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SEMISMOOTHNESS AND DIRECTIONAL SUBCONVEXITY OF FUNCTIONS

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Dedicated to Terry Rockafellar on the occasion of his seventieth birthday

Abstract: The relationships between semismoothness of a function and submonotonicity of its subdifferentials at some given point are studied. A notion of approximate starshapedness at that point is introduced and compared with these properties.

Key words: *generalized convexity, nonsmooth analysis, semismoothness, smoothness, starshaped function, subconvexity, subdifferential*

Mathematics Subject Classification: *49J52, 46N10, 46T20, 90C30*

1 Introduction

The present paper is devoted to the study of semismoothness of continuous or lower semicontinuous functions. Semismoothness is an important property, in particular for algorithms and inverse mapping theorems (see [33, 34, 48] and, for the vector-valued case, [8, 9, 49, 39, 47, 50, 51, 61, 62, 64]....) and optimality conditions ([7]). In particular, it is shown in [60] that the metric projection operator to the semidefinite cone is strongly semismooth; such a result provides a foundation for Newton's methods for second-order cone optimization ([9]) and semidefinite cone optimization ([61]). But, up to our knowledge, the study of semismoothness has been limited to the locally Lipschitzian case. Still, the case of lower semicontinuous functions is important in optimization theory: for instance indicator functions, eigenvalue functions and optimal value functions are of wide use. It is often a delicate matter to determine whether a marginal or performance function is locally Lipschitz or not (see [23, 46] for such a question). Thus, it may be useful to enlarge the framework for semismoothness to the case of continuous or lower semicontinuous functions on a Banach space. We also tackle the relationships between semismoothness of a function and submonotonicity of its subdifferentials, in the sense of Spingarn ([56]) and Rockafellar ([54]); this last property is also important for algorithms ([57]). In particular, we extend some results of [5, 16, 43, 56] to the case of non Lipschitzian functions. Among the functions which can be encompassed in the new class of semismoothness are the functions of the form $f = g + h$, where g is an arbitrary closed proper convex function and h is a finite-valued function of class C^1.

We illustrate our results in pointing out the links with a notion of approximate starshapedness