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STRONG CONVERGENCE OF AN ITERATIVE METHOD FOR HIERARCHICAL FIXED-POINT PROBLEMS

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Abstract: This paper deals with a viscosity-like method for approximating a specific solution of the following fixed-point problem: find $\tilde{x} \in \mathcal{H}$; $\tilde{x} = (proj_{Fix(T)} \circ P)\tilde{x}$, where $\mathcal{H}$ is a Hilbert space, P and T are two nonexpansive mappings on a closed convex subset D and $proj_{Fix(T)}$ denotes the metric projection on the set of fixed-points of T. This amounts to saying that $\tilde{x}$ is the fixed-point of T which satisfies a variational inequality depending on a given criterion P, namely: find $\tilde{x} \in \mathcal{H}$; $0 \in (I-P)\tilde{x} + N_{Fix(T)}\tilde{x}$, where $N_{Fix(T)}$ denotes the normal cone to the set of fixed-points of T. Strong convergence results for the viscosity-like method are proved. It should be noticed that the proposed method can be regarded, for instance, as a generalized version of Halpern's algorithm.

Key words: *viscosity method, nonexpansive mapping, fixed point, variational inequality*

Mathematics Subject Classification: *47H09, 47H10, 65J15*

1 Introduction

In nonlinear analysis a common approach to solving a problem with multiple solutions is to replace it by a family of perturbed problems admitting a unique solution, and to obtain a particular solution as the limit of these perturbed solutions when the perturbation vanishes. In this paper, we introduce a more general approach which consists in finding a particular part of the solution set of a given fixed-point problem, i.e. fixed-points which solve a variational inequality *criterion*. More precisely, the goal of this paper is to present a method for finding hierarchically a fixed-point of a nonexpansive mapping T with respect to a nonexpansive mapping P, namely

$$\text{Find } \tilde{x} \in Fix(T) \quad \text{such that} \quad \langle \tilde{x} - P(\tilde{x}), x - \tilde{x} \rangle \geq 0 \quad \forall x \in Fix(T), \tag{1.1}$$

i.e., $0 \in (I-P)\tilde{x} + N_{Fix(T)}(\tilde{x})$, where $Fix(T) = \{\bar{x} \in D; \bar{x} = T(\bar{x})\}$ is the set of fixed-points of T and D is a closed convex subset of a real Hilbert space $\mathcal{H}$.

It is not hard to check that solving (1.1) is equivalent to the fixed-point problem

$$\text{Find } \tilde{x} \in D \quad \text{such that} \quad \tilde{x} = proj_{Fix(T)} \circ P(\tilde{x}), \tag{1.2}$$

where $proj_{Fix(T)}$ stands for the metric projection on the closed convex set $Fix(T)$.

It is worth mentioning that when the solution set S of (1.1) is a singleton (which is the case for example where Q is a contraction), the problem reduces to the viscosity fixed-point

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solution introduced in [15] and further developed in [18].

Throughout, $\mathcal{H}$ is a real Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the associated scalar product and $\| \cdot \|$ stands for the corresponding norm. To begin with, let us recall the following concepts which are of common use in the context of convex and nonlinear analysis, see for example Brézis [3]. An operator, $A : \mathcal{H} \to 2^{\mathcal{H}}$, is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0 \quad \text{whenever} \quad u \in A(x), v \in A(y).$$

It is said to be maximal monotone if, in addition, the graph, $graphA := \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in A(x)\}$, is not properly contained in the graph of any other monotone operator. It is well-known that for each $x \in \mathcal{H}$ and $\lambda > 0$ there is a unique $z \in \mathcal{H}$ such that $x \in (I + \lambda A)z$. The single-valued operator $J_\lambda^A := (I+\lambda A)^{-1}$ is called the resolvent of A of parameter λ. It is a nonexpansive mapping which is everywhere defined. Let us also recall that a mapping P is nonexpansive if for all $x, y \in \mathcal{H}$, one has $\|P(x)-P(y)\| \leq \|x-y\|$, and finally that an operator sequence A_n is said to be graph convergent to A if $(graph(A_n))$ converges to $graph(A)$ in the Kuratowski-Painlevé's sense, i.e. $\limsup_n graph(A_n) \subset graph(A) \subset \liminf_n graph(A_n)$.

