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STABILITY OF INEQUALITY SYSTEMS INVOLVING MAX-TYPE FUNCTIONS

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Abstract: We study the stability of certain inequality systems that arise in monotonic analysis and are defined by certain classes of *abstract linear functions*. We consider the non-negative orthant $\mathbb{R}^n_+$ as a base space and the class of abstract linear functions consisting of the family of the max-type functions of the form $a(x) := \langle a, x\rangle = \max_{i=1,2,...,n} a_i x_i$, with a and x in $\mathbb{R}^n_+$. The stability, under perturbations of all the coefficients, of the solution set mapping of systems of infinitely many max-type inequalities, $\{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ is studied from different points of view (lower semicontinuity, continuity in the Bouligand sense, metric regularity, the existence of strong Slater points, adapted Robinson-Ursescu condition). Some Farkas and Gale type solvability results are also presented.

Key words: *stability, inequality systems, monotonic analysis, max-type functions, increasing concave-along-rays functions*

Mathematics Subject Classification: *90C34, 90C31,52A01*

1 Introduction

In the preface of his remarkable book "Abstract Convexity and Global Optimization" [16], A. Rubinov claims for the need of new tools in the analysis of the today complex optimization problems. He also states that local approximation, and the subsequent techniques of nonsmooth analysis, are of limited utility when one faces *global optimization* problems. They require particular global tools and, so, generalizations of concepts like the *convex subdifferential*, should be addressed by means of notions as global affine support and its extensions. The idea behind is the so-called *convexity without linearity*, theory known as *abstract convexity* (see, also, [13]). This topic has an impressive number of applications, even in theoretical fields, but its development has been mainly driven by applications in optimization. Very recent papers as [3], [7], [11], [12], [17], [18], [19], [20], etc., show the maturity level reached by the subject as well as the large number of applications.

Monotonic analysis is an advanced part of abstract convex analysis based on the use of elementary functions which are monotone on cones; it has many applications in mathematical economics (see e.g. [11]). In this paper we study the stability of inequality systems in $\mathbb{R}^n_+$, involving *max-type functions* of the form

$$\langle a, x\rangle := \max_{i\in I} a_i x_i, \qquad x \in \mathbb{R}^n_+,$$

where $a \in \mathbb{R}^n_+$ and $I := \{1, 2, ..., n\}$. Semi-infinite systems $\sigma := \{\langle a_t,x\rangle \geq b_t,\ t \in T\}$ of *max-type inequalities* arise in monotonic analysis [16] describing the so-called *co-normal* subsets of $\mathbb{R}^n_+$. If f is any increasing function defined on $\mathbb{R}^n_+$, then the upper level sets $\left\{x \in \mathbb{R}^n_+ : f(x) \geq c\right\}$ are co-normal. We study the stability of the upper level set, $F = \left\{x \in \mathbb{R}^n_+ : f(x) \geq 0\right\}$, of an abstract concave function depending on its representation as a solution set of some associated abstract linear semi-infinite system. For instance, a continuous increasing positively homogeneous function f can be represented as the pointwise infimum of a subset of functions of the form $\langle a,x\rangle - b$, for some $a \in \mathbb{R}^n_+, b \in \mathbb{R}$, so it is an abstract concave function, e.g. $f(x) = \inf\{\langle a_t,x\rangle - b_t,\ t \in T\}$, for some (possibly infinite) index set T. The upper level set F of the function f is the solution set of the max-type system $\sigma := \{\langle a_t,x\rangle \geq b_t,\ t \in T\}$.

The main objective of this paper is to study the *stability* of the solution set of $\sigma = \{\langle a_t,x\rangle \geq b_t,\ t \in T\}$, F, under small perturbations of *all* the coefficients involved in the system. Perturbations of the coefficients in the *nominal system* σ yield a new *perturbed system* $\sigma_1 := \{\langle a^1_t,x\rangle \geq b^1_t,\ t \in T\}$, and the associated *perturbed solution set* F_1. The perturbations should be sufficiently small to guarantee $\{a^1_t, t \in T\} \subset \mathbb{R}^n_+$. It is also assumed that b_t and b^1_t are non-negative scalars, for all $t \in T$.

We consider as a *parameter space* the set Θ of all the max-type inequality systems on $\mathbb{R}^n_+$, with a fixed index set T. This parameter space can be identified with $\left(\mathbb{R}^{n+1}_+\right)^T$, since each system σ can alternatively be represented by $\left\{\binom{a_t}{b_t}\right\}_{t\in T}$. The system $\sigma \in \Theta$ is *consistent* if its solution set F is non-empty. The subset of Θ formed by all the consistent systems will be denoted by Θ_c.

The parameter space Θ is endowed with the topology of the uniform convergence of the coefficient vectors, via the *extended distance* $d : \Theta \times \Theta \to [0,+\infty]$ given by

$$d(\sigma_1,\sigma) := \sup_{t\in T} \left\| \begin{pmatrix} a^1_t \\ b^1_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|, \tag{1.1}$$

where $\|\cdot\|$ is the $l^\infty-$norm in $\mathbb{R}^{n+1}$ (i.e., $\|x\| = \max_{i\in I} |x_i|$). The condition that σ belongs to the interior set of Θ_c is referred to as *stability for the consistency.*

Recently, López et al.,[9], have studied the stability of inequality systems involving *min-type functions* of the form $\langle a,x\rangle_{\min} := \min_{i\in I} a_i x_i$, for $a, x \in \mathbb{R}^n_{++}$. They have found results about solvability and stability in the environment of $\mathbb{R}^n_{++}$ in some sense similar to the ones we present here. One might think of a possible duality scheme by taking into account that the max-type inequality $\langle a,x\rangle_{\max} \equiv \langle a,x\rangle \geq b$ can be written as a min-type inequality $\langle \frac{1}{a}, \frac{1}{x}\rangle_{\min} \leq \frac{1}{b}$ through the variable transformation $x \mapsto \frac{1}{x}$, where $\frac{1}{x}$ is the vector defined by $\left(\frac{1}{x}\right)_i = \frac{1}{x_i}$ if $x_i \neq 0$ and by $\left(\frac{1}{x}\right)_i = 0$ if $x_i = 0$. Nonetheless the results there and the ones here cannot be obtained from one another as the following systems show: Consider the max-type system

$$\sigma = \{\langle a_t,x\rangle \geq b_t,\ t \in T\} \equiv \{\langle a_t,x\rangle_{\max} \geq b_t,\ t \in T\} \tag{1.2}$$

and the corresponding min-type system

$$\tau = \left\{\left\langle \frac{1}{a_t}, \frac{1}{x}\right\rangle_{\min} \leq \frac{1}{b_t},\ t \in T\right\}, \tag{1.3}$$

where $T =]0,\infty[$, $a_t = (1/t, 1/t)^T$ for $t \in T$, $b_t = 1/t$, for $t \geq 1$, and $b_t = 1$ for $0 < t < 1$. Then, both are consistent systems but τ is stable with respect to the consistency while σ is not (see Remark 4.4 in Section 4 below).

