$ is second order (F,ρ_0)-pseudoconvex and $-\sum_{i\in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1,2,\cdots,r$ is second order (F,ρ_α)-quasiconvex, and $\sum_{\alpha=1}^r \rho_\alpha + \rho_0 \geq 0$, then x^* is an optimal solution to (P).

Proof. Since (x^*,y^*,w^*,p^*) is an optimal solution of (GD_2), by the generalized Fritz John necessary conditions [6], there exist $\tau_0 \in \mathbb{R}$, $v \in \mathbb{R}^n$, $\tau_\alpha \in \mathbb{R}, \alpha = 1,2,\cdots,r$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^m$, such that

$$\begin{aligned}\tau_0\{-\nabla f(x^*) + \sum_{i\in I_0} \nabla y^*{}_i g_i(x^*) - w^* + \frac{1}{2}p^{*T}\nabla[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} y^*{}_i g_i(x^*)]p^*\}\\ + v^T\{\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*) + \nabla[\nabla^2 f(x^*)p^* - \nabla^2 y^{*T} g(x^*)p^*]\}r\\ + \sum_{\alpha=1}^r \tau_\alpha\{\nabla \sum_{i\in I_\alpha} y^*{}_i g_i(x^*) - \frac{1}{2}p^{*T}\nabla[\nabla^2 \sum_{i\in I_\alpha} y^*{}_i g_i(x^*)p^*]\} = 0, \quad (3.7)\end{aligned}$$

$$\tau_0\{g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_i(x^*)p^*\} - v^T\{g_i(x^*) + \nabla^2 g_i(x^*)p^*\} - \gamma_i = 0,\ i \in I_0, \quad (3.8)$$

$$\begin{aligned}\tau_\alpha\{g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_i(x^*)p^*\} - v^T\{\nabla g_i(x^*) + \nabla^2 g_i(x^*)p^*\} - \gamma_i = 0,\\ i \in I_\alpha, \alpha = 1,2,\cdots,r, \quad (3.9)\end{aligned}$$

$$\tau_0 x^* - v = \beta \in N_C(w^*), \quad (3.10)$$

$$(\tau_0 p^* + v)^T\{\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} y^*{}_i g_i(x^*)\} - \sum_{\alpha=1}^r (\tau_\alpha p^* + v)^T\{\nabla^2 \sum_{i\in I_\alpha} y^*{}_i g_i(x^*)\} = 0, \quad (3.11)$$

$$\tau_\alpha\{\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 \sum_{i\in I_\alpha} y^*{}_i g_i(x^*)p^*\} = 0, \alpha = 1,2,\cdots,r, \quad (3.12)$$

$$\gamma^T y^* = 0, \quad (3.13)$$

$$(\tau_0,\tau_1,\tau_2,\cdots,\tau_r,\beta,\gamma) \geq 0, \quad (3.14)$$

$$(\tau_0,\tau_1,\tau_2,\cdots,\tau_r,\beta,\gamma,v) \neq 0. \quad (3.15)$$

Because of Assumption (B2), (3.11) gives

$$\tau_\alpha p^* + v = 0 \qquad \alpha = 0,1,2,\cdots,r. \quad (3.16)$$

Multiplying (3.9) by $y^*{}_i, i \in I_\alpha, \alpha = 1,2,\cdots,r$ and using (3.12), we have

$$\begin{aligned}\tau_\alpha\{y^*{}_i g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 y^*{}_i g_i(x^*)p^*\} - v^T\{\nabla y^*{}_i g(x^*) + \nabla^2 y^*{}_i g(x^*)p^*\} = 0,\\ i \in I_\alpha, \alpha = 1,2,\cdots,r,\end{aligned}$$

thus

$$\tau_\alpha\{\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) - \frac{1}{2}p^{*T}\sum_{i\in I_\alpha}\nabla^2 y^*{}_i g_i(x^*)p^*\} \\ - v^T\{\sum_{i\in I_\alpha}\nabla y^*{}_i g(x^*) + \sum_{i\in I_\alpha}\nabla^2 y^*{}_i g(x^*)p^*\} = 0, \quad \alpha = 1, 2, \cdots, r.$$

From (3.12), it follows that

$$v^T\{\sum_{i\in I_\alpha}\nabla y^*{}_i g(x^*) + \sum_{i\in I_\alpha}\nabla^2 y^*{}_i g(x^*)p^*\} = 0, \alpha = 1, 2, \cdots, r. \tag{3.17}$$