From now on, we assume that

$$S := \left\{ \tilde{x} \in D \quad | \quad \tilde{x} = (proj_{Fix(T)} \circ P)\tilde{x} \right\} \neq \emptyset. \tag{1.3}$$

Indeed in a large number of variational or optimization problems the solution fails to be unique, for example when considering problems arising in plasticity theory, phase transitions and linear mathematical programming. In such a situation it is important, both for theoretical and numerical reasons, to describe methods which allow us to reach some particular solutions.

To this end, given a contraction $C : D \to D$, namely $||Cx - Cy|| \leq \varrho||x - y||$ for all $x, y \in D$, where $\varrho \in [0, 1)$, to approximate a point in S, we propose the following viscosity algorithm:

$$x_{n+1} = \lambda_n C x_n + (1 - \lambda_n) \left(\alpha_n P x_n + (1 - \alpha_n) T x_n \right), \quad \text{for } n \geq 0, \tag{1.4}$$

where $x_0 \in D$, (λ_n) and $(\alpha_n) \subset (0, 1)$.

Our main purpose is to study the asymptotic convergence of the sequence (x_n) generated by scheme (1.4). Under suitable conditions on the parameters, we establish the convergence in norm of (x_n) to the unique fixed-point of the mapping $proj_S \circ C$. It is worth noting that when $\alpha_n \equiv 0$, scheme (1.4) reduces to the well-known viscosity method for finding fixed points of nonexpansive mappings initially proposed in [7, 19, 2] and further studied in a general context in [15, 18]. We would also like to emphasize that when $T = I$, problem (1.1) reduces to the problem of finding fixed-points of the mapping P and algorithm (1.4) is nothing but a regularized version of the Mann iteration method. Moreover, it should be noticed that the same scheme has been investigated in [9] in the case where $Fix(P) \cap Fix(T) \neq \emptyset$ with several control conditions on the parameters (α_n) and (λ_n). It turns out in all cases that the iteration converges strongly to an element in $Fix(P) \cap Fix(T)$, which has become a classical result. Our approach is completely different and our interest is in finding a part of the fixed-point set of T satisfying a variational criterion. So, in our analysis, no assumption is required on the intersection of the fixed-point sets of the maps P and T.

2 Preliminaries

Before going over some preliminary results, we wish to point out the link with some monotone inclusions and convex programming problems. In these contexts the proposed algorithm looks like a generalized viscosity Mann iteration method:

Example 2.1 (Monotone inclusions). By setting $P = I - \gamma\mathcal{F}$, where $\mathcal{F}$ is κ-Lipschitzian and η-strongly monotone with $\gamma \in (0, 2\kappa/\eta]$, (1.1) reduces to

$$\text{find } \tilde{x} \in Fix(T) \quad \text{such that } \langle x - \tilde{x}, \mathcal{F}(\tilde{x})\rangle \geq 0 \;\forall\; x \in Fix(T),$$

a variational inequality studied in Yamada [20].

Example 2.2 (Convex programming). Let φ be a lower semicontinuous convex function, by setting $T = prox_{\lambda\varphi} := argmin\{\varphi(y) + \frac{1}{2\lambda}\|\cdot - y\|^2\}$, and $P = I - \gamma\nabla\psi$, ψ a convex function such that $\nabla\psi$ is κ-strongly monotone and η-Lipschitzian (which is equivalent to the fact that $\nabla\psi$ is η^{-1} cocoercive) with $\gamma \in (0, 2/\eta]$, and thanks to the fact that $Fix(prox_{\lambda\varphi}) = (\partial\varphi)^{-1}(0) = Argmin\varphi$, (1.1) reduces to the hierarchical minimization problem:

$$\min_{x \in Argmin\varphi} \psi(x),$$

a problem considered in Cabot [5].

Example 2.3 (Minimization on a fixed-point set). Let T be a nonexpansive mapping, by setting $P = I - \gamma\nabla\varphi$, φ a convex function; $\nabla\varphi$ is κ-strongly monotone and η-Lipschitzian (thus η^{-1} cocoercive) with $\gamma \in (0, 2/\eta]$, (1.1) reduces to $\min\limits_{x \in Fix(T)} \varphi(x)$, a problem studied in Yamada [20]. On the other hand, when $P = I - \tilde{\gamma}(A - \gamma f)$, A being a linear bounded $\bar{\gamma}$-strongly monotone operator, f a given α-contraction and $\gamma > 0$ with $\tilde{\gamma} \in (0, 1/\|A\| + \bar{\gamma}]$, (1.1) reduces to the problem of minimizing a quadratic function over the set of fixed-points of a nonexpansive mapping studied in Marino and Xu [14], namely

$$\langle (A - \gamma f)\bar{x}, x - \bar{x}\rangle \geq 0, \;\forall x \in Fix(T).$$

In these cases, our approach permits to relax the assumptions on the data.

