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SEMICONTINUITY OF THE SOLUTION MAPPING OF ε-VECTOR EQUILIBRIUM PROBLEM*

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Abstract: In this paper, we study the ε-vector equilibrium problem (ε-VEP). Several existence results for ε-VEP were established. We also investigate continuities of the solution mapping of ε-VEP. In particular, a result concerning the lower semicontinuity of the solution mapping of ε-VEP is presented.

Key words: *ε-vector equilibrium problem, vector equilibrium problem, C-quasiconvexity, C-continuity, C-compactness*

Mathematics Subject Classification: *49J40, 49K40, 90C31*

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1 Introduction

The concept of ε-solution is very adaptable for the cases that the feasible regions are non-convex or non closed sets. In fact, the original problems are the special cases of ε- approximate problems such as the famous Ekland's variational principle, which is an ε-solution rule for optimization problem. The concept of ε-solution also is the basis of numerical computing, e.g., stability, well-posedness and so on. The interesting example of the ε-equilibrium problem is the generalized game for ε-strategy in economics.

The notion of approximate solutions adapted in this paper follows from the concept of ε-efficiency originally introduced in multiple objective programming by Loridan [16] in 1984. Two years later, White [27] introuduced six alternative definitions of ε-efficent solutions and established the relationships between these concepts. ε-efficiency for more general vector optimization problems are considered in [19, 21]. For the concept of ε-solution for (vector) variational inequality problem, Tammer [22, 23] studied the existence and the generalization of Ekeland's variational priciple. Since vector equilibrium problem is a very general mathematical model covering vector optimization, vector variational inequalities and so on as special cases, the main motivation of this paper is to study the behavior of the solution map of the parametric vector equlibrium problems by following the idea of Loridan [16].

Let X be a real Hausdorff topological vector space and Z a real topological vector space. A set $C \subset Z$ is said to be a cone if $\lambda x \in C$ for any $\lambda \geq 0$ and for any $x \in C$. The cone C is called solid if it has nonempty interior, i.e., $\operatorname{int} C \neq \emptyset$. A cone C is said to be pointed if $C \cap (-C) = \{\theta_Z\}$ where θ_Z denotes the zero vector of Z. For any set $A \subset Z$, we let $\operatorname{bd} A$ and $\operatorname{cl} A$ denote the boundary and closure of A, respectively. Also, we denote A^c the

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complement of the set A. For any set A of a real vector space, the convex hull of A, denoted by $\operatorname{co} A$, is the smallest convex set containing A.

Let $f : X \times X \to Z$ and $\emptyset \neq K \subset X$. For fixed $\varepsilon \in \operatorname{int} C$, the ε-vector equilibrium problem (ε-VEP, for short) is to find $x \in K$ such that

$$(\varepsilon\text{-VEP}) \qquad f(x, y) + \varepsilon \notin -\operatorname{int} C, \text{ for all } y \in K.$$

Let $\Omega : \operatorname{int} C \to 2^X$ be the set-valued mapping such that $\Omega(\varepsilon)$ is the solutions set of ε-VEP for $\varepsilon \in \operatorname{int} C$, i.e.,

$$\Omega(\varepsilon) = \{x \in K : f(x, y) + \varepsilon \notin -\operatorname{int} C, \text{ for all } y \in K\}.$$

We remark that ε-VEP is closely related to the vector equilibrium problem (VEP) which is to find $x \in \operatorname{cl} K$ such that

$$(\text{VEP}) \qquad f(x, y) \notin -\operatorname{int} C, \text{ for all } y \in K.$$

Let S denote the solution set of VEP, i.e.,

$$S = \{x \in \operatorname{cl} K : f(x, y) \notin -\operatorname{int} C \text{ for all } y \in K\}.$$

If K is closed, then VEP becomes the ordinary vector equilibrium problem. The classical vector equilibrium problems and its extensions have been extensively studied in the literature. See, [1, 2, 3, 4, 6, 7, 8, 14, 20, 28] and the references therein. We may regard solutions of ε-VEP as approximate solutions of the problem VEP. We remark that $S \neq \emptyset$ does not imply $\Omega(\varepsilon) \neq \emptyset$, for all $\varepsilon \in \operatorname{int} C$.

Example 1.1. Let $X = \mathbb{R}$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $K = \left(0, \frac{\pi}{2}\right)$. Suppose that $f : X \times X \to Z$ is defined by

$$f(x, y) = \begin{pmatrix} -|x \cdot \tan y| \\ -|x^2 \cdot \tan y| \end{pmatrix}.$$

Then $0 \in S$ but $\Omega(\varepsilon) = \emptyset$ for each $\varepsilon > 0$.

The purpose of this paper is to establish relationship between the sets $\Omega(\varepsilon)$ and S for $\varepsilon \in \operatorname{int} C$. We also investigate continuities of the solution mapping $\Omega : \operatorname{int} C \to 2^X$. In particular, a result concerning the lower semicontinuity of Ω is presented.

We observe that our results in this paper can be employed to study the behavior of solution maps of parametric vector optimization, parametric vector variational inequalities, parametric generalized games and so on.