Next, we summarize the structure of the paper. Section 2 is devoted to notation and preliminaries. Section 3 includes some *solvability results* relative to max-type systems which are versions of Farkas and Gale alternative theorems. Section 4 provides different characterizations of the stability of the *feasible set mapping*; i.e., the set valued function $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n_+$ that assigns to each $\sigma \in \Theta$ its solution set F. In this section we prove that the condition that σ belongs to the interior set of Θ_c is equivalent to the *lower semicontinuity* of $\mathcal{F}$ at σ (also equivalent in this case to the *continuity in the Bouligand sense* of $\mathcal{F}$ at σ), to the existence of *strong Slater points* and to the adapted *Robinson-Ursescu condition.* In Section 5 we discuss some error bounds for the solution set F, *linear regularity* of the collection $\{F_t,\ t \in T\}$ of solution sets of the systems $\sigma_t = \{\langle a_t, x\rangle \geq b_t\},\ t \in T$, and the *metric regularity* of certain associated mapping.

2 Notation and Preliminary Results

As it is usual, given a non-empty set X of a topological space the symbols $\operatorname{int} X$, $\operatorname{cl} X$ and $\operatorname{bd} X$ stand for the *interior*, the *closure* and the *boundary* of X, respectively. If X is a subset of a vector space then $\operatorname{conv} X$ and $\operatorname{cone} X$ will represent the *convex hull* and the *conical convex hull* of X. Consider the n-dimensional vector space $\mathbb{R}^n$ with the $l^\infty-$norm. This norm is symbolized by $\|\cdot\|$, whereas $\mathbb{B}$ is the unit open ball for this norm. If $n = 1$ we use $\mathbb{R}$ instead of $\mathbb{R}^1$. We also use the following notation:

- x_i is the i-th coordinate of a vector $x \in \mathbb{R}^n$;
- if $x, y \in \mathbb{R}^n$, then $x \geq y \Leftrightarrow x_i \geq y_i$ for all $i = 1, \ldots, n$;
- if $x, y \in \mathbb{R}^n$, then $x > y \Leftrightarrow x_i > y_i$ for all $i = 1, \ldots, n$;
- $\mathbf{0}$ is the null-vector in $\mathbb{R}^n$;
- $\mathbf{1}$ is the vector in $\mathbb{R}^n$ whose coordinates are all equal to 1;
- $\lim_k$ will be interpreted as $\lim_{k\to\infty}$;
- $\{x^k\}$ denotes the sequence $x^1, x^2, ..., x^k, ...$;
- $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$;
- $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : x > \mathbf{0}\}$;
- $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$ and $\overline{\mathbb{R}}^n_+ \equiv (\overline{\mathbb{R}}_+)^n$;
- $\mathbb{R}^T_+$ is the set of all the functions defined from the set T into $\mathbb{R}_+$;
- if $a \in \mathbb{R}^n_+$ and $b \in \mathbb{R}_+$ then $\frac{b}{a}$ is the vector in $\overline{\mathbb{R}}^n_+$ whose i-th coordinate is $\frac{b}{a_i}$, where we adopt the convention
$$\frac{b}{0} = \begin{cases} 0, & \text{if } b = 0, \\ +\infty, & \text{if } b > 0; \end{cases} \tag{2.1}$$
- if $h : X \to \overline{\mathbb{R}}_+$ then the *effective domain* of h is $\operatorname{dom} h := \{x \in X \mid h(x) < +\infty\}$.

A set $U \subset \mathbb{R}^n_+$ is called *co-normal* if

$$x \in U,\ y \in \mathbb{R}^n_+ \text{ and } y \geq x \ \Rightarrow\ y \in U.$$

The empty set is co-normal by definition. If f is an arbitrary increasing function defined on $\mathbb{R}^n_+$ then the level sets $\{x \in \mathbb{R}^n_+ : f(x) \geq c\}$ are co-normal (possibly empty) for all $c \in \mathbb{R}$.

The intersection and the union of a family of co-normal sets are co-normal. The solution set of $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ is closed and co-normal.

It is obvious that a nonempty co-normal set U is *co-radiant*; i.e., if $x \in U$ and $\lambda \geq 1$, then $\lambda x \in U$. Moreover, for each $y > \mathbf{0}$ a positive scalar λ exists such that $\lambda y \in U$.

3 Solvability Results for Max-type Systems

Dual characterizations of the solvability of a max-type inequality system, $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$, are provided in this section. In the ordinary linear case there are well-known alternative theorems of Farkas and Gale; here we present non-convex versions of them. In [16, §8.2], Rubinov presents a very general non-linear extension of the classical Farkas lemma for finite systems of linear inequalities and also a dual characterization of inconsistency. For max-type systems we provide straightforward proofs which have the advantage of avoiding the use of conjugation theory.

If

$$F_t := \left\{x \in \mathbb{R}^n_+ \mid \langle a_t, x\rangle \geq b_t\right\},\ t \in T,$$

one easily observes, taking into account (2.1), that

$$F_t = \mathbb{R}^n_+ \diagdown \left\{x \mid \mathbf{0} \leq x < \frac{b_t}{a_t}\right\},$$

and, then, $F_t = \mathbb{R}^n_+$ if $b_t = 0$, and $F_t = \emptyset$ if $a_t = \mathbf{0}$ and $b_t > 0$. The solution set of σ is, accordingly,

$$F = \bigcap_{t\in T} F_t = \mathbb{R}^n_+ \diagdown \bigcup_{t\in T} \left\{x \mid \mathbf{0} \leq x < \frac{b_t}{a_t}\right\}$$

and, so,

$$\sigma \in \Theta_c \Leftrightarrow \bigcup_{t\in T} \left\{x \mid \mathbf{0} \leq x < \frac{b_t}{a_t}\right\} \neq \mathbb{R}^n_+. \tag{3.1}$$

The following proposition can be considered a Gale-type theorem:

Proposition 3.1. *The system* $\sigma := \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$ *if and only if* $\sup\left\{\frac{b_t}{\|a_t\|}, t \in T\right\} < \infty$.

(Here the supremum is taken in $\overline{\mathbb{R}}_+$ as a consequence of (2.1).)

Proof. ($\Rightarrow$) If $z \in F$ one has

$$b_t \leq \langle a_t, z\rangle \leq \|a_t\| \, \|z\|\,,\ \text{for every } t \in T.$$

If we divide by $\|a_t\|$ (always having in mind (2.1)), we get

$$\frac{b_t}{\|a_t\|} \leq \|z\|\,,\ \text{for every } t \in T.$$

($\Leftarrow$) Reasoning by contradiction, $\sigma \notin \Theta_c$ and (3.1) would yield the existence of a sequence $\{t_k\} \subset T$ such that, in $\overline{\mathbb{R}}^n_+$,

$$\frac{b_{t_k}}{a_{t_k}} > k\mathbf{1},\ k = 1, 2, \ldots\,.$$

Thus, $\frac{b_{t_k}}{\|a_{t_k}\|} > k$ and we arrive to the contradiction

$$\lim_k \frac{b_{t_k}}{\|a_{t_k}\|} = \infty.$$

□

The following proposition is a semi-infinite abstract version of Farkas' lemma.

