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WEAK CONVERGENCE THEOREMS BY CESÀRO MEANS FOR A NONEXPANSIVE MAPPING AND AN EQUILIBRIUM PROBLEM

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Abstract: In this paper, we introduce an iterative scheme by Cesàro means for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Then, we show that the sequence converges weakly to a common element of these two sets. Using this result, we obtain a generalization of the well-known nonlinear ergodic theorem by Baillon. Further, we consider the problem of finding a common point of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone mapping in a Hilbert space.

Key words: *nonexpansive mapping, equilibrium problem, fixed point*

Mathematics Subject Classification: *47H05, 47J05, 47J25*

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1 Introduction

In this paper, we deal with the equilibrium problem, abbreviated EP, as follows: Given a nonempty set K and a function f of $K \times K$ into $\mathbb{R}$ with $f(x,x) = 0$ for all $x \in K$,

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0 \text{ for all } y \in K.$$

Such a point $\bar{x} \in K$ is called a solution of EP. According to Blum and Oettli [3], EP is deeply related to optimization problems, saddle point problems, Nash equilibria in noncooperative games, fixed point problems, variational inequality problems, complementary problems and so on; see [3, 2, 8] for more details. Many researchers have widely studied EP. For instance, Blum and Oettli [3] showed the existence of solutions of EP. Iusem and Sosa [8] discussed the relation between EP and convex feasibility problems. They [9] also studied iterative algorithms for solving EP. Combettes and Hirstoaga [4] proved some convergence theorems for EP by using the resolvents of the function f; see also Moudafi [10].

On the other hand, Takahashi and Toyoda [16] introduced an iteration process to find a common solution of a fixed point problem of a nonexpansive mapping and a variational inequality problem for an inverse-strongly-monotone mapping; see also [6, 7]. Recently, Tada and Takahashi [13] considered the problem of finding a common solution of an equilibrium problem and a fixed point problem and they obtained two convergence theorems.

In this paper, motivated by these results, especially [7, 13], we introduce an iterative method by Cesàro means in order to approximate a common solution of an equilibrium problem and a fixed point problem for a nonexpansive mapping in Hilbert spaces. Then we prove that the iterative sequence converges weakly to some common solution of the equilibrium problem and the fixed point problem. Using this result, we improve the result of Baillon [1], which is the well-known nonlinear ergodic theorem. Further we consider the problem of finding a common point of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem for a monotone mapping.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence of H and $x \in H$. We write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and $\{x_n\}$ converges strongly to x, respectively. The set of all positive integers is denoted by $\mathbb{N}$.

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping T of C into H is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping T of C into H is also said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$; see [5] for more details. The set of fixed points of T is denoted by $\mathrm{F}(T)$. For each $x \in H$, there exists a unique point $z \in C$ such that

$$\|x - z\| = \min\{\|x - y\| : y \in C\}.$$

This nearest point z is denoted by Px and P is called the metric projection of H onto C. We know that P is nonexpansive, and moreover, firmly nonexpansive. Let $x \in H$ and $z \in C$ be given. We also know that $z = Px$ if and only if

$$\langle x - z, z - y\rangle \ge 0 \tag{2.1}$$

for all $y \in C$; see [15] for more details. We know the following [16]:

Lemma 2.1 (Takahashi-Toyoda [16]). *Let C be a nonempty closed convex subset of a real Hilbert space H, let P be the metric projection of H onto C, and let $\{x_n\}$ be a sequence in H. If*

$$\|x_{n+1} - u\| \le \|x_n - u\|$$

for all $u \in C$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a function of $C \times C$ into $\mathbb{R}$ that satisfies the following conditions:

(F1) $f(x, x) = 0$ for all $x \in C$;

(F2) f is monotone, that is, $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;

(F3) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$;

(F4) f is upper hemicontinuous, that is, for any $x, y \in C$, a function τ of $[0, 1]$ into $\mathbb{R}$ defined by $\tau(t) = f((1 - t)x + ty, y)$ for all $t \in [0, 1]$ is upper semicontinuous.