2 Preliminaries

Definition 2.1 (C-quasiconvexity, [10, 17, 24]). Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-quasiconvex on K* if it satisfies one of the following two equivalent conditions:

(i) for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \in z - C, \text{ for all } z \in C(f(x_1), f(x_2)),$$

where $C(f(x_1), f(x_2))$ is the set of upper bounds of $f(x_1)$ and $f(x_2)$, i.e.,

$$C(f(x_1), f(x_2)) := \{z \in Z : z \in f(x_1) + C \text{ and } z \in f(x_2) + C\}.$$

(ii) for each $z \in Z$,
$$A(z) := \{x \in K : f(x) \in z - C\}$$
is convex or empty.

Definition 2.2 (C-proper quasiconvexity, [24]). Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-properly quasiconvex on K* if for every $x_1, x_2 \in K$ and $\lambda \in [0, 1]$ we have either

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_1) - C,$$

or

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_2) - C;$$

f is said to be *strictly C-properly quasiconvex on K* if for every $x_1, x_2 \in K$ and $\lambda \in (0, 1)$ we have either

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_1) - \operatorname{int} C,$$

or

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_2) - \operatorname{int} C.$$

Usually we define cone concavity of f by the fact that $-f$ is cone convex. However the following definition for cone concavity is also natural.

Definition 2.3 (C-quasiconcavity). Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-quasiconcave on K* if for each $z \in Z$, the following set:

$$\{x \in K : f(x) \notin z - \operatorname{int} C\}$$

is convex or empty; f is said to be *strictly C-quasiconcave on K* if for each $z \in Z$, the following set:

$$\{x \in K : f(x) \notin z - \operatorname{cl} C\}$$

is convex or empty.

Proposition 2.4. *Let X be a nonempty compact subset of a real topological vector space and Z a real topological vector space with a proper solid convex cone C. Suppose that $f : X \to Z$ is $(-C)$-properly quasiconvex on X. Then f is C-quasiconcave on X.*

Proof. Let $z \in Z$ and $x_1, x_2 \in \{x \in X : f(x) \notin z - \operatorname{int} C\}$. Since f is $(-C)$-properly quasiconvex on X, for each $x' \in [x_1, x_2]$

$$f(x') \in \{f(x_1), f(x_2)\} + \operatorname{int} C.$$

Hence $f(x') \notin z - \operatorname{int} C$. Therefore $\{x \in X : f(x) \notin z - \operatorname{int} C\}$ is convex on X, i.e., f is C-quasiconcave on X. □

Definition 2.5 (C-continuity, [17, 25]). Let X be a topological space, and Z a topological vector space with a partial ordering defined by a solid pointed convex cone C. Suppose that f is a vector-valued function from X to Z. Then, f is said to be *C-continuous at $x \in X$* if it satisfies one of the following two equivalent conditions:

(i) For any neighbourhood $V_{f(x)} \subset Z$ of $f(x)$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in V_{f(x)} + C$ for all $u \in U_x$.

(ii) For any $k \in \operatorname{int} C$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in f(x) - k + \operatorname{int} C$ for all $u \in U_x$.

Moreover a vector-valued function f is said to be *C-continuous in X* if f is C-continuous at every x on X.

Remark 2.6. Whenever $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, C-continuity and $(-C)$-continuity are the same as ordinary lower and upper semicontinuity, respectively. In [25, Definition 2.1], C-continuous function is called C-lower semicontinuous function, and $(-C)$-continuous function is called C-upper semicontinuous function.

Proposition 2.7 ([24, Proposition 2.1]). *Let X be a topological space, and Z a topological vector space with a partial ordering defined by a solid pointed convex cone C. Suppose that f is a vector-valued function from X to Z. Then f is C-continuous on X if and only if for each $z \in Z$, $f^{-1}(z + \operatorname{int} C)$ is an open subset of X.*

Definition 2.8 ([5]). Let X and Y be two topological spaces, $T : X \to 2^Y$ a set-valued mapping.

(i) T is said to be upper semicontinuous (u.s.c. for short) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $z \in U$, $T(z) \subset V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) T is said to be *lower semicontinuous* (l.s.c. for short) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $z \in U$, $T(z) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) T is said to be *continuous* at $x \in X$ if $T(x)$ is both u.s.c. and l.s.c.; T is said to be continuous on X if it is both u.s.c. and l.s.c. at each $x \in X$.

From [11, 18], we obtain the following lemma.

Lemma 2.9. *Let X and Y be two topological spaces, $T : X \to 2^Y$ a multivalued mapping. If for any $x \in X$, $T(x)$ is compact, then T is u.s.c. on X if and only if for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \to x$ and for every $y_\alpha \in T(x_\alpha)$, there exist $y \in T(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$, such that $y_\beta \to y$.*

Definition 2.10 (C-compactness, [17]). Let C be a nonempty convex cone in a Hausdorff topological space Z. We say $E \subset Z$ is C-compact if any cover of E of the form

$$\{\mathcal{U}_\alpha + C : \alpha \in I, \mathcal{U}_\alpha \text{ are open}\}$$

admits a finite subcover.