Proposition 3.2. *The inequality* $\langle a, x\rangle \geq b$ *is a* consequence *of the system* $\sigma = \{\langle a_t, x\rangle \geq b_t, t \in T\}$ *if and only if*

$$\frac{b}{a} \in \text{cl} \bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u \leq \frac{b_t}{a_t} \right\}, \tag{3.2}$$

where the closure is taken in $\overline{\mathbb{R}}^n_+$.

Proof. Let us start with the inconsistent case; i.e., $F = \emptyset$. In this case every inequality $\langle a, x\rangle \geq b$ is a trivial consequence of σ. According to (3.1), we also have

$$\bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u < \frac{b_t}{a_t} \right\} = \mathbb{R}^n_+,$$

and this entails

$$\begin{aligned}\overline{\mathbb{R}}^n_+ &= \text{cl}(\mathbb{R}^n_+)\\ &= \text{cl} \bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u < \frac{b_t}{a_t} \right\}\\ &\subset \text{cl} \bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u \leq \frac{b_t}{a_t} \right\} \subset \overline{\mathbb{R}}^n_+.\end{aligned}$$

Hence (3.2) trivially holds.

Now we approach the consistent case, $F \neq \emptyset$. Assume, first, that $\langle a, x\rangle \geq b$ is a consequence of the system σ and that, reasoning by contradiction, (3.2) fails. Next we make the following discussion:

i) The possibility $a = \mathbf{0}$ and $b > 0$ is excluded because $\langle a, x\rangle \geq b$ is a consequence of the consistent system σ.

ii) $b = 0$ yields $b/a = \mathbf{0}$, which trivially belongs to the set $\bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u \leq \frac{b_t}{a_t} \right\}$, and this possibility is also precluded by assumption.

iii) If $a \neq \mathbf{0}$ and (3.2) fails b must be positive (otherwise, $b/a = \mathbf{0}$ and the same contradiction that in ii) arises).

The unfulfillment of (3.2) entails the existence of a sufficiently large scalar $M > 0$ and a sufficiently small $\varepsilon > 0$ such that the vector z whose components are

$$z_i := \begin{cases} \frac{b}{a_i} - \varepsilon, & \text{if } a_i > 0,\\ M, & \text{if } a_i = 0,\end{cases}$$

satisfy $z \geq \mathbf{0}$ and

$$z \notin \bigcup_{t\in T} \left\{ x \mid \mathbf{0} \leq x < \frac{b_t}{a_t} \right\}.$$

Therefore, $z \in F$ and the current assumption implies $\langle a, z\rangle \geq b$. At the same time,

$$\langle a, z\rangle = \max_{\{i:a_i>0\}} a_i \left(\frac{b}{a_i} - \varepsilon\right) = \max_{\{i:a_i>0\}} \{b - a_i\varepsilon\} < b,$$

and we get a new contradiction.

Conversely, let us assume that (3.2) holds and proceed with the following discussion:

i) If $b = 0$ it is obvious that $\langle a, x\rangle \geq b$ is a consequence of the system σ.

ii) If $a = \mathbf{0}$ and $b > 0$, (3.2) reads

$$\infty\mathbf{1} \in \operatorname{cl} \bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u \leq \frac{b_t}{a_t} \right\},$$

and there exist sequences $\{t_k\} \subset T$ and $\{u^k\} \subset \overline{\mathbb{R}}^n_+$ such that

$$u^k \leq \frac{b_{t_k}}{a_{t_k}},\ k = 1, 2, ...,\ \text{and}\ u^k \geq k\mathbf{1}.$$

Therefore,

$$\lim_k \frac{b_{t_k}}{\|a_{t_k}\|} = \infty,$$

and this contradicts $\sigma \in \Theta_c$ by virtue of Proposition 3.1.

iii) If $a \neq \mathbf{0}$, $b > 0$, and (3.2) is satisfied, there will exist sequences $\{t_k\} \subset T$ and $\{u^k\} \subset \overline{\mathbb{R}}^n_+$ such that

$$u^k \leq \frac{b_{t_k}}{a_{t_k}},\ k = 1, 2, ...,\ \text{and}\ \frac{b}{a} = \lim_k u^k.$$

Because $b > 0$ there must exist k_0 such that $b_{t_k} > 0$ for every $k \geq k_0$.

For each $z \in F$ one has

$$\langle a_{t_k}, z\rangle \geq b_{t_k},\ k \geq k_0,$$

and there will exist an associated $i_k \in I$ such that the i_k-th coordinate of the vector a_{t_k}, which we denote here by $a_{t_k i_k}$, satisfies

$$a_{t_k i_k} z_{i_k} \geq b_{t_k},\ k \geq k_0,$$

or, equivalently,

$$z_{i_k} \geq \frac{b_{t_k}}{a_{t_k i_k}},\ k \geq k_0.$$

Then, there must exist $\widehat{i} \in I$ appearing infinitely many times in the inequalities above; i.e., there is a subsequence $\{k_l\}$ such that $i_{k_l} = \widehat{i}$, and $k_l \geq k_0$. Therefore,

$$z_{\widehat{i}} \geq \frac{b_{t_{k_l}}}{a_{t_{k_l}\widehat{i}}} \geq u^{k_l}_{\widehat{i}},\ k_l \geq k_0.$$

Taking limits for $l \to \infty$ we obtain $z_{\widehat{i}} \geq b/a_{\widehat{i}}$. Consequently,

$$\langle a, z\rangle \geq a_{\widehat{i}} z_{\widehat{i}} \geq b$$

and, certainly, the inequality $\langle a, x\rangle \geq b$ is a consequence of σ. □

4 Stability of a Semi-infinite Max-type Inequality System

In this section we study the stability of the system $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$. At this point, we have to emphasize the idea that stability is relative to the *representation* of the set $F = \{x \in \mathbb{R}^n_+ \mid x$ is solution of $\sigma\}$, not to the set F itself. So, the same set F can have "good" (stable) and "bad" (unstable) representations. To look for stable representations of any relevant set in optimization underlies every pre-conditioning strategy in numerics. Here we analyze some stability criteria studied in [1], in the context of ordinary nonlinear programming, and in [4], [5], [6] and [8] relatively to semi-infinite linear programming. The stability approach inspired in [14] does not apply here because the image sets of the mappings considered in this section are in $\mathbb{R}^T_+$ (which is not a Banach space) and their graphs are not convex.

Since we start from a consistent nominal system ($\sigma \in \Theta_c$), a first stability criterion to be considered is the stability with respect to the consistency. This property means that small perturbations in the coefficients do not affect the consistency.

Definition 4.1. The system $\sigma \in \Theta_c$ is *stable with respect to the consistency (stable*, in brief*)* if $\sigma \in \operatorname{int} \Theta_c$.

The following example shows that Θ_c is not an open set in our parameter space (Θ, d), in which case the stability with respect to the consistency would be trivially fulfilled.