The following lemma summarizes some properties of graph convergence which will be needed in our analysis, see for example [3, 11].

Lemma 2.4. *i) Let B be a maximal monotone operator, then $(t_n^{-1}B)$ graph converges to $N_{A^{-1}(0)}$ as $t_n \to 0$ provided that $A^{-1}(0) \neq \emptyset$ and (t_nB) graph converges to $N_{\overline{domA}}$ as $t_n \to 0$.*

ii) Let (B_n) be a sequence of maximal monotone operators. If A is a Lipschitz maximal monotone operator, then $A + B_n$ is maximal monotone. Furthermore, if B_n graph converges to B, then B is maximal monotone and $(A + B_n)$ graph converges to $A + B$.

Remark 2.1. It is well-known that since T is a nonexpansive mapping on D, $I - T$ is a Lipschitz continuous maximal monotone operator on D. Moreover, T is demiclosed on D in the sense that, if (x_n) converges weakly to x in D and $(x_n - Tx_n)$ strongly converges to 0, then x is a fixed-point of T.

The following lemma will be needed in the proof of the main theorem.

Lemma 2.5. *Let* $(a_n) \subset (0,1)$, $(b_n) \subset \mathbb{R}$ *and* $(s_n) \subset \mathbb{R}_+$ *such that*

$$s_{n+1} \leq (1-a_n)s_n + b_n, \quad \forall n \geq 0. \tag{2.1}$$

If the following conditions are satisfied:

$$\limsup_{n\to\infty} \frac{b_n}{a_n} \leq 0, \; a_n \to 0 \text{ and } \sum_{n\geq 0} a_n = \infty, \tag{2.2}$$

then $\lim_{n\to\infty} s_n = 0$

Proof. Given any $\epsilon > 0$, by the condition $\limsup_{n\to\infty} \frac{b_n}{a_n} \leq 0$ we know that there exists $p \in \mathbb{N}$ such that $b_n \leq \epsilon a_n$ for $n \geq p$, so that $s_{n+1} \leq (1-a_n)s_n + a_n\epsilon, \quad \forall n \geq p$. Setting $c_{n,k} = \prod_{j=k}^{n}(1-a_j)$ for any integers n and k such that $n \geq k$, by induction we deduce

$$\begin{aligned} s_{n+1} &\leq c_{n,p}s_p + \epsilon \sum_{k=p}^{n-1} a_k c_{n,k+1} + a_n\epsilon, \\ &= c_{n,p}s_p + \epsilon\left(a_n + (c_{n,n-1} - c_{n,p})\right) \\ &= c_{n,p}s_p + \epsilon\left(1 - c_{n,p}\right). \end{aligned}$$

As a consequence, from the additional conditions $a_n \to 0$ and $\sum_{n\geq 0} a_n = \infty$, we obtain the desired result, because $c_{n,p} \to 0$ as $n \to \infty$ □

3 The Main Results

To prove some useful lemmas related to the strong convergence of the method (1.4) to a solution of (1.1), we need the following conditions on the sequences $(\lambda_n), (\alpha_n) \in (0,1)$:

$$(\text{P1}): \quad \lambda_n = o(\alpha_n);$$

$$(\text{P2}): \quad \sum \lambda_n = \infty;$$

$$(\text{P3}): \quad \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2 \lambda_n} \to 0, \quad \frac{\lambda_n - \lambda_{n-1}}{\alpha_n \lambda_n} \to 0.$$

It is easily checked that all these conditions are satisfied, for instance, in the case when $\alpha_n = \dfrac{1}{n^\gamma}$ and $\lambda_n := \dfrac{1}{n^\beta}$ provided that $\gamma \in (0, 1/2)$ and $\beta \in (\gamma, 1-\gamma)$.

Now, let us establish the following key preliminary results.