Recall that $x \in C$ is said to be a solution of EP if $f(x,y) \geq 0$ for all $y \in C$. The set of solutions of EP is denoted by $\mathrm{EP}(C,f)$. According to [3, Corollary 1], we may define a mapping J_r of H into C for each $r > 0$ as follows:

$$J_r(x) = \{z \in C : 0 \leq f(z,y) + \frac{1}{r} \langle y - z, z - x\rangle \text{ for all } y \in C\}. \tag{2.2}$$

Such a mapping J_r is said to be the resolvent of f for $r > 0$, which has the following property [3]; see also [4].

Lemma 2.2 ([3], [4, Lemma 2.12]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a function of $C \times C$ into $\mathbb{R}$ that satisfies the conditions from (F1) to (F4) above. Let J_r be the resolvent of f for $r > 0$. Then*

1. $J_r(x) \neq \emptyset$ *for all* $x \in H$*;*
2. J_r *is single-valued and firmly nonexpansive;*
3. $\mathrm{F}(J_r) = \mathrm{EP}(C,f)$*;*
4. $\mathrm{EP}(C,f)$ *is closed and convex.*

We need the following:

Lemma 2.3 ([3]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a function of $C \times C$ into $\mathbb{R}$ that satisfies the conditions from (F1) to (F4) above. Let $x \in C$. If $f(z,x) \leq 0$ for all $z \in C$, then $f(x,y) \geq 0$ for all $y \in C$.*

Proof. Assume $f(z,x) \leq 0$ for all $z \in C$. Let $y \in C$ be given. For $t \in (0,1)$, define $x_t = (1-t)x + ty$. By assumption, it is clear that $x_t \in C$ and $f(x_t, x) \leq 0$ for all $t \in (0,1)$. Then, from (F1) and (F3), we have

$$\begin{aligned} 0 &= f(x_t, x_t) \\ &\leq (1-t) f(x_t, x) + t f(x_t, y) \\ &\leq t f(x_t, y) \end{aligned}$$

and hence $f(x_t, y) \geq 0$ for all $t \in (0,1)$. Since f is upper hemicontinuous, we conclude that

$$0 \leq \limsup_{t \to +0} f(x_t, y) \leq f(x,y).$$

This completes the proof. □

We also need the following:

Lemma 2.4 ([7], [15, p. 59]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence of H, let $\{z_n\}$ be a sequence of H defined by*

$$z_n = \frac{1}{n} \sum_{k=1}^{n} x_k$$

for every $n \in \mathbb{N}$, and let T be a mapping of C into H. Suppose that $z \in C$ and $\|x_n - z\| \geq \|x_{n+1} - Tz\|$ for every $n \in \mathbb{N}$. If there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ with $z_{n_i} \rightharpoonup z$, then $z = Tz$.

Proof. By assumption, we have

$$\begin{aligned} 0 &\le \|x_k - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz + Tz - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz\|^2 - \|x_{k+1} - Tz\|^2 + 2\langle x_k - Tz, Tz - z\rangle + \|Tz - z\|^2 . \end{aligned}$$

Summing these inequalities from $k = 1$ to n and dividing by n, we have

$$\begin{aligned} 0 &\le \frac{1}{n}(\|x_1 - Tz\|^2 - \|x_{n+1} - Tz\|^2) + 2\langle z_n - Tz, Tz - z\rangle + \|Tz - z\|^2 \\ &\le \frac{1}{n}\|x_1 - Tz\|^2 + 2\langle z_n - Tz, Tz - z\rangle + \|Tz - z\|^2 . \end{aligned}$$

Therefore, for each $i \in \mathbb{N}$, we have

$$0 \le \frac{1}{n_i}\|x_1 - Tz\|^2 + 2\langle z_{n_i} - Tz, Tz - z\rangle + \|Tz - z\|^2 .$$