Example 4.2. Consider the system in $\mathbb{R}^2_+$

$$\sigma = \left\{\max\{tx_1, tx_2\} \geq t^2,\ t \in [0,1]\right\}.$$

Observe that $\mathbf{1}$ is a solution of σ and, so, $\sigma \in \Theta_c$. Consider the perturbed system

$$\sigma_k = \left\{\max\left\{a^k_{t_1}x_1, a^k_{t_2}x_2\right\} \geq t^2,\ t \in [0,1]\right\},\ k = 1, 2, ...,$$

where

$$a^k_{t_1} = a^k_{t_2} = \begin{cases} t, & \text{if } t \in]\frac{1}{k}, 1], \\ 0, & \text{if } t \in [0, \frac{1}{k}]. \end{cases}$$

We have

$$d(\sigma_k, \sigma) = \sup_{0 \leq t \leq 1/k} t = \frac{1}{k},$$

and $\lim_k d(\sigma_k, \sigma) = 0$. Since $\sigma_k \notin \Theta_c$ we must conclude that $\sigma \in \operatorname{bd} \Theta_c$.

The following proposition provides a very simple criterion to recognize the stability with respect to the consistency.

Proposition 4.3. *Let* $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$. *Then* $\sigma \in \operatorname{int} \Theta_c$ *if and only if*

$$\mathbf{0} \notin \operatorname{cl}\{a_t,\ t \in T\}.$$

Proof. ($\Rightarrow$) Reasoning by contradiction, let us assume that

$$\mathbf{0} \in \operatorname{cl}\{a_t,\ t \in T\}.$$

In this case we will see how to construct a sequence $\{\sigma_k\} \subset \Theta \diagdown \Theta_c$ converging to σ, which contradicts our present hypothesis.

Because $\mathbf{0} \in \operatorname{cl}\{a_t,\ t \in T\}$, there will exist a sequence $\{t_k\} \subset T$ such that $\lim_k a_{t_k} = \mathbf{0}$. (If there is a $t_0 \in T$ such that $a_{t_0} = \mathbf{0}$, we shall take $t_k = t_0$, for every k.)

Let $\sigma_k = \{\langle a_t^k, x\rangle \geq b_t^k,\ t \in T\}$, $k = 1, 2, ...$, with

$$a_t^k := \begin{cases} \mathbf{0}, & \text{if } t = t_k, \\ a_t, & \text{if } t \in T \diagdown \{t_k\}, \end{cases}$$

and

$$b_t^k := \begin{cases} \max\{\frac{1}{k}, b_{t_k}\} & \text{if } t = t_k, \\ b_t, & \text{if } t \in T \diagdown \{t_k\}. \end{cases}$$

Therefore,

$$d(\sigma_k, \sigma) \leq \max\{\|a_{t_k}\|, \frac{1}{k}\},\ k = 1, 2, ...,$$

and $\sigma_k \to \sigma$. At the same time, $\sigma_k \notin \Theta_c$ because the inequality

$$\langle a_{t_k}^k, x\rangle = \langle \mathbf{0}, x\rangle \geq b_{t_k}^k \geq \frac{1}{k},$$

is itself inconsistent.

($\Leftarrow$) Since $\mathbf{0} \notin \mathrm{cl}\{a_t,\ t \in T\}$, there will exist $\delta > 0$ such that

$$\|a_t\| > \delta, \text{ for every } t \in T.$$

Consider any possible system $\sigma_1 := \{\langle a_t^1, x\rangle \geq b_t^1,\ t \in T\} \in \Theta$ such that $d(\sigma_1, \sigma) < \delta/2$. We shall prove that $\sigma_1 \in \Theta_c$, which implies $\sigma \in \operatorname{int}\Theta_c$.

Since $\sigma \in \Theta_c$, and by Proposition 3.1, there exists $M > 0$ such that

$$\frac{b_t}{\|a_t\|} \leq M, \text{ for all } t \in T.$$

The definition of $\|a_t\|$ entails the existence of $i_t \in I$, associated with each $t \in T$, such that $\|a_t\| = a_{t i_t}$. Then we obtain, for any $t \in T$,

$$\begin{aligned} \frac{b_t^1}{\|a_t^1\|} \leq \frac{b_t^1}{a_{t i_t}^1} &< \frac{b_t + \frac{\delta}{2}}{a_{t i_t} - \frac{\delta}{2}} \\ &= \frac{\frac{b_t}{\|a_t\|} + \frac{\delta}{2\|a_t\|}}{1 - \frac{\delta}{2\|a_t\|}} \\ &\leq \frac{M + \frac{1}{2}}{1 - \frac{1}{2}} = 2M + 1, \end{aligned}$$

and hence $\sigma_1 \in \Theta_c$, again by Proposition 3.1. □

Remark 4.4. We can apply this proposition to the consistent system σ (1.2) described in the introduction to see easily that it is not stable with respect to the consistency. The associated min-type system τ (1.3) is stable in this sense by a direct application of the condition given in the Proposition 4.1 in [9] because $\inf\{1/b_t, t \in T\} > 0$.

Generically, in optimization, the stability of the feasible set is related to the existence of *strict feasible solutions* which, in the semi-infinite context, are called strong Slater points.

Definition 4.5. The system $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ is said to satisfy the *strong Slater condition (SS condition*, in short*)* if there exist x^0 and $\eta > 0$ such that $\langle a_t, x^0\rangle \geq b_t + \eta$, for all $t \in T$. In this case x^0 is called an *SS point of* σ.

Proposition 4.6. *$\sigma \in \operatorname{int} \Theta_c$ if and only if σ satisfies the strong Slater condition.*

Proof. ($\Rightarrow$) By Proposition 3.1 there exists $M > 0$ such that

$$\frac{b_t}{\|a_t\|} \leq M, \text{ for all } t \in T, \tag{4.1}$$

and Proposition 4.3 provides the existence of $\delta > 0$ such that

$$\|a_t\| \geq \delta, \text{ for all } t \in T. \tag{4.2}$$

Then, for every $t \in T$,

$$\langle a_t, (M+1)\mathbf{1} \rangle = (M+1)\|a_t\| \geq b_t + \delta,$$

and $(M+1)\mathbf{1}$ is an SS point with positive slack $\eta = \delta$.

($\Leftarrow$) If x^0 is an SS point with slack $\eta > 0$, we have

$$\langle a_t, x^0 \rangle \geq b_t + \eta, \text{ for every } t \in T,$$

and this entails $x^0 \neq \mathbf{0}$. Then,

$$\|a_t\| \|x^0\| \geq \langle a_t, x^0 \rangle \geq b_t + \eta \geq \eta.$$

Thus,

$$\|a_t\| \geq \frac{\eta}{\|x^0\|}, \text{ for every } t \in T,$$

and Proposition 4.3 applies to conclude that $\sigma \in \operatorname{int} \Theta_c$. □

The following propositions provide additional information about the SS points.

Proposition 4.7. *If $\sigma \in \operatorname{int} \Theta_c$ and $z > \mathbf{0}$ is feasible for σ, then λz is an SS point for all $\lambda > 1$.*

Proof. The reasoning used in the proof of the direct statement in Proposition 4.6 guarantees the existence of $K > 0$ such that $K\mathbf{1}$ is an SS point. Moreover, for ρ sufficiently large we can be sure that $K\mathbf{1} \leq \rho z$, and ρz will be also an SS point. We shall deal with the non-trivial case $\rho > \lambda > 1$.