Lemma 3.1. *Suppose in addition to* $\lambda_n \to 0$ *and* [(P2)-(P3)] *that the sequence* (x_n) *given by scheme (1.4) is bounded, then*

$$\lim_{n\to+\infty} \frac{1}{\alpha_n} ||x_{n+1} - x_n|| = 0. \tag{3.1}$$

Proof. By relation (1.4), we have

$$\begin{aligned} x_{n+1} - x_n &= \lambda_n C x_n - \lambda_{n-1} C x_{n-1} \\ &+ (1-\lambda_n)(\alpha_n P x_n + (1-\alpha_n) T x_n) \\ &- (1-\lambda_{n-1})(\alpha_{n-1} P x_{n-1} + (1-\alpha_{n-1}) T x_{n-1}), \end{aligned}$$

that is

$$\begin{aligned} x_{n+1} - x_n &= \lambda_n (C x_n - C x_{n-1}) \\ &+ (1-\lambda_n)(\alpha_n (P x_n - P x_{n-1}) + (1-\alpha_n)(T x_n - T x_{n-1})) \\ &+ (\lambda_{n-1} - \lambda_n)(-C x_{n-1} + \alpha_{n-1} P x_{n-1} + (1-\alpha_{n-1}) T x_{n-1}) \\ &+ (1-\lambda_n)(\alpha_n - \alpha_{n-1})(P x_{n-1} - T x_n), \end{aligned}$$

so that

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq \lambda_n \varrho ||x_n - x_{n-1}|| + (1-\lambda_n)||x_n - x_{n-1}|| \\ &+ |\lambda_{n-1} - \lambda_n| \times || - C x_{n-1} + \alpha_{n-1} P x_{n-1} + (1-\alpha_{n-1}) T x_{n-1}|| \\ &+ (1-\lambda_n)|\alpha_n - \alpha_{n-1}| \times ||P x_{n-1} - T x_n||. \end{aligned}$$

Since (x_n) is assumed to be bounded so are the sequences (Px_n), (Tx_n) and (Cx_n). Consequently, we deduce that there exists a positive constant M_1 such that

$$||x_{n+1} - x_n|| \leq (1 - (1-\varrho)\lambda_n)||x_n - x_{n-1}|| + M_1(|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}|).$$

It is then immediate that

$$\begin{aligned} \left(\frac{1}{\alpha_n}||x_{n+1} - x_n||\right) &\leq (1-(1-\varrho)\lambda_n)\left(\frac{1}{\alpha_{n-1}}||x_n - x_{n-1}||\right) \\ &+ M_2\left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| + M_2 \frac{1}{\alpha_n}|\alpha_{n-1} - \alpha_n| \\ &+ M_1 \frac{1}{\alpha_n}|\lambda_n - \lambda_{n-1}|, \end{aligned}$$

where M_2 is a positive constant which does not depend on n. In the light of Lemma 2.5, we infer that $\frac{1}{\alpha_n}||x_{n+1} - x_n|| \to 0$ under the conditions (P2) and $\lambda_n \to 0$, provided that

$$\frac{1}{\lambda_n}\left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| \to 0, \quad \frac{1}{\alpha_n \lambda_n}|\alpha_{n-1} - \alpha_n| \to 0 \quad \text{and} \quad \frac{1}{\alpha_n \lambda_n}|\lambda_n - \lambda_{n-1}| \to 0.$$

Clearly, the above conditions are satisfied under assumption (P3), which completes the proof □

Throughout the rest of the paper we will assume that the following qualification condition, which will assure the additivity of the normal cones N_D and $N_{Fix(T)}$, holds true

$$Fix(T) \cap intD \neq \emptyset. \tag{3.2}$$

Lemma 3.2. *Suppose that* $\alpha_n \to 0$ *and assume in addition to the conditions* [(P1)-(P3)] *and (3.2) that the sequence* (x_n) *given by scheme (1.4) is bounded. Then every weak cluster-point of* (x_n) *given by (1.4) belongs to* S*, the solution set of (1.1).*