Tending $i \to \infty$, we obtain

$$0 \le 2\langle z - Tz, Tz - z\rangle + \|Tz - z\|^2 = -\|Tz - z\|^2$$

and hence $Tz = z$. □

3 Weak Convergence Theorem

We prove the following theorem which is the main result of this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into H and let f be a function of $C \times C$ into $\mathbb{R}$ that satisfies the conditions from (F1) to (F4). Suppose that $\mathrm{F}(T) \cap \mathrm{EP}(C, f) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be two sequences defined by*

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = TJ_{r_n}x_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence of positive real numbers with $\liminf_{n\to\infty} r_n > 0$ and J_{r_n} is the resolvent of f for $r_n > 0$. Then $\{z_n\}$ converges weakly to some point $z \in \mathrm{F}(T) \cap \mathrm{EP}(C, f)$. Moreover, $z = \lim_{n\to\infty} Px_n$, where P is the metric projection of H onto $\mathrm{F}(T) \cap \mathrm{EP}(C, f)$.

Proof. Put $y_n = J_{r_n}x_n$. First let us prove that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded and $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Let $u \in \mathrm{F}(T) \cap \mathrm{EP}(C, f)$. Since $\mathrm{EP}(C, f) = \mathrm{F}(J_{r_n})$ and J_{r_n} is firmly nonexpansive for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{n+1} - u\|^2 &= \|TJ_{r_n}x_n - TJ_{r_n}u\|^2 \\ &\le \|J_{r_n}x_n - J_{r_n}u\|^2 \\ &\le \|x_n - u\|^2 - \|(I - J_{r_n})x_n - (I - J_{r_n})u\|^2 \end{aligned}$$

$$\leq \|x_n - u\|^2 - \|x_n - J_{r_n} x_n\|^2$$
$$\leq \|x_n - u\|^2 \leq \|x_1 - u\|^2 .$$

This implies that $\lim_{n\to\infty} \|x_n - u\|$ exists and $\{x_n\}$ is bounded and hence both $\{y_n\}$ and $\{z_n\}$ are also bounded. Further, we have

$$\|x_n - J_{r_n} x_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 .$$

So, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - J_{r_n} x_n\| = 0. \tag{3.1}$$

Let $\{w_n\}$ be a sequence in C defined by

$$w_n = \frac{1}{n} \sum_{k=1}^{n} y_k$$

for $n \in \mathbb{N}$. It is clear that

$$\|z_n - w_n\| = \frac{1}{n} \left\| \sum_{k=1}^{n} (x_k - y_k) \right\| \leq \frac{1}{n} \sum_{k=1}^{n} \|x_k - y_k\| .$$

From (3.1), we obtain

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.2}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$. It follows from (3.2) that the subsequence $\{w_{n_i}\}$ of $\{w_n\}$ also converges weakly to z. We show that $z \in \mathrm{EP}(C, f)$. By (2.2) and (F2), we have

$$\begin{aligned} f(y, y_k) &\leq -f(y_k, y) \\ &\leq \frac{1}{r_k} \langle y - y_k, y_k - x_k \rangle \\ &\leq \frac{1}{r_k} \|y - y_k\| \|y_k - x_k\| \\ &\leq L \|y_k - x_k\| \end{aligned}$$

for all $y \in C$ and $k \in \mathbb{N}$, where $L = \sup\{\|y - y_k\| / r_k : k \in \mathbb{N}\}$. Thus, from (F3), we have

$$\begin{aligned} f(y, w_n) &= f(y, \frac{1}{n} \sum_{k=1}^{n} y_k) \\ &\leq \frac{1}{n} \sum_{k=1}^{n} f(y, y_k) \\ &\leq \frac{1}{n} \sum_{k=1}^{n} \|y_k - x_k\| L \end{aligned}$$

for every $n \in \mathbb{N}$. Then it follows from (3.1) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|y_k - x_k\| L = 0.$$