Using (3.1), we may deduce from (3.7) have

$$\begin{aligned}
&(\tau_\alpha p^* + v)^T\{\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*) + \nabla[\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)]p^*\} \\
&-\sum_{\alpha=1}^{r}(\tau_\alpha p^* + v)^T\{\nabla^2[\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla[\nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)]p^*\} \\
&-\tau_0\{\nabla\sum_{i\in M\setminus I_0} y^*{}_i g_i(x^*) + \nabla^2\sum_{i\in M\setminus I_0} y^*{}_i g_i(x^*)p^*\} \\
&-\frac{1}{2}\tau_0 p^{*T}\{\nabla[\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)]p^*\} \\
&+\sum_{\alpha=1}^{r}\tau_\alpha\{\nabla\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla^2[\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)]p^*\} \\
&+\sum_{\alpha=1}^{r}\frac{1}{2}\tau_\alpha p^{*T}\{\nabla[\nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)]p^*\} = 0.
\end{aligned}$$

From (3.16), it follows that

$$\sum_{\alpha=1}^{r}(\tau_\alpha - \tau_0)\{\nabla\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)p^*\} \\ + \frac{1}{2}v^T\{\nabla[\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)]p^* - \nabla[\nabla^2\sum_{i\in M\setminus I_0} y^*{}_i g_i(x^*)]p^*)\} = 0.$$

That is

$$\sum_{\alpha=1}^{r}(\tau_\alpha - \tau_0)\{\nabla\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)p^*\} \\ + \frac{1}{2}v^T\{\nabla[\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*)]p^*\} = 0. \tag{3.18}$$

If for all $\alpha = 0, 1, 2, \cdots, r$, $\tau_\alpha = 0$, then $v = 0$ from (3.16), $\gamma = 0$ from (3.8) and (3.9), and $\beta = 0$ from (3.10); that is, $(\tau_0, \tau_1, \tau_2, \cdots, \tau_r, \beta, \gamma, v) = 0$, contradicts (3.15). Thus, there exists an $\overline{\alpha} \in \{0, 1, 2, \cdots, r\}$, such that $\tau_{\overline{\alpha}} > 0$.

We claim that $p^* = 0$. Indeed, if $p^* \neq 0$, then (3.16) gives

$$(\tau_\alpha - \tau_{\overline{\alpha}})p^* = 0, \alpha = 1, 2, \cdots, r.$$

This implies $\tau_\alpha = \tau_{\overline{\alpha}} > 0, \alpha = 1, 2, \cdots, r$. So, (3.17) and (3.16) yield

$$p^{*T}\{\sum_{i\in I_\alpha} \nabla {y^*}_i g(x^*) + \sum_{i\in I_\alpha} \nabla^2 {y^*}_i g(x^*)p^*\} = 0, \alpha = 1, 2, \cdots, r,$$

which contradicts assumption (B1). Hence, $p^* = 0$. Based on (3.17) and $p^* = 0$, we have $v = 0$. In view of (B3), (3.16), $p^* = 0$ and $\tau_{\overline{\alpha}} > 0$ for some $\overline{\alpha} \in \{0, 1, 2, \cdots, r\}$, (3.18) implies $\tau_\alpha = \tau_{\overline{\alpha}} > 0$, for all $\alpha = 0, 1, 2, \cdots, r$. Now from (3.8) and (3.9), it follows that

$$\tau_0 \nabla g_i(x^*) - \gamma_i = 0, i \in I_0, \tag{3.19}$$

$$\tau_\alpha \nabla g_i(x^*) - \gamma_i = 0, i \in I_\alpha, \alpha = 1, 2, \cdots, r, \tag{3.20}$$

Therefore $g(x^*) \geq 0$ since $\gamma \geq 0$ and $\tau_\alpha > 0, \alpha = 0, 1, 2, \cdots, r$. Thus, x^* is feasible for (P), and the objective functions of (P) and (GD_2) are equal.