Proof. Consider a subsequence of (x_n) (again labelled (x_n)) which converges weakly to some element $\bar{x}$ in $\mathcal{H}$. Under conditions (P1), (P2) and $\alpha_n \to 0$, it is easily deduced from (1.4) that $x_{n+1} - Tx_n \to 0$, because (x_n) is bounded by hypothesis and $\lambda_n \to 0$ by (P1). Moreover, by Lemma 3.1, we clearly have $x_{n+1} - x_n \to 0$. Consequently, we obtain $x_n - Tx_n \to 0$, so that $\bar{x} = T\bar{x}$ since T is demiclosed (see Remark 2.1). Again by (1.4), we also have

$$x_{n+1} - x_n = \lambda_n(Cx_n - x_n) + (1-\lambda_n)\left(\alpha_n(Px_n - x_n) + (1-\alpha_n)(Tx_n - x_n)\right), \tag{3.3}$$

that is

$$\frac{1}{(1-\lambda_n)\alpha_n}(x_n - x_{n+1}) = \left((I-P) + \frac{1-\alpha_n}{\alpha_n}(I-T) + \frac{\lambda_n}{(1-\lambda_n)\alpha_n}(I-C)\right)x_n. \tag{3.4}$$

Lemma 2.4 assures that the operator sequence $(\frac{1-\alpha_n}{\alpha_n}(I-T))$ graph converges to $N_{Fix(T)}$ and $(\frac{\lambda_n}{(1-\lambda_n)\alpha_n}(I-C))$ graph converges to N_D which in the light of a result in ([1]) allows us to deduce that the operator $(I-P) + \frac{1-\alpha_n}{\alpha_n}(I-T) + \frac{\lambda_n}{(1-\lambda_n)\alpha_n}(I-C)$ graph converges to $(I-P) + N_D + N_{Fix(T)}$. The latter coincides with $(I-P) + N_{Fix(T)}$ thanks to the qualification condition (3.2).

Now, by passing to the limit in (3.4), as $n \to \infty$ and by taking into account the fact that $\frac{1}{(1-\lambda_n)\alpha_n}||x_{n+1} - x_n|| \to 0$ and that the graph of $(I-P) + N_{Fix(T)}$ is weakly-strongly closed, we finally obtain $0 \in (I-P)\bar{x} + N_{Fix(T)}\bar{x}$, in other words $\bar{x}$ solves problem (1.1). This completes the proof □

As this work was done in the same spirit as that developed by Cabot ([5]) in the context of minimization problems, we are going to use the same type of hypothesis which amounts to assuming that there exist two positive constants θ and κ such that

$$\forall x \in D \quad \|x - Tx\| \geq \kappa \text{dist}(x, Fix(T))^{\theta}, \tag{3.5}$$

where $\text{dist}(x, Fix(T)) := \inf_{q \in Fix(T)} ||q - x||$.

This kind of hypothesis was used in ([16]) by Senter and Dotson so as to obtain a strong convergence result for Mann iterates. Later Maiti and Ghosh ([13]), Tan and Xu ([17]) studied the approximation of fixed-points of a nonexpansive mapping T by Ishikawa iterates under the condition introduced in ([16]) and pointed out that this assumption is weaker than the requirement that the mapping T is demi-compact.

In view of establishing our main convergence result, we will need the following condition on the parameters (α_n) and (λ_n):

$$\text{(P4)}: \quad \alpha_n^{1+1/\theta} = o(\lambda_n).$$

In the special case when $\alpha_n = \dfrac{1}{n^{\gamma}}$ and $\lambda_n := \dfrac{1}{n^{\beta}}$, (P4) is satisfied for $\beta < \gamma(1 + \frac{1}{\theta})$. As a consequence, one can check that conditions [(P1)-(P4)] hold true, for example, for β and γ satisfying

$$\beta \in (0, \gamma(1+1/\theta)) \;\; \text{with } \gamma \in \left(0, \frac{1}{2+1/\theta}\right] \quad \text{or} \quad \beta \in (\gamma, 1-\gamma) \;\; \text{with } \gamma \in \left(\frac{1}{2+1/\theta}, 1/2\right).$$

Now, we are in position to state the main convergence

Theorem 3.3. *Assume that the assumptions* [(P1)-(P4)], *(3.2) and (3.5) hold true and suppose that the sequence given by scheme (1.4) is bounded. Then* (x_n) *converges strongly to the unique fixed point,* $\bar{x}$*, of* $P_S \circ C$*, where* P_S *is the metric projection from* $\mathcal{H}$ *onto* S*.*