Since $w_{n_i} \rightharpoonup z$ and $f(y,\cdot)$ is weakly lower semicontinuous from (F3), we conclude that

$$f(y,z) \leq \liminf_{i\to\infty} f(y,w_{n_i}) \leq 0$$

for all $y \in C$. By virtue of Lemma 2.3, this implies that $f(z,y) \geq 0$ for all $y \in C$. Consequently, we have $z \in \mathrm{EP}(C,f)$. Next we show $z \in \mathrm{F}(T)$. Since both T and J_{r_n} are nonexpansive and $z \in \mathrm{EP}(C,f)$, we have

$$\|x_{n+1} - Tz\| = \|Ty_n - Tz\| \leq \|y_n - z\| = \|J_{r_n}x_n - J_{r_n}z\| \leq \|x_n - z\|.$$

Thus it follows from Lemma 2.4 that $z \in \mathrm{F}(T)$. On the other hand, since $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in \mathrm{F}(T) \cap \mathrm{EP}(C,f)$, Lemma 2.1 implies that $Px_n \to w \in \mathrm{F}(T) \cap \mathrm{EP}(C,f)$ as $n \to \infty$. So, to complete the proof, it is enough to prove $z = w$. From $z \in \mathrm{F}(T) \cap \mathrm{EP}(C,f)$ and (2.1), it holds that

$$\begin{aligned}\langle z - w, x_k - Px_k\rangle &= \langle z - Px_k, x_k - Px_k\rangle + \langle Px_k - w, x_k - Px_k\rangle \\ &\leq \langle Px_k - w, x_k - Px_k\rangle \\ &\leq \|Px_k - w\| \, \|x_k - Px_k\| \\ &\leq \|Px_k - w\| \, M\end{aligned}$$

for every $k \in \mathbb{N}$, where $M = \sup\{\|x_k - Px_k\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle z - w, z_{n_i} - \frac{1}{n_i}\sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i}\sum_{k=1}^{n_i} \|Px_k - w\| \, M.$$

Since $z_{n_i} \rightharpoonup z$ as $i \to \infty$ and $Px_n \to w$ as $n \to \infty$, we obtain $\langle z - w, z - w\rangle \leq 0$. This means $z = w$. This completes the proof. □

4 Applications

In this section, we prove some weak convergence theorems in a Hilbert space by using Theorem 3.1. We first prove the following theorem that is a generalization of the result by Baillon [1].

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a nonexpansive mapping of C into H with $\mathrm{F}(T) \neq \emptyset$, and let $x \in H$. Let $\{x_n\}$ and $\{z_n\}$ be two sequences defined by*

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = TP_C x_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n \in \mathbb{N}$, where P_C is the metric projection of H onto C. Then $\{z_n\}$ converges weakly to $z = \lim_{n\to\infty} P_{\mathrm{F}(T)}x_n$, where $P_{\mathrm{F}(T)}$ is the metric projection of H onto $\mathrm{F}(T)$.

Proof. Let f be a function of $C \times C$ into $\mathbb{R}$ defined by $f(x,y) = 0$ for $x, y \in C$. Then it is obvious that f satisfies the conditions from (F1) to (F4) in Theorem 3.1, $\mathrm{EP}(C,f) = C$, and the resolvent J_r of f for $r > 0$ is the metric projection P_C of H onto C. Therefore

$\mathrm{F}(T) \cap \mathrm{EP}(C, f) = \mathrm{F}(T) \neq \emptyset$. Hence Theorem 3.1 implies that $\{z_n\}$ converges weakly to $z \in \mathrm{F}(T)$ and $z = \lim_{n\to\infty} P_{\mathrm{F}(T)} x_n$. □

In Theorem 4.1, if T is a mapping of C into itself and $x \in C$, then we see that $x_{n+1} = Tx_n = T^n x$ and the sequence $\{z_n\}$ defined by

$$z_n = \frac{1}{n} \sum_{k=1}^{n} T^{k-1} x$$

for every $n \in \mathbb{N}$ converges weakly to $z \in \mathrm{F}(T)$. So, we obtain the first nonlinear ergodic theorem which was proved by Baillon [1]; see [15, Theorem 3.2.1].