Let us introduce now the function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$h(\mu) := \inf_{t \in T} (\langle a_t, z \rangle \mu - b_t).$$

h is a increasing concave function (infimum of affine functions) and, so, for any possible $\lambda \in]1, \rho[$,

$$\begin{aligned} h(\lambda) &= h\left(\frac{\rho-\lambda}{\rho-1}1 + \frac{\lambda-1}{\rho-1}\rho\right) \\ &\geq \frac{\rho-\lambda}{\rho-1}h(1) + \frac{\lambda-1}{\rho-1}h(\rho) \\ &\geq \frac{\lambda-1}{\rho-1}h(\rho). \end{aligned}$$

Since ρz is an SS point, $h(\rho) > 0$ and, consequently, λz is an SS point with slack $(\lambda - 1)h(\rho)/(\rho-1)$.

Finally, if $\lambda \geq \rho$ it is absolutely obvious that λz is again an SS point. □

Remark 4.8. The condition $z > \mathbf{0}$ cannot be omitted in the last proposition, as the following example shows.

Example 4.9. Consider the system in $\mathbb{R}^2_+$

$$\sigma = \{\max\{tx_1, (1-t)x_2\} \geq t,\ t \in [0,1]\}.$$

Observe that $z = \binom{1}{0}$ is a solution of σ, but λz with $\lambda > 1$ is not an SS point, because the inequality associated with $t = 0$ does not allow for a positive slack at this point

$$\max\{0 \cdot \lambda, 1 \cdot 0\} = 0 \geq 0.$$

Corollary 4.10. *If $\sigma \in \operatorname{int}\Theta_c$ and $z \in \operatorname{int} F$, then z is an SS point for σ.*

Proof. $z \in \operatorname{int} F$ entails $z > \mathbf{0}$ and the existence of $\lambda \in]0,1[$ such that $\lambda z \in F$. Then, we write $z = \frac{1}{\lambda}(\lambda z)$ and Proposition 4.7 applies. □

Remark 4.11. If $x^0 > \mathbf{0}$ is SS point for σ, then x^0 does not need to belong to $\operatorname{int} F$.

Example 4.12. Let us consider the system in $\mathbb{R}^2_+$

$$\sigma = \{\max\{tx_1, tx_2\} \geq t-1,\ t \in [1,\infty[\}.$$

Observe that $z = \binom{1}{1}$ is an SS point (with positive slack $\eta = 1$), but $z \notin \operatorname{int} F$, because $\binom{\lambda}{\lambda} \notin F$, for every $\lambda \in [0,1[$.

The following characterizations of the stability of σ are formulated as properties of the *feasible set mapping* $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n_+$, which assigns to each $\sigma \in \Theta$ its solution set F; i.e., $\mathcal{F}(\sigma) = F$. Remember that $\mathcal{F}$ is *lower semicontinuous in the Berge sense* (*B-lsc*, in brief) at $\sigma \in \Theta_c$ if, for each open set relative to $\mathbb{R}^n_+$, W, such that $\mathcal{F}(\sigma) \cap W \neq \emptyset$, there exists an open set U, relative to Θ, containing σ and such that $\mathcal{F}(\sigma_1) \cap W \neq \emptyset$ for every $\sigma_1 \in U$.

Proposition 4.13. *Let $\sigma \in \Theta_c$. Then $\sigma \in \operatorname{int}\Theta_c$ if and only if the feasible set mapping $\mathcal{F}$ is B-lsc at σ.*

Proof. ($\Leftarrow$) It is obvious from the very definition of the lower semicontinuity.

($\Rightarrow$) Suppose that $\sigma \in \operatorname{int}\Theta_c$ and that W is an open set relative to $\mathbb{R}^n_+$ such that $\mathcal{F}(\sigma) \cap W \neq \emptyset$. We shall prove the existence of $\delta > 0$ such that $\sigma_1 \in \Theta$ and $d(\sigma_1, \sigma) < \delta$ imply $\mathcal{F}(\sigma_1) \cap W \neq \emptyset$.

Pick a point $y \in \mathcal{F}(\sigma) \cap W$. It is obvious that we can find $z \in \mathcal{F}(\sigma) \cap W$ such that $z > \mathbf{0}$ (take $z = y + \varepsilon\mathbf{1}$ with $\varepsilon > 0$ small enough). If $\lambda > 1$ is sufficiently close to 1 we can be sure that $\lambda z \in W$ and, by Proposition 4.7, λz is an SS point for σ with some positive slack η.

If we define $\delta := \frac{\eta}{1+\lambda\|z\|}$, and $\sigma_1 = \{\langle a_t^1, x\rangle \geq b_t^1,\ t \in T\} \in \Theta$ verifies $d(\sigma_1, \sigma) < \delta$, we can write, for every $t \in T$,

$$\begin{aligned}
\langle a_t^1, \lambda z\rangle &= \lambda \max_{i \in I} a_{t_i}^1 z_i > \lambda \max_{i \in I}(a_{t_i} - \delta) z_i \\
&\geq \lambda\{(\max_{i \in I} a_{t_i} z_i) - \delta \|z\|\} = \langle a_t, \lambda z\rangle - \lambda\delta \|z\| \\
&\geq b_t + \eta - \frac{\lambda\eta \|z\|}{1 + \lambda \|z\|} \\
&= b_t + \frac{\eta}{1 + \lambda \|z\|} = b_t + \delta > b_t^1.
\end{aligned}$$

Hence, $\lambda z \in \mathcal{F}(\sigma_1) \cap W$, and this set is obviously non-empty. □

In relation to our feasible set mapping $\mathcal{F}$, we shall consider the *inner limit* $\liminf_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma})$ which is the set of points that are limit points of sequences $\{x^r\}$, $x^r \in \mathcal{F}(\sigma_r)$, for all possible sequences $\{\sigma_r\}$, $\sigma_r \to \sigma$; whereas the *outer limit* $\limsup_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma})$ consists of all possible cluster points of such sequences. When $\mathcal{F}(\sigma) = \limsup_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma})$ it is said that $\mathcal{F}$ is *outer semicontinuous* (*osc*) at σ and, similarly, $\mathcal{F}$ is *inner semicontinuous* (*isc*) at σ if $\mathcal{F}(\sigma) = \liminf_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma})$. Following [15] we say that $\mathcal{F}$ is *continuous in the Bouligand sense* at $\sigma \in \Theta_c$ if

$$\liminf_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma}) = \limsup_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma}) = \mathcal{F}(\sigma);$$

i.e., if $\mathcal{F}$ is simultaneously osc and isc at σ. The continuity in the Bouligand sense is equivalent to require that $\lim_{\widehat{\sigma}\to\sigma} \mathcal{F}(\widehat{\sigma}) = \mathcal{F}(\sigma)$ in the sense of Painlevé-Kuratowski.

According to [15], the inner semicontinuity of $\mathcal{F}$ at σ is equivalent to the lower semicontinuity of $\mathcal{F}$ at σ (in the sense of Berge). Moreover, the outer semicontinuity of $\mathcal{F}$ at σ is equivalent to the *closedness* of $\mathcal{F}$ at σ, property that is defined in the following terms, relatively to $\mathcal{F}$:

$\mathcal{F}$ is *closed* at $\sigma \in \Theta_c$ if, for all sequences $\{\sigma_k\} \subset \Theta_c$ and $\left\{z^k\right\} \subset \mathbb{R}^n_+$ satisfying $\lim_k \sigma_k = \sigma$, $\lim_k z^k = z$ and $z^k \in \mathcal{F}(\sigma_k)$, one has $z \in \mathcal{F}(\sigma)$.