Proof. Thanks to (1.4), we can write

$$\begin{aligned} x_{n+1} - \bar{x} &= \lambda_n(Cx_n - \bar{x}) + (1-\lambda_n)\left(\alpha_n(Px_n - \bar{x}) + (1-\alpha_n)(Tx_n - \bar{x})\right) \\ &= \left(\lambda_n(Cx_n - C\bar{x}) + (1-\lambda_n)\left(\alpha_n(Px_n - P\bar{x}) + (1-\alpha_n)(Tx_n - \bar{x})\right)\right) \\ &+ \left(\lambda_n(C\bar{x} - \bar{x}) + \alpha_n(1-\lambda_n)(P\bar{x} - \bar{x})\right). \end{aligned}$$

An elementary computation yields

$$||a+b||^2 - 2\langle b, a+b\rangle = ||a||^2 - ||b||^2 \ \forall a, b \in \mathcal{H}, \tag{3.6}$$

so that

$$\begin{aligned} &||x_{n+1} - \bar{x}||^2 - 2\langle \lambda_n(C\bar{x} - \bar{x}) + \alpha_n(1-\lambda_n)(P\bar{x} - \bar{x}), x_{n+1} - \bar{x}\rangle \\ &\quad \leq ||\lambda_n(Cx_n - C\bar{x}) + (1-\lambda_n)\left(\alpha_n(Px_n - P\bar{x}) + (1-\alpha_n)(Tx_n - \bar{x})\right)||^2. \end{aligned}$$

By convexity of the mapping $x \to ||x||^2$, we get

$$\begin{aligned} &||x_{n+1} - \bar{x}||^2 - 2\langle \lambda_n(C\bar{x} - \bar{x}) + \alpha_n(1-\lambda_n)(P\bar{x} - \bar{x}), x_{n+1} - \bar{x}\rangle \\ &\quad \leq \lambda_n||Cx_n - C\bar{x}||^2 + (1-\lambda_n)||\alpha_n(Px_n - P\bar{x}) + (1-\alpha_n)(Tx_n - \bar{x})||^2 \\ &\quad \leq \lambda_n||Cx_n - C\bar{x}||^2 + (1-\lambda_n)\left(\alpha_n||Px_n - P\bar{x}||^2 + (1-\alpha_n)||Tx_n - \bar{x}||^2\right). \end{aligned}$$

As a straightforward consequence, we obtain

$$\begin{aligned} &||x_{n+1} - \bar{x}||^2 - 2\langle \lambda_n(C\bar{x} - \bar{x}) + \alpha_n(1-\lambda_n)(P\bar{x} - \bar{x}), x_{n+1} - \bar{x}\rangle \\ &\quad \leq \lambda_n\varrho^2||x_n - \bar{x}||^2 + (1-\lambda_n)\left(\alpha_n||x_n - \bar{x}||^2 + (1-\alpha_n)||x_n - \bar{x}||^2\right) \\ &\quad = (1-(1-\varrho^2)\lambda_n)||x_n - \bar{x}||^2, \end{aligned}$$

which yields

$$\begin{aligned} ||x_{n+1} - \bar{x}||^2 &\leq (1-(1-\varrho^2)\lambda_n)||x_n - \bar{x}||^2 \\ &+2\lambda_n\langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &+2\alpha_n(1-\lambda_n)\langle P\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle. \end{aligned} \tag{3.7}$$

On the one hand, observing that $\alpha_n \to 0$ thanks to (P4), by Lemma 3.2 we have that any weak cluster-point of (x_n) is in S. Consequently, since $\bar{x} = P_S(C\bar{x})$, it is easily checked that

$$\limsup_{n\to\infty} \langle C\bar{x} - \bar{x}, x_n - \bar{x})\rangle \leq 0. \tag{3.8}$$

One the other hand, we will estimate the last term in the right hand side of the inequality (3.7). Clearly, we have

$$\langle P\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle = \left\langle P\bar{x} - \bar{x}, P_{Fix(T)}x_{n+1} - \bar{x}\right\rangle + \left\langle P\bar{x} - \bar{x}, x_{n+1} - P_{Fix(T)}x_{n+1}\right\rangle.$$