Lemma 2.11 ([17, Theorem 7.2]). *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that f is a vector-valud function from X to Z. If f is C-continuous, then $\bigcup_{x \in X}\{f(x)\}$ is C-compact.*

Definition 2.12 (KKM-map). Let X be a topological vector space, and K a nonempty subset of X. Suppose that F is a multifunction from K to 2^X. Then F is said to be a *KKM-map*, if

$$\operatorname{co}\{x_1,\ldots,x_n\}\subset\bigcup_{i=1}^{n}F(x_i)$$

for each finite subset $\{x_1,\ldots,x_n\}$ of X.

Remark 2.13. Obviously if F is a KKM-map, then $x\in F(x)$ for each $x\in K$.

Lemma 2.14 ([9]). *Let X be a Hausdorff topological vector space, and K a nonempty subset of X, and let G be a multifunction from K to 2^X. Suppose that G is a KKM-map and that $G(x)$ is a closed subset of X for each $x\in K$. If $G(\hat{x})$ is compact for at least one $\hat{x}\in K$, then $\bigcap_{x\in K}G(x)\neq\emptyset$.*

Proposition 2.15 ([26]). *Let Z be a real topological vector space. Suppose that C is a solid pointed convex cone in Z. Then $\operatorname{cl}C+\operatorname{int}C=\operatorname{int}C$.*

Proposition 2.16. *Let Z be a real topological vector space, A a subset of Z, and C a solid convex cone in Z. If $A\cap(-\operatorname{int}C)=\emptyset$, then*

$$(A+\operatorname{cl}C)\cap(-\operatorname{int}C)=\emptyset.$$

Proof. Suppose to the contrary that there exists $z\in(A+\operatorname{cl}C)\cap(-\operatorname{int}C)$. Then there exist $a\in A$, $c'\in\operatorname{cl}C$ and $c\in\operatorname{int}C$ such that $z=a+c'=-c$. Hence $a=-(c'+c)\in-\operatorname{int}C$, by Proposition 2.15. This contradicts to the fact that $A\cap(-\operatorname{int}C)=\emptyset$. □

Proposition 2.17. *Let Z be a real topological vector space and C a solid pointed convex cone in Z with $k\in\operatorname{int}C$. Then the following properties hold:*

(i) *for every $z\in Z$ there exists $t\in\mathbb{R}$ such that $z\in t\cdot k+\operatorname{int}C$;*

(ii) *for every $z\in\operatorname{int}C$ there exists $t>0$ such that $z-t\cdot k\in\operatorname{int}C$.*

Proof. (i). Let $z\in Z$. $-k+\operatorname{int}C$ is a neighborhood of $theta_Z$. Since Z is a topological vector space, each neighborhood of θ_Z is absorbing. Hence there exists $\alpha>0$ such that $z\in\alpha(-k+\operatorname{int}C)$, i.e., $z\in(-\alpha\cdot k+\operatorname{int}C)$.

(ii). Let $z\in\operatorname{int}C$. Then there exists a neighborhood U of θ_Z such that $z+u\subset\operatorname{int}C$. Since Z is t.v.s., there exists $\alpha>0$ such that $k\in\alpha\cdot U$. Hence $z-\dfrac{1}{\alpha}\cdot k\in(z+U)\subset\operatorname{int}C$. □

3 Main Results

In this section, we will establish several results for ε-vector equilibrium problems. First we derive that $\Omega(\varepsilon)$ is not empty for $\varepsilon\in\operatorname{int}C$ under suitable conditions.

Theorem 3.1. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C\subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valud function from $X\times X$ to Z. Also we assume that the following conditions:*

(i) $S':=\{x\in\operatorname{cl}K : f(x,y)\notin-\operatorname{int}C$ *for all* $y\in\operatorname{cl}K\}\neq\emptyset$*;*

(ii) $\mathrm{cl}\, K$ *is compact;*

(iii) f *is C-continuous on $X \times X$.*

Then ε-VEP has at least one solution for each $\varepsilon \in \mathrm{int}\, C$.

Proof. Let $\varepsilon \in \mathrm{int}\, C$ and $x \in S'$. Then by condition (iii), for each $y \in \mathrm{cl}\, K$ there are neighborhoods $\mathcal{U}_y$ of x and $\mathcal{V}_y$ of y such that

$$f(u,v) \in (f(x,y) - \varepsilon) + \mathrm{int}\, C \text{ for all } (u,v) \in \mathcal{U}_y \times \mathcal{V}_y.$$

Since $\bigcup_{y \in \mathrm{cl}\, K} V_y \supset \mathrm{cl}\, K$ and $\mathrm{cl}\, K$ is compact, we can choose $y_i \in \mathrm{cl}\, K,\ i = 1, \ldots, n$, such that $\bigcup_{i=1}^{n} V_{y_i} \supset \mathrm{cl}\, K$. Then for $\mathcal{U} := \bigcap_{i=1}^{n} \mathcal{U}_{y_i}$, we have

$$f(u,y) \in \bigcup_{i=1}^{n} ((f(x,y_i) - \varepsilon) + \mathrm{int}\, C) \ \text{ for all } u \in \mathcal{U} \text{ and } y \in \mathrm{cl}\, K.$$