Next we deal with the problem of finding a common solution of a variational inequality problem and a fixed point problem. Such a problem was discussed in [16, 6, 7]. Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a mapping of C into H. The variational inequality problem is formulated as follows:

$$\text{Find } x \in C \text{ such that } \langle y - x, Ax \rangle \geq 0 \text{ for all } y \in C.$$

Such a point $x \in C$ is called a solution of this problem. The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$, that is,

$$\mathrm{VI}(C, A) = \{x \in C : \langle y - x, Ax \rangle \geq 0 \text{ for all } y \in C\}.$$

A mapping A of C into H is said to be monotone if

$$\langle x - y, Ax - Ay \rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into H is also said to be hemicontinuous if for any $x, y, z \in C$, a function τ of $[0, 1]$ into $\mathbb{R}$ defined by $\tau(t) = \langle z, A((1 - t)x + ty) \rangle$ for all $t \in [0, 1]$ is continuous; see, for example, [15]. Let A be a monotone and hemicontinuous mapping of C into H and let f be a function of $C \times C$ into $\mathbb{R}$ defined by

$$f(x, y) = \langle y - x, Ax \rangle$$

for $x, y \in C$. Then, it is obvious that f satisfies (F1) and (F3). From the definition, we have

$$f(x, y) + f(y, x) = \langle y - x, Ax - Ay \rangle = - \langle x - y, Ax - Ay \rangle .$$

The monotonicity of A implies that $f(x, y) + f(y, x) \leq 0$. Further we have

$$f((1 - t)x + ty, y) = (1 - t) \langle y - x, A((1 - t)x + ty) \rangle .$$

Since A is hemicontinuous, f satisfies (F4) in Theorem 3.1. Consequently we may define the resolvent J_r of f for each $r > 0$. In this case we know that $\mathrm{EP}(C, f) = \mathrm{VI}(C, A)$. For convenience, we call J_r the resolvent of A for r. From observations above, we obtain the following:

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone hemicontinuous mapping of C into H and T a nonexpansive mapping of C into H. Suppose that $\mathrm{F}(T) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be two sequences defined by*

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = TJ_{r_n} x_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n \in \mathbb{N}$*, where* $\{r_n\}$ *is a sequence of positive real numbers with* $\liminf_{n\to\infty} r_n > 0$ *and* J_{r_n} *is the resolvent of* A *for* r_n*. Then* $\{z_n\}$ *converges weakly to some point* $z \in \mathrm{F}(T) \cap \mathrm{VI}(C, A)$*. Moreover,* $z = \lim_{n\to\infty} Px_n$*, where* P *is the metric projection of* H *onto* $\mathrm{F}(T) \cap \mathrm{VI}(C, A)$*.*

Let B be a multi-valued mapping of H into H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operators B' on H. By virtue of Rockafellar's theorem [12], for a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $(I + \lambda B)^{-1} \colon H \to D(B)$, which is called the resolvent of B for λ. Let $B^{-1}0 = \{x \in H : Bx \ni 0\}$. It is known that the resolvent $(I + \lambda B)^{-1}$ is nonexpansive and $B^{-1}0 = \mathrm{F}((I + \lambda B)^{-1})$ for all $\lambda > 0$. As a direct consequence of Theorem 4.2, we immediately obtain the following. A similar result was given by Passty [11].

Corollary 4.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* A *be a monotone hemicontinuous mapping of* C *into* H *and* B *a maximal monotone operator on* H*. Suppose that* $B^{-1}0 \cap \mathrm{VI}(C, A) \neq \emptyset$*. Let* $\{x_n\}$ *and* $\{z_n\}$ *be two sequences defined by*

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = (I + \lambda B)^{-1} J_{r_n} x_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n \in \mathbb{N}$*, where* $\lambda > 0$*,* $\{r_n\}$ *is a sequence of positive real numbers with* $\liminf_{n\to\infty} r_n > 0$*, and* J_{r_n} *is the resolvent of* A *for* r_n*. Then* $\{z_n\}$ *converges weakly to some point* $z \in B^{-1}0 \cap \mathrm{VI}(C, A)$*. Moreover,* $z = \lim_{n\to\infty} Px_n$*, where* P *is the metric projection of* H *onto* $B^{-1}0 \cap \mathrm{VI}(C, A)$*.*

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