Since $P_{Fix(T)}x_{n+1} \in Fix(T)$, by (1.1) we have

$$\left\langle P\bar{x} - \bar{x}, P_{Fix(T)}x_{n+1} - \bar{x}\right\rangle \leq 0$$

and therefore

$$\begin{aligned} \langle P\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle &\leq \left\langle P\bar{x} - \bar{x}, x_{n+1} - P_{Fix(T)}x_{n+1}\right\rangle \\ &\leq ||P\bar{x} - \bar{x}|| \times ||x_{n+1} - P_{Fix(T)}x_{n+1}|| \\ &= ||P\bar{x} - \bar{x}|| \times \text{dist}(x_{n+1}, Fix(T)). \end{aligned}$$

Consequently, by (3.5) we obtain

$$\langle P\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \leq \kappa^{-1/\theta}||P\bar{x} - \bar{x}|| \times ||Tx_{n+1} - x_{n+1}||^{1/\theta}. \tag{3.9}$$

Furthermore, as T is nonexpansive, we obviously have

$$||x_{n+1} - Tx_{n+1}|| \leq ||x_{n+1} - Tx_n|| + ||x_{n+1} - x_n||.$$

Then, in view of (1.4), we immediately infer the existence of a positive constant κ_1 such that, for all $n \geq 0$,

$$||x_{n+1} - Tx_{n+1}|| \leq \kappa_1(\alpha_n + \lambda_n + ||x_{n+1} - x_n||).$$

This combined with (3.9) implies that

$$\langle P\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \leq \kappa_2(\alpha_n + \lambda_n + ||x_{n+1} - x_n||)^{1/\theta}, \tag{3.10}$$

for a positive constant κ_2.

Now, in the light of assumption (P4), Lemma 3.1 and taking into account the fact that $\lambda_n/\alpha_n \to 0$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{\alpha_n}{\lambda_n}(\alpha_n + \lambda_n + ||x_{n+1} - x_n||)^{1/\theta} &= \lim_{n\to\infty} \frac{\alpha_n^{1+1/\theta}}{\lambda_n}\left(1 + \frac{\lambda_n}{\alpha_n} + \frac{||x_{n+1} - x_n||}{\alpha_n}\right)^{1/\theta}, \\ &= \lim_{n\to\infty} \frac{\alpha_n^{1+1/\theta}}{\lambda_n} = 0,\end{aligned}$$

which by (3.10) leads to

$$\limsup_{n\to\infty} \frac{\alpha_n}{\lambda_n} \langle P\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \leq 0. \tag{3.11}$$

Finally, by (3.7), (3.8), (3.11) and using Lemma 2.5, we conclude that the sequence (x_n) converges strongly to $\bar{x}$, which completes the proof □

We would like to point out the following interesting remarks.

Remark 3.1. i) Since any weak-cluster point of (x_n) is in $Fix(T)$, we would like to emphasize that it is enough to assume that (3.5) holds true in a neighborhood of $Fix(T)$.

ii) We would also like to note that, thanks to a result by Lemaire ([10]), (3.5) is in the convex minimization setting equivalent to

$$\forall x \in \mathcal{H} \quad \psi(x) - \min \psi \geq \kappa \mathrm{d}ist(x, \mathrm{A}rgmin\psi)^{\theta+1},$$

which is exactly one of the assumptions used in ([5]) to obtain convergence results (proposition 3.4 and proposition 4.3) of a proximal method for hierarchical minimization problems. In ([5]), the convergence results are valid only in finite dimensional case.

iii) Finally, it is worth noticing that the result of Lemma 2.5 is still valid if we replace (2.2) by $a_n \to 0$, $\sum_{n\geq 0} a_n = \infty$ and $\sum_{n\geq 0} b_n < \infty$ (see for instance [12]). Through the proof of Theorem 3.3, one may then observe that if, in addition to conditions (P2) and (P3), we just assume that $\lambda_n \to 0$ and $\sum_{n\geq 0} \alpha_n < \infty$, then the sequence (x_n) converges strongly to the unique fixed-point of the mapping $P_{Fix(T)} \circ C$. It turns out that in this case (i.e. (α_n) *is supposed to converge quickly to zero*) the limit attains by (x_n) is independent of the mapping P.

4 Conclusion

A new and promising algorithm in hierarchical fixed-point problems is presented. The strong convergence of the corresponding sequence is investigated. The limit attained by this sequence is the solution of a variational inequality involving fixed-point sets.

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