Hence

$$f(u,y) + \varepsilon \in \bigcup_{i=1}^{n} (f(x,y_i) + \mathrm{int}\, C) \text{ for all } u \in \mathcal{U} \text{ and } y \in \mathrm{cl}\, K.$$

Since $x \in S'$ and $y_1, \ldots, y_n \in \mathrm{cl}\, K$ we have

$$(f(x,y_i) + \mathrm{int}\, C) \cap (-\mathrm{int}\, C) = \emptyset, \text{ for all } i = 1, \ldots, n,$$

from which it follows that

$$\left(\bigcup_{i=1}^{n} (f(x,y_i) + \mathrm{int}\, C) \right) \cap (-\mathrm{int}\, C) = \emptyset.$$

Consequently,

$$f(u,y) + \varepsilon \notin -\mathrm{int}\, C \text{ for all } u \in \mathcal{U} \text{ and } y \in \mathrm{cl}\, K.$$

Moreover $K \cap \mathcal{U} \neq \emptyset$ because of $x \in \mathrm{cl}\, K$. Let $\bar{x} \in K \cap \mathcal{U}$. Then $f(\bar{x}, y) + \varepsilon \notin -\mathrm{int}\, C$ for all $y \in \mathrm{cl}\, K$. In particular, $\bar{x} \in \Omega(\varepsilon)$. Therefore the problem ε-VEP has at least one solution. □

Example 3.2. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} \left(\frac{y}{x}, -1\right) & \text{if } x \geq y > 0, \\ \left(\frac{x}{y}, -2\right) & \text{if } y > x > 0, \\ (x+y, -3) & \text{if } 0 > x + y, \\ (0, -4) & \text{otherwise,} \end{cases}$$

$C = \mathbb{R}^2_+$ and $K = (-1, 1)$. Then $1 \in S'$, i.e., $S' \neq \emptyset$, $\mathrm{cl}\, K$ is compact, and f is C-continuous on $\mathbb{R} \times \mathbb{R}$. Thus ε-VEP has at least one solution for each $\varepsilon > 0$ by Theorem 3.1. Actually, $\hat{x} = 1 - \frac{\varepsilon}{2}$ is a solution of ε-VEP.

Example 3.3. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined by $f(x, y) = \begin{pmatrix} -1 \\ -|x-y| \end{pmatrix}$, $K = [-1, 1]$ and $C = \mathbb{R}^2_+$. Then $S' = \emptyset$ and also for each $\varepsilon \in (0, 1)$, $\Omega(\varepsilon) = \emptyset$. We observe that we need not only conditions (ii) and (iii) but also (i) in Theorem 3.1.

Corollary 3.4. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X and that f is a vector-valud function from $X \times X$ to Z. Also we assume that the following conditions:*

(i) $S := \{x \in \operatorname{cl} K : f(x, y) \notin -\operatorname{int} C$ *for all* $y \in K\} \neq \emptyset$;

(ii) $\operatorname{cl} K$ *is compact;*

(iii) f *is C-continuous on $X \times X$;*

(iv) $f(x, \cdot)$ *is $(-C)$-continuous on* $\operatorname{bd} K$ *for some* $x \in S$.

Then ε-VEP has at least one solution for each $\varepsilon \in \operatorname{int} C$.

Proof. Let $\hat{x} \in S$ which satisfies condition (iv). Then for every $y \in K$,

$$f(\hat{x}, y) \notin -\operatorname{int} C. \tag{3.1}$$

Suppose to the contrary that there exists $\hat{y} \in \operatorname{bd} K$ such that

$$f(\hat{x}, \hat{y}) \in -\operatorname{int} C.$$

By condition (iv), $f(\hat{x}, \cdot)$ is $(-C)$-continuous at $\hat{y} \in \operatorname{bd} K$. Hence there exists a neighborhood $\mathcal{V}$ of $\hat{y}$ such that

$$f(\hat{x}, v) \in \left(\frac{f(\hat{x}, \hat{y})}{2} - \operatorname{int} C \right), \text{ for all } v \in \mathcal{V}.$$

Because of $\hat{y} \in \operatorname{bd} K$, $\mathcal{V} \cap K \neq \emptyset$. Hence there exists $y' \in \mathcal{V} \cap K \neq \emptyset$ such that

$$f(\hat{x}, y') \in \left(\frac{f(\hat{x}, \hat{y})}{2} - \operatorname{int} C \right) \subset -\operatorname{int} C.$$

This contradicts to (3.1). Therefore for each $y \in \operatorname{bd} K$,

$$f(\hat{x}, y) \notin -\operatorname{int} C,$$

i.e., for every $y \in \operatorname{cl} K$,

$$f(\hat{x}, y) \notin -\operatorname{int} C.$$

Hence $\hat{x} \in S'$, i.e., $S' \neq \emptyset$. $\square$

Now the result follows from Theorem 3.1.