The following result has its convex semi-infinite counterpart in Theorem 4.1 in [10].

Proposition 4.14. *Let $\sigma \in \Theta_c$. Then $\sigma \in \operatorname{int}\Theta_c$ if and only if the feasible set mapping $\mathcal{F}$ is continuous in the Bouligand sense at σ.*

Proof. According to the comments above the only thing that is still to be proved is that $\mathcal{F}$ is always closed at any $\sigma \in \Theta_c$, but this property is a straightforward consequence of the fact that all the coefficients of the systems in $\{\sigma_k\}$ are pointwise convergent to the coefficients of σ and also of the continuity of $\langle \cdot, \cdot \rangle$. □

The last characterizations of the stability of $\mathcal{F}$ given in this paper are related with the relevant property of metric regularity. They also bring the appealing conclusion that $\sigma \in \operatorname{int}\Theta_c$ is equivalent to the fact that σ remains consistent for small perturbations of the intercepts $b_t,\ t \in T$.

Associated with the *fixed* function $a : T \to \mathbb{R}^n_+$, let us introduce the set valued function $\mathcal{M} : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^T_+$ defined by

$$\mathcal{M}(x) := \{f \in \mathbb{R}^T_+ \mid \langle a(\cdot), x\rangle \geq f(\cdot)\}.$$

Given the function $b \in \mathbb{R}^T_+$, we have $b \in \mathcal{M}(z)$ if and only if z is a solution of the system $\{\langle a_t, x\rangle \geq b_t,\ t \in T\}$. In other words, $\mathcal{M}^{-1}$ is the feasible set mapping restricted to those systems with fixed $a_t,\ t \in T$. The following definition is based on [14].

Definition 4.15. The system $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ satisfies the *Robinson-Ursescu condition* if $b \in \operatorname{int}\mathcal{M}(\mathbb{R}^n_+)$.

Proposition 4.16. *Given $\sigma \in \Theta_c$, $\sigma \in \operatorname{int}\Theta_c$ if and only if σ satisfies the Robinson-Ursescu condition.*

Proof. $b \in \operatorname{int}\mathcal{M}(\mathbb{R}^n_+)$ if there is $\delta > 0$ such that $\sup_{t\in T} |h(t)| \leq \delta$ and $b + h \in \mathbb{R}^T_+$ conjointly entail $b + h \in \mathcal{M}(\mathbb{R}^n_+)$.

($\Rightarrow$) Let $\delta > 0$ be a scalar such that $\sigma_1 \in \Theta$ and $d(\sigma_1, \sigma) \leq \delta$ entail $\sigma_1 \in \Theta_c$. Let $h : T \to \mathbb{R}$ be an arbitrary function satisfying $|h(t)| \leq \delta$, for every $t \in T$, and such that $b_t + h(t) \geq 0$, for all $t \in T$. Then

$$\sigma_1 = \{\langle a_t, x\rangle \geq b_t + h(t),\ t \in T\} \in \Theta_c,$$

and we can pick a point $z^1 \in F_1$ and observe that

$$\left\langle a_t, z^1 \right\rangle \geq b_t + h(t), \text{ for every } t \in T,$$

entails $b + h \in \mathcal{M}(z^1) \subset \mathcal{M}(\mathbb{R}^n_{++})$. Since h is arbitrary, we conclude that $b \in \operatorname{int} \mathcal{M}(\mathbb{R}^n_+)$.

($\Leftarrow$) If $b \in \operatorname{int} \mathcal{M}(\mathbb{R}^n_+)$, and $\eta > 0$ is sufficiently small, the constant function $h(t) = \eta$, for every $t \in T$, satisfies $b + h \in \mathcal{M}(\mathbb{R}^n_+)$. Hence a point x^0 exists such that $\left\langle a_t, x^0 \right\rangle \geq b_t + \eta$, for every $t \in T$, and x^0 is an SS point. If we apply Proposition 4.6, the proof is finished. □

Remark 4.17. From a theoretical point of view all the Propositions 4.3, 4.6, 4.13, 4.14 and 4.16 could be used to analyze the stability of the inequality system. Nonetheless, for practical reasons, Proposition 4.3 is very simple to apply just by looking into the set of the coefficients a_t; also, by finding a strong Slater point of the system σ one can directly apply Proposition 4.6. So, these two propositions are quite useful in order to establish the stability of the systems; in fact they can be checked to conclude the validity of the B-lsc or Bouligand-continuity of the feasible set mapping $\mathcal{F}$ or of the Robinson-Ursescu condition of the system σ.

5 Metric regularity and error bounds

Definition 5.1. Given $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$ and $z \in \mathcal{F}(\sigma)$, we say that $\mathcal{M}$ is *metrically regular at z for b* if there exist two positive scalars k and ε such that

$$d(y, \mathcal{M}^{-1}(b^1)) \leq k d(b^1, \mathcal{M}(y)), \tag{5.1}$$

provided that $y \in \mathbb{R}^n_+$, $b^1 \in \mathbb{R}^T_+$, and

$$\|y - z\| < \varepsilon,\ \sup_{t \in T} \left|b^1_t - b_t\right| < \varepsilon. \tag{5.2}$$

In our context, it can be easily checked that

$$d(b^1, \mathcal{M}(y)) = \sup_{t \in T} (b^1_t - \langle a_t, y\rangle)^+,$$

with $a^+ := \max\{a, 0\}$.

Proposition 5.2. *Let $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$. Then $\sigma \in \operatorname{int} \Theta_c$ if $\mathcal{M}$ is metrically regular at z for b, for every $z \in F$.*

Proof. We shall prove that σ satisfies the Robinson-Ursescu condition and, then, apply Proposition 4.16.

Take $z \in F$ and let k and ε be a pair of positive scalars such that (5.1) holds provided that the conditions (5.2) are fulfilled. Reasoning by contradiction, assume that $b \in \operatorname{bd} \mathcal{M}(\mathbb{R}^n_+)$, which entails the existence of a function $h : T \to \mathbb{R}$ satisfying $|h(t)| \leq \frac{\varepsilon}{2}$, for every $t \in T$, such that $b + h \in \mathbb{R}^T_+$ and

$$b + h \notin \mathcal{M}(\mathbb{R}^n_+). \tag{5.3}$$

If $b^1 := b+h$, it is evident that $\sup_{t\in T}\left|b_t^1 - b_t\right| \le \frac{\varepsilon}{2} < \varepsilon$ and that $\mathcal{M}^{-1}(b^1) = \emptyset$. Nevertheless,

$$\begin{aligned}\sup_{t\in T}(b_t^1 - \langle a_t, z\rangle)^+ &= \sup_{t\in T}(b_t + h(t) - \langle a_t, z\rangle)\\ &\le \sup_{t\in T}|h(t)| + \sup_{t\in T}(b_t - \langle a_t, z\rangle)\\ &\le \frac{\varepsilon}{2},\end{aligned}$$

whereas $d(z, \mathcal{M}^{-1}(b^1)) = d(z, \emptyset) = +\infty$, and this contradicts (5.1) for $y = z$.