Multiplying (3.19) by ${y^*}_i, i \in I_0$ and using (3.13), it follows that

$$\tau_0 {y^*}_i g_i(x^*) = 0, i \in I_0.$$

By $\tau_0 > 0$, it follows that

$${y^*}_i g_i(x^*) = 0, i \in I_0. \tag{3.21}$$

Also, $v = 0$, $\tau_0 > 0$ and (3.10) give

$$x^* \in N_C(w^*).$$

Hence

$$s(x^*|C) = {x^*}^T w^*. \tag{3.22}$$

Therefore, from (3.21), (3.22) and $p^* = 0$, we have

$$f(x^*) + s(x^*|C) = f(x^*) - \sum_{i\in I_0} {y^*}_i g_i(x^*) + {u^*}^T w^* - \frac{1}{2}{p^*}^T[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} {y^*}_i g_i(x^*)]p^*.$$

If, for all feasible (x, u, y, w, p), $f(\cdot) - \sum_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T w$ is second order (F, ρ_0)-pseudoconvex and $-\sum_{i\in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \cdots, r$ is second order (F, ρ_α)-quasiconvex, and $\sum_{\alpha=1}^r \rho_\alpha + \rho_0 \geq 0$, by Theorem 3.3, then x^* is an optimal solution to (P). □

4 Special Cases and Some Remarks

Let us consider $C = \{Bw : w^T Bw \leq 1\}$. It is easily shown that $(x^T Bx)^{1/2} = s(x|C)$ and that the set C is compact and convex. Then the primal problem (P) and the dual problems

(GD_1) and (GD_2) in this paper become the primal problem (P_1) and dual problems $(\mathrm{GP})_1$ and $(2\mathrm{GP})_1$ by Zhang and Mond [11], respectively, where

$$
\begin{aligned}
(GD)_1 \; : \; \text{Maximize} \quad & f(u) + u^T Bw - \sum_{i \in I_0} y_i g_i(u) \\
\text{subject to} \quad & \nabla f(u) + Bw - y^T \nabla g(u) = 0, \\
& \sum_{i \in I_\alpha} y_i g_i(u) \le 0, \alpha = 1, 2, \cdots, r, \\
& y \ge 0, \\
& w^T Bw \le 1.
\end{aligned}
$$

and

$$
\begin{aligned}
(2GD)_2 \; : \; \text{Maximize} \quad & f(u) + u^T Bw - \sum_{i \in I_0} y_i g_i(u) - \frac{1}{2} p^T [\nabla^2 f(u) - \nabla^2 \sum_{i \in I_0} y_i g_i(u)] p \\
\text{subject to} \quad & \nabla f(u) + Bw - y^T \nabla g(u) + \nabla^2 f(u) p - \nabla^2 y^T g(u) p = 0, \\
& \sum_{i \in I_\alpha} y_i g_i(u) - \frac{1}{2} p^T \nabla^T \sum_{i \in I_\alpha} y_i g_i(u) p \le 0, \alpha = 1, 2, \cdots, r, \\
& y \ge 0, \\
& w^T Bw \le 1.
\end{aligned}
$$

It is obvious that the notations for the generalized first order and second order (F, ρ)-convexity in this paper are generalizations of the notations of first order and second order invexity in Zhang and Mond [11]. So our results in this paper improve and extend the main works in [11].

In [11], Zhang and Mond obtained the following second order converse duality result:

Theorem 4.1 (see Theorem 6 in [11]). *Let (x^*, y^*, w^*, p^*) be an optimal solution of (GD_2) at which*

(C1) *the $n \times n$ Hessian matrix $\nabla[\nabla^2 f(x^*) - \nabla^2 ({y^*}^T g(x^*))] p^*$ is positive or negative definite,*

(C2) *the vectors $\{[\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} {y^*}_i g_i(x^*)]_j$, $[\nabla^2 \sum_{i \in I_\alpha} {y^*}_i g_i(x^*)]_j$, $\alpha = 1, 2, \cdots, r$, $j = 1, 2, \cdots, n\}$ are linearly independent, where $[\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} {y^*}_i g_i(x^*)]_j$ is the j^{th} row of $[\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} {y^*}_i g_i(x^*)]$ and $[\nabla^2 \sum_{i \in I_\alpha} {y^*}_i g_i(x^*)]_j$ is the j^{th} row of $[\nabla^2 \sum_{i \in I_\alpha} y_i g_i(x^*)]$.*

If, for all feasible (x, u, y, w, p), $f(\cdot) - \sum_{i \in I_0} y_i g_i(\cdot) + (\cdot)^T Bw$ is second order pseudoinvex and $\sum_{i \in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \cdots, r$ is second order quasincave with respect to the same η, then x^* is an optimal solution to (P).

We note that the matrix $\nabla[\nabla^2 f(x^*) - \nabla^2 ({y^*}^T g(x^*))] p^*$ is positive or negative definite in the assumption (C1) of Theorem 4.1, and the result of Theorem 4.1 implies $p^* = 0$ (see the proof of Theorem 6 in [11]). It is obvious that the assumption and the result are inconsistent. In our paper, we give an appropriate modification for this deficiency.

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