Theorem 3.5. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X and that f is a vector-valud function from $X \times X$ to Z with $f(x, x) \notin -\operatorname{int} C$ for all $x \in X$. Also we assume that the following conditions:*

(i) $\operatorname{cl} K$ *is compact convex;*

(ii) *$f(x, \cdot)$ is C-quasiconvex on X for each $x \in X$;*

(iii) *$f(\cdot, y)$ is $(-C)$-continuous on X for each $y \in X$;*

(iv) *f is C-continuous on $X \times X$.*

Then the problem ε-VEP has at least one solution, i.e., $\Omega(\varepsilon)$ is nonempty for each $\varepsilon \in \operatorname{int} C$.

Proof. Let for each $y \in \operatorname{cl} K$,

$$G(y) := \{x \in \operatorname{cl} K : f(x, y) \notin -\operatorname{int} C\}.$$

First, we show that $G(y)$ is a KKM-map. Suppose to the contrary that there exists $\alpha_i \in [0, 1]$, $x_i \in \operatorname{cl} K$ $(i = 1, \dots, n)$ such that

$$\sum_{i=1}^{n} \alpha_i x_i = x \notin \bigcup_{i=1}^{n} G(x_i).$$

Then

$$f(x, x_i) \in -\operatorname{int} C, \; i = 1, \dots, n.$$

Moreover $x \in \operatorname{cl} K$ because of the convexity of $\operatorname{cl} K$. Hence by condition (ii),

$$f(x, x) = f(x, \sum_{i=1}^{n} \alpha_i x_i) \in -\operatorname{int} C,$$

which contradicts to the fact that $f(x, x) \notin -\operatorname{int} C$ for all $x \in X$.

Next by condition (iv) and Proposition 2.7, $A := \{y \in X : f(x, y) \in -\operatorname{int} C\}$ is an open subset of X. Then $G(y) = \operatorname{cl} K \cap (A^c)$ is a closed subset of X. Hence $G(y)$ is closed for each $y \in K$. Also $\operatorname{cl} K$ is compact. Hence $G(y)$ is compact for each $y \in K$. Thus we can apply Lemma 2.14, and so

$$S' = \bigcap_{y \in K} G(y) \neq \emptyset.$$

Hence by Theorem 3.1, the problem ε-VEP has at least one solution. □

Remark 3.6. We observe that the condition that f is both C-continuous and $(-C)$-continuous doesn't imply that f is continuous. See, e.g., [17, Theorem 5.3 and Remark 5.4].

Remark 3.7. Theorem 3.5 is only one of the variations of Theorem 3.1. Using various existance results for VEP, we may obtain conditions of nonemptyness of S'. Then we can derive another existence results for ε-VEP easily. If we assume closedness of C, we may utilize existence results for generalized VEP in [4, 12, 13, 15].

Next, we show that the solution mapping Ω of ε-VEP is upper semicontinuous on $\operatorname{int} C$ under some suitable conditions.

Theorem 3.8. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valued function from $X \times X$ to Z. Also we assume that the following conditions:*

(i) *K is compact;*

(ii) *$f(\cdot, y)$ is $(-C)$-continuous on X for each $y \in X$;*

(iii) $\Omega(\varepsilon)$ *is nonempty for each* $\varepsilon \in \operatorname{int} C$.

Then Ω *is u.s.c. on* $\operatorname{int} C$.

Proof. Let $\varepsilon_\lambda \to \varepsilon$ and $x_\lambda \in \Omega(\varepsilon_\lambda)$. Since K is compact, we can assume, without loss of generality, $x_\lambda \to x \in K$. Suppose to the contrary that $x \notin \Omega(\varepsilon)$. Then there exists $y \in K$ such that $f(x,y) + \varepsilon \in -\operatorname{int} C$. Since Z is a topological vector space, there exists a neighborhood U of θ_Z such that $f(x,y)+\varepsilon+U+U \subset -\operatorname{int} C$. Then $f(x,y)+\varepsilon+U+U-\operatorname{int} C \subset (-\operatorname{int} C - \operatorname{int} C) \subset -\operatorname{int} C$. Because of $\varepsilon_\lambda \to \varepsilon$, $x_\lambda \to x$, and condition (ii), there exists a $\hat{\lambda}$ such that for every $\lambda \geq \hat{\lambda}$, $f(x_\lambda, y)+\varepsilon_\lambda \in -\operatorname{int} C$. This contradicts the fact that $x_\lambda \in \Omega(\varepsilon_\lambda)$. Hence $x \in \Omega(\varepsilon)$. Therefore by Lemma 2.9, $\Omega(\varepsilon)$ is u.s.c. on $\operatorname{int} C$. □

Corollary 3.9. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valued function from $X \times X$ to Z with $f(x,x) \notin -\operatorname{int} C$ for all $x \in X$. Also we assume that the following conditions:*

(i) *K is compact convex;*

(ii) *$f(x,\cdot)$ is C-quasiconvex on X for each $x \in X$;*

(iii) *$f(\cdot,y)$ is $(-C)$-continuous on X for each $y \in X$;*

(iv) *f is C-continuous on $X \times X$.*

Then Ω *is u.s.c. on* $\operatorname{int} C \cup \{\theta_Z\}$.