(Notice that (5.1) can be valid when $\mathcal{M}^{-1}(b^1) = \emptyset$, but in this case the value of $\sup_{t\in T}(b_t^1 - \langle a_t, y\rangle)^+$ must be $+\infty$.) □

We have just shown that the metric regularity at any point of F for b implies the stability with respect to the consistency of the system σ . With respect to the reverse implication, we show the existence of a global error bound on any cone K in $\mathbb{R}_+^n$ defined by the condition $\frac{\min_i y_i}{\max_i y_i} > \gamma$, where γ is any fixed positive real number. This property will allow us to prove the converse implication to Proposition 5.2 at any $z > \mathbf{0}$.

Theorem 5.3. *Let δ and γ be any pair of positive real numbers. If $\sigma = \{\langle a_t, x\rangle \ge b_t,\ t \in T\} \in \Theta_c$ is such that $\|a_t\| \ge \delta$ for all $t \in T$, then*

$$d\left(y, \mathcal{M}^{-1}\left(b^1\right)\right) \le \frac{1}{\delta\gamma} d\left(b^1, \mathcal{M}\left(y\right)\right), \tag{5.4}$$

for any $b^1 \in \mathbb{R}_+^T$ and for all $y \in \mathbb{R}_+^n$ with $\frac{\min_i y_i}{\max_i y_i} > \gamma$. Moreover, for $y = \mathbf{0}$ it holds:

$$d\left(\mathbf{0}, \mathcal{M}^{-1}\left(b^1\right)\right) \le \frac{1}{\delta} d\left(b^1, \mathcal{M}\left(\mathbf{0}\right)\right).$$

Proof. *i*) The case $\mathcal{M}^{-1}\left(b^1\right) \ne \varnothing$, $\frac{\min_i y_i}{\max_i y_i} > \gamma$. Without loss of generality we may assume that $d\left(y, \mathcal{M}^{-1}\left(b^1\right)\right) > 0$ and $d\left(b^1, \mathcal{M}\left(y\right)\right) < +\infty$. Since $d(y, \mathcal{M}^{-1}(b^1)) > 0$ we have

$$0 < \sup_{t\in T}(b_t^1 - \langle a_t, y\rangle)^+ = \sup_{t\in T}(b_t^1 - \langle a_t, y\rangle) < \infty. \tag{5.5}$$

Let us introduce the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(\lambda) := \sup_{t\in T}(b_t^1 - \langle a_t, y\rangle\,\lambda).$$

We have $0 < h(1) < \infty$, by (5.5). Moreover, h is a lsc decreasing convex function (supremum of affine functions) and, so, it is continuous on $[1, +\infty[$. (It is evident that $[1, +\infty[\subset \operatorname{dom} h$.)

Suppose that $\lambda_1 > 1$ is taken large enough to guarantee $\lambda_1 y \in \mathcal{M}^{-1}(b^1)$ and, so, $h(\lambda_1) \le 0$ (this is possible because $y > \mathbf{0}$, as a consequence of the condition $\frac{\min_i y_i}{\max_i y_i} > \gamma > 0$). The Bolzano theorem provides $\lambda_0 \in]1, \lambda_1]$ satisfying $h(\lambda_0) = 0$ and, accordingly, $\lambda_0 y \in \mathcal{M}^{-1}(b^1)$.

Now

$$d(y, \mathcal{M}^{-1}(b^1)) \le \|y - \lambda_0 y\| = (\lambda_0 - 1)\,\|y\|,$$

and thus,

$$\begin{aligned} d\left(b^1,\mathcal{M}\left(y\right)\right) &= \sup_{t\in T}(b_t^1-\langle a_t,y\rangle)^+ \\ &= \sup_{t\in T}(b_t^1-\langle a_t,y\rangle) \\ &= \sup_{t\in T}\left\{b_t^1-\langle a_t,\lambda_0 y\rangle+(\lambda_0-1)\langle a_t,y\rangle\right\} \\ &\geq \sup_{t\in T}\left\{b_t^1-\langle a_t,\lambda_0 y\rangle+(\lambda_0-1)\left\|a_t\right\|\min_{i\in I}y_i\right\} \\ &\geq h(\lambda_0)+(\lambda_0-1)\delta\min_{i\in I}y_i \\ &= 0+(\lambda_0-1)\left\|y\right\|\delta\frac{\min_{i\in I}y_i}{\max_{i\in I}y_i} \\ &\geq \delta\gamma d(y,\mathcal{M}^{-1}(b^1)). \end{aligned}$$

ii) The case $\mathcal{M}^{-1}\left(b^1\right)=\varnothing$, $y\in\mathbb{R}_+^n$. Obviously, $d(y,\mathcal{M}^{-1}(b^1))=+\infty$. We will show that $d\left(b^1,\mathcal{M}\left(y\right)\right)=+\infty$ as well. Observe that $\mathcal{M}^{-1}\left(b^1\right)=\varnothing$ gives that $\sup_{t\in T}\left(b_t^1/\left\|a_t\right\|\right)=+\infty$ by virtue of Proposition 3.1. Now,

$$\begin{aligned} d\left(b^1,\mathcal{M}\left(y\right)\right) &= \sup_{t\in T}(b_t^1-\langle a_t,y\rangle)^+ \\ &= \sup_{t\in T}(b_t^1-\langle a_t,y\rangle) \\ &\geq \sup_{t\in T}\left(b_t^1-\left\|a_t\right\|\left\|y\right\|\right) \\ &= \sup_{t\in T}\left\|a_t\right\|\left(\frac{b_t^1}{\left\|a_t\right\|}-\left\|y\right\|\right) \\ &\geq \delta\sup_{t\in T'}\left(\frac{b_t^1}{\left\|a_t\right\|}-\left\|y\right\|\right) \\ &= \delta\sup_{t\in T'}\left(\frac{b_t^1}{\left\|a_t\right\|}\right)-\left\|y\right\| \\ &= +\infty, \end{aligned}$$

where $T'=\left\{t\in T:\frac{b_t^1}{\left\|a_t\right\|}>\left\|y\right\|\right\}$.

iii) The case $y=\mathbf{0}$. Now,

$$d\left(b^1,\mathcal{M}\left(\mathbf{0}\right)\right)=\sup_{t\in T}\left(b_t^1-\langle a_t,\mathbf{0}\rangle\right)^+=\sup_{t\in T}b_t^1=B^1.$$

If $B^1=+\infty$ there is nothing to prove. For $B^1<+\infty$ put $x=\frac{B^1}{\delta}\mathbf{1}$ and observe that

$$\langle a_t,x\rangle=\frac{B^1}{\delta}\left\|a_t\right\|\geq B^1\geq b_t,$$

for all $t\in T$. Hence $x\in\mathcal{M}^{-1}(b^1)$ and, so

$$d\left(\mathbf{0},\mathcal{M}^{-1}\left(b^1\right)\right)\leq\left\|x\right\|=\frac{B^1}{\delta}=\frac{1}{\delta}d\left(b^1,\mathcal{M}\left(\mathbf{0}\right)\right).$$

□

Remark 5.4. Notice that we have actually proved that

$$d(y, \mathcal{M}^{-1}(b^1)) = +\infty = d\left(b^1, \mathcal{M}\left(y\right)\right),$$

for all $y \in \mathbb{R}^n_+$, whenever the system $\sigma_1 = \{\langle a_t, x\rangle \geq b^1_t,\ t \in T\}$ is not consistent.