Proof. The result follows from Theorems 3.5 and 3.8. □

We now establish that the solution mapping Ω of ε-VEP is lower semicontinuous on $\operatorname{int} C$ under suitable assumptions.

Theorem 3.10. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valud function from $X \times X$ to Z. Also we assume that the following conditions:*

(i) *K is compact convex;*

(ii) *$f(x,\cdot)$ is C-continuous on X for each $x \in X$;*

(iii) *$f(\cdot,y)$ is strictly C-quasiconcave on K for each $y \in K$;*

(iv) *$\Omega(\varepsilon)$ is nonempty for each $\varepsilon \in \operatorname{int} C$.*

Then Ω *is l.s.c. on* $\operatorname{int} C$.

Proof. Let $\varepsilon \in \operatorname{int} C$. Let $\mathcal{V}$ be an open set of X with $\mathcal{V} \cap \Omega(\varepsilon) \neq \emptyset$. Suppose that $x \in \mathcal{V} \cap \Omega(\varepsilon)$ and that $\hat{x} \in \Omega(\alpha \cdot \varepsilon)$, where $\alpha \in (0,1)$. We choose $x' \in (x, \hat{x}) \cap \mathcal{V}$, where (a,b) denotes the line segment between a and b.

Obviously $\hat{x} \in \Omega(\varepsilon)$. Because of condition (iii),

$$f(x', v) \notin -\varepsilon - \operatorname{cl} C, \text{ for all } v \in X.$$

Since $-\varepsilon - \operatorname{cl} C$ is a closed set, for each $v \in X$ there exist a positive number $t_v > 0$ such that

$$f(x', v) - t_v \cdot \varepsilon \notin -\varepsilon - \operatorname{cl} C.$$

Because of conditions (i) and (ii), by Lemma 2.11, $\bigcup_{v\in X} f(x',v)$ is C-compact. Clearly $f(x',v) - t_v \cdot \varepsilon + \operatorname{int} C$ is a neighborhood of $f(x',v)$ and

$$\bigcup_{v\in X} \{f(x',v) - t_v \cdot \varepsilon + \operatorname{int} C\} \supset \bigcup_{v\in X} f(x',v).$$

Hence there exist $v_1, \ldots, v_n \in X$ such that

$$\bigcup_{i=1}^{n} \{f(x',v_i) - t_{v_i} \cdot \varepsilon + \operatorname{int} C\} \supset \bigcup_{v\in X} f(x',v). \tag{3.2}$$

Since $f(x',v_i) - t_{v_i} \cdot \varepsilon \notin -\varepsilon - \operatorname{cl} C,\ i = 1, \ldots, n$, there exist corresponding numbers $t^1, \ldots, t^n \in (0,1)$ such that

$$f(x',v_i) - (t_{v_i} + t^i) \cdot \varepsilon \notin -\varepsilon - \operatorname{cl} C,\ i = 1, \ldots, n.$$

Let $\tau = \min\{t_1, \ldots, t_n\}$. Then by Proposition 2.16,

$$\left(\bigcup_{i=1}^{n} f(x',v_i) - (t_{v_i} + \tau) \cdot \varepsilon \right) \bigcap -\varepsilon - \operatorname{cl} C = \emptyset.$$

Because of (3.2),

$$f(x',v) - \tau \cdot \varepsilon \in \left(\bigcup_{i=1}^{n} f(x',v_i) - (t_{v_i} + \tau) \cdot \varepsilon \right), \text{ for all } v \in X.$$

Accordingly

$$f(x',v) - \tau \cdot \varepsilon \notin -\varepsilon - \operatorname{int} C, \text{ for all } v \in X,$$

i.e.,

$$x' \in \Omega((1-\tau) \cdot \varepsilon).$$

Therefore $x' \in \Omega(\varepsilon')$ for all $\varepsilon' \in (1-\tau)\varepsilon + \operatorname{int} C$. Hence Ω is l.s.c. on $\operatorname{int} C$. □

Theorem 3.11. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valud function from $X \times X$ to Z. Also we assume that the following conditions:*

(i) *K is compact convex;*

(ii) *$f(x,\cdot)$ is C-continuous on X for each $x \in X$;*

(iii) *$f(\cdot,y)$ is strictly $(-C)$-properly quasiconvex on K for each $y \in K$;*

(iv) *$\Omega(\varepsilon)$ is nonempty for each $\varepsilon \in \operatorname{int} C$.*

Then Ω is l.s.c. on $\operatorname{int} C$.

Proof. Let $\hat{\varepsilon} \in \operatorname{int} C$ be arbitrary but fixed and $\mathcal{V}$ be an open set with $\mathcal{V} \cap \Omega(\hat{\varepsilon}) \neq \emptyset$. Let $\hat{x} \in \mathcal{V} \cap \Omega(\hat{\varepsilon})$. Then we show there exist $\bar{x} \in \mathcal{V}$ and $\mu > 0$ such that for all $\varepsilon \in (1-\mu)\hat{\varepsilon} + \operatorname{int} C$, we have

$$f(\bar{x},y) + \varepsilon \notin -\operatorname{int} C, \qquad \text{for all } y \in K.$$

We note that $(1-\mu)\hat{\varepsilon} + \operatorname{int} C$ is a neighborhood of $\hat{\varepsilon}$.

First we select $\bar{x} \in \mathcal{V}$ in the following way. Let $\alpha \in (0,1),\ x_0 \in \Omega(\alpha\hat{\varepsilon})$ and

$$\bar{x} \in \mathcal{V} \bigcap \{x \in K : x = \lambda \hat{x} + (1-\lambda)x_0,\ 0 < \lambda < 1\}.$$

Next we find corresponding $\mu \in (0, 1-\alpha)$. Because of the way in selecting $\bar{x}$, we have

$$f(\bar{x}, y) \in f(\hat{x}, y) + \operatorname{int} C,$$

or

$$f(\bar{x}, y) \in f(x_0, y) + \operatorname{int} C.$$

Let $K' := \{y \in K : f(\bar{x}, y) \notin f(x_0, y) + \operatorname{int} C\}$. By condition (ii) and Proposition 2.7, $A := \{y \in X : f(\bar{x}, y) \in f(x_0, y) + \operatorname{int} C\}$ is an open set of X. Then $A^{\mathrm{c}} = \{y \in X : f(\bar{x}, y) \notin f(x_0, y) + \operatorname{int} C\}$ is a closed set of X. Hence $K' = (K \cap A^{\mathrm{c}})$ is a closed set, i.e., compact set. Because of Proposition 2.17, we have $f(\bar{x}, v) \in f(\hat{x}, v) + \operatorname{int} C$ for all $v \in K'$. Thus, for each $v \in K'$ there exists $\mu_v \in (0, 1-\alpha)$ such that

$$f(\bar{x}, v) \in f(\hat{x}, v) + \mu_v \cdot \hat{\varepsilon} + \operatorname{int} C.$$

Hence

$$\mathfrak{M}^{\bar{x}} \subset \mathfrak{M}^{\hat{x}} + \bigcup_{v \in K'} (\mu_v \cdot \hat{\varepsilon} + \operatorname{int} C),$$

where $\mathfrak{M}^{\bar{x}}$ and $\mathfrak{M}^{\hat{x}}$ denote $\bigcup_{v \in K'} \{f(\bar{x}, v)\}$ and $\bigcup_{v \in K'} \{f(\hat{x}, v)\}$, respectively. Because of compactness of K' and condition (ii), $\mathfrak{M}^{\bar{x}}$ is C-compact by Lemma 2.11. In addition,

$$\bigcup_{v \in K'} \{\mu_v \cdot \hat{\varepsilon} + \operatorname{int} C\} = \bigcup_{v \in K'} \{\mu_v \cdot \hat{\varepsilon}\} + \operatorname{int} C = \bigcup_{v \in K'} \{\mu_v \cdot \hat{\varepsilon}\} + \operatorname{int} C + \operatorname{int} C,$$

and $\mathfrak{M}^{\hat{x}} + \bigcup_{v \in K'} (\mu_v \cdot \hat{\varepsilon} + \operatorname{int} C)$ is an open covering of $\mathfrak{M}^{\bar{x}}$. Hence we can choose a finite subset $\{\mu_{v_1}, \dots, \mu_{v_n}\} \subset \{\mu_v : v \in K'\}$ such that

$$\mathfrak{M}^{\bar{x}} \subset \mathfrak{M}^{\hat{x}} + \bigcup_{i=1}^{n} (\mu_{v_i} \cdot \hat{\varepsilon} + \operatorname{int} C).$$

Putting $\mu = \min\{\mu_{v_1}, \dots, \mu_{v_n}\}$, we have

$$\mathfrak{M}^{\bar{x}} \subset \mathfrak{M}^{\hat{x}} + \mu \cdot \hat{\varepsilon} + \operatorname{int} C.$$

Hence

$$\mathfrak{M}^{\bar{x}} - \mu \cdot \hat{\varepsilon} \subset \mathfrak{M}^{\hat{x}} + \operatorname{int} C. \tag{3.3}$$

Because of $\hat{x} \in \Omega(\hat{\varepsilon})$

$$\big(\mathfrak{M}^{\hat{x}} + \hat{\varepsilon}\big) \cap (-\operatorname{int} C) = \emptyset.$$

Hence by Proposition 2.16,

$$\big(\mathfrak{M}^{\hat{x}} + \hat{\varepsilon} + \operatorname{int} C\big) \cap (-\operatorname{int} C) = \emptyset.$$

Therefore by (3.3),

$$\big(\mathfrak{M}^{\bar{x}} + (1-\mu)\hat{\varepsilon}\big) \cap (-\operatorname{int} C) = \emptyset.$$