Corollary 5.5. *Let $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \operatorname{int}\Theta_c$, then $\mathcal{M}$ is metrically regular at z for b, for every $z \in F$ such that $z > \mathbf{0}$.*

Proof. Let $z > \mathbf{0},$ take $\varepsilon = \frac{1}{2}\min_{i\in I} z_i > 0$ and $\gamma = \frac{\varepsilon}{\|z\|+\varepsilon}$. Since $\sigma \in \operatorname{int}\Theta_c$, Proposition 4.3 gives the existence of $\delta > 0$ such that $\|a_t\| \geq \delta$, for every $t \in T$. Moreover, if $y \in \mathbb{R}^n_+$ and $\|y - z\| < \varepsilon$, then

$$\min_{i\in I} y_i \geq \min_{i\in I} z_i - \varepsilon = 2\varepsilon - \varepsilon = \varepsilon$$

and

$$\max_{i\in I} y_i \leq \max_{i\in I} z_i + \varepsilon = \|z\| + \varepsilon,$$

so

$$\frac{\min_{i\in I} y_i}{\max_{i\in I} y_i} \geq \gamma.$$

By the last theorem we get

$$d\left(y, \mathcal{M}^{-1}\left(b^1\right)\right) \leq kd\left(b^1, \mathcal{M}\left(y\right)\right),$$

for $k = \frac{1}{\delta\gamma} = \frac{\|z\|+\varepsilon}{\delta\varepsilon}$. Therefore, (5.1) holds at z for b. (Notice that we do not establish any condition on b^1.) □

The following property, which is called *linear regularity* (see, for instance, [2] will allow us to prove, under an extra condition on the coefficients $a_t's$, the existence of a global error bound in $\mathbb{R}^n_+$.

Proposition 5.6. *If $F_t = \{x \in \mathbb{R}^n_+ / \langle a_t, x\rangle \geq b_t\ \},\ t \in T,\ F = \cap_{t\in T} F_t \neq \varnothing$ and $y \in \mathbb{R}^n_+$, then*

$$d\left(y, F\right) = \sup_{t\in T} d\left(y, F_t\right) = \sup_{t\in T} \min_{i\in I_+(a_t)} \left(\frac{b_t}{a_{t_i}} - y_i\right)^+,$$

where $I_+\left(a_t\right) = \{i \in I : a_{t_i} \neq 0\}$. (In case of $a_t = \mathbf{0}$ we consider $\min_{i\in I_+(a_t)} \left(\frac{b_t}{a_{t_i}} - y_i\right)^+ =$ 0.)

Proof. Notice that the second equality follows immediately from the definition of F_t. We will show the first one. Let $y \in \mathbb{R}^n_+$ and $r_t = d\left(y, F_t\right)$ for all $t \in T$. Put $r := \sup_{t\in T} r_t$. Since $F_t \supset F$ it follows that $r_t \leq d\left(y, F\right)$; hence

$$r = \sup_{t\in T} d\left(y, F_t\right) \leq d\left(y, F\right).$$

Now, due to the fact that each F_t is co-normal, $y + r_t\mathbf{1} \in F_t$. From $r \geq r_t$ we have $y + r\mathbf{1} \in F_t$ for all $t \in T$, i.e. $y + r\mathbf{1} \in F$ and so $d\left(y, F\right) \leq r$. Therefore $d\left(y, F\right) = r = \sup_{t\in T} d\left(y, F_t\right)$. □

Remark 5.7. The very definition of the linear regularity property of the collection $\{F_t,\ t \in T\}$ reads as $d\left(y, F\right) \leq k\sup_{t\in T} d\left(y, F_t\right)$ for some positive constant k. In our case this family of co-normal sets is strongly linearly regular in the sense that actually the equality $d\left(y, F\right) = \sup_{t\in T} d\left(y, F_t\right)$ holds true.

Proposition 5.8. *If* $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$ *and*

$$\inf\{a_{t_i} : i \in I_+(a_t), t \in T\} = \gamma > 0,$$

then the solution set F *has a global error bound on* $\mathbb{R}^n_+$ *with bound* $k \leq \gamma^{-1}$.

Proof. Take $y \in \mathbb{R}^n_+$. If $y \in F$ there is nothing to prove; assume that $y \notin F$, then

$$\begin{aligned}
d(y, F) &= \sup_{t\in T} d(y, F_t) \\
&= \sup_{t\in T} \min_{i\in I_+(a_t)} \left(\frac{b_t}{a_{t_i}} - y_i\right)^+ \\
&= \sup_{t\in T} \min_{i\in I_+(a_t)} \left(\frac{b_t}{a_{t_i}} - y_i\right) \\
&= \sup_{t\in T} \min_{i\in I_+(a_t)} \frac{1}{a_{t_i}}(b_t - a_{t_i}y_i) \\
&\leq \frac{1}{\gamma}\sup_{t\in T} \min_{i\in I_+(a_t)} (b_t - a_{t_i}y_i) \\
&= \frac{1}{\gamma}\sup_{t\in T}\left(b_t - \max_{i\in I_+(a_t)} a_{t_i}y_i\right) \\
&= \frac{1}{\gamma}\sup_{t\in T}(b_t - \langle a_t, y\rangle) \\
&= \frac{1}{\gamma} d(b, \mathcal{M}(y)).
\end{aligned}$$

□

Finally, we show that, under this last condition on the coefficients, the stability of σ with respect to the consistency yields the metric regularity, i.e., the reverse implication of (5.1).

Theorem 5.9. *If* $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \operatorname{int}\Theta_c$ *and*

$$\inf\{a_{t_i} : i \in I_+(a_t), t \in T\} = \gamma > 0,$$

then

$$d\left(y, \mathcal{M}^{-1}\left(b^1\right)\right) \leq \frac{1}{\gamma} d\left(b^1, \mathcal{M}(y)\right),$$

for any $b^1 \in \mathbb{R}^T_+$ *and all* $y \in \mathbb{R}^n_+$. *In particular,* $\mathcal{M}$ *is metrically regular at any* $y \in F$ *for* b.

Proof. Our present assumption leads, by virtue of Proposition 4.3, to the existence of $\delta > 0$ such that $\|a_t\| \geq \delta$, for every $t \in T$. If $b^1 \in \mathbb{R}^T_+$ is such that $\mathcal{M}^{-1}\left(b^1\right) = \varnothing$, then the Remark 5.4 gives that both terms are $+\infty$. If $b^1 \in \mathbb{R}^T_+$ is such that its associated system $\sigma_1 = \{\langle a_t, x\rangle \geq b^1_t,\ t \in T\}$ is consistent, an application of Proposition 5.8 to σ_1 yields the result. □

In view of Proposition 5.2 we get the following characterization for finite systems.

Corollary 5.10. *If* T *is finite and* σ *is consistent, then* $\sigma \in \operatorname{int}\Theta_c$ *if and only if* $\mathcal{M}$ *is metrically regular at any* $z \in F$ *for* b.

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