On the other hand for each $v \in (K \setminus K')$, $f(\bar{x}, v) \in f(x_0, v) + \operatorname{int} C$. Since $x_0 \in \Omega(\alpha\hat{\varepsilon})$, we have $f(\bar{x}, v) + \alpha\hat{\varepsilon} \notin -\operatorname{int} C$. Because of $\alpha < (1-\mu)$,

$$\left(\bigcup_{v \in (K \setminus K')} \{f(\bar{x}, v) + (1-\mu)\hat{\varepsilon}\} \right) \cap -\operatorname{int} C = \emptyset,$$

from which it follows that

$$\left(\bigcup_{v \in K} \{f(\bar{x}, v) + (1-\mu)\hat{\varepsilon}\} \right) \cap -\operatorname{int} C = \emptyset.$$

Let $\mathcal{U} = (1-\mu)\hat{\varepsilon} + \operatorname{int} C$. Then $\mathcal{U}$ is an open set containing $\hat{\varepsilon}$. For every $\varepsilon \in \mathcal{U}$,

$$\bigcup_{v \in K} \{f(\bar{x}, v) + (1-\mu)\hat{\varepsilon}\} + \operatorname{int} C \supset \bigcup_{v \in K} \{f(\bar{x}, v) + \varepsilon\}.$$

Therefore by Proposition 2.16,

$$\left(\bigcup_{v \in K} \{f(\bar{x}, v) + \varepsilon\} \right) \cap -\operatorname{int} C = \emptyset,$$

from which it follows $f(\bar{x}, v) + \varepsilon \notin -\operatorname{int} C$ for all $v \in K$, i.e., $\bar{x} \in \Omega(\varepsilon)$ for all $\varepsilon \in \mathcal{U}$. Hence Ω is l.s.c. at $\hat{\varepsilon}$. Since $\hat{\varepsilon}$ is arbitrary, Ω is l.s.c. on $\operatorname{int} C$. □

Corollary 3.12. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valud function from $X \times X$ to Z with $f(x, x) \notin -\operatorname{int} C$ for all $x \in X$. Also we assume that the following conditions:*

(i) *K is compact convex;*

(ii) *$f(x, \cdot)$ is C-quasiconvex on X for each $x \in X$;*

(iii) *$f(\cdot, y)$ is strictly $(-C)$-properly quasiconvex on K for each $y \in K$;*

(iv) *$f(\cdot, y)$ is $(-C)$-continuous on X for each $y \in X$*

(v) *f is C-continuous on $X \times X$.*

Then Ω is continuous on $\operatorname{int} C$.

Proof. The result follows from Theorems 3.5 and 3.11. □

Corollary 3.13. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valud function from $X \times X$ to Z. Also we assume that the following conditions:*

(i) *K is compact convex;*

(ii) *$f(\cdot, y)$ is $(-C)$-continuous on X for each $y \in X$;*

(iii) *f is C-continuous on $X \times X$;*

(iv) $f(\cdot, y)$ *is strictly* $(-C)$*-properly quasiconvex on* K *for each* $y \in K$*;*

(v) $\Omega(\varepsilon)$ *is nonempty for each* $\varepsilon \in \operatorname{int} C$.

Then Ω *is continuous on* $\operatorname{int} C$.

Proof. The result follows from Theorems 3.8 and 3.11. □

Finally we show that nonemptyness of the solutions sets of ε-VEP implies that there exists a solution to VEP under mild conditions.

Theorem 3.14. *Let* X *be a real Hausdorff topological vector space. Let* Z *be a real topological vector space with a solid pointed convex cone* $C \subset Z$. *Suppose that* K *is a nonempty subset of* X*, that* f *is a vector-valud function from* $X \times X$ *to* Z. *Also we assume that the following conditions:*

(i) $\operatorname{cl} K$ *is compact;*

(ii) $f(\cdot, y)$ *is* $(-C)$*-continuous on* X *for each* $y \in X$*;*

(iii) $\Omega(\varepsilon) \neq \emptyset$ *for all* $\varepsilon \in \operatorname{int} C$.

Then S *is nonempty.*

Proof. Let $\{\varepsilon_\lambda\} \subset \operatorname{int} C$, $\varepsilon_\lambda \to \theta_Z$, and $x_\lambda \in \Omega(\varepsilon_\lambda)$. Then by condition (i), without loss of generality, we assume $x_\lambda \to x$ and $x \in \operatorname{cl} K$. Suppose to the contrary that $f(x, y) \in -\operatorname{int} C$ for some $y \in K$. Then by condition (ii), there is a λ_0 such that for all $\lambda \geq \lambda_0$

$$f(x_\lambda, y) \in -\operatorname{int} C.$$

This contradicts to the fact that $x_\lambda \in \Omega(\varepsilon_\lambda)$. Hence $f(x, y) \notin -\operatorname{int} C$ for all $y \in K$ and thus $x \in S$ from which the result follows. □

We remark that from the proof of Theorem 3.5, one can see that Condition (iii) in Theorem 3.5 can be replaced by the condition that $\Omega(\varepsilon) \neq \emptyset$ for some net $\{\varepsilon_\lambda\} \subset \operatorname{int} C$ such that $\varepsilon_\lambda \to \theta_Z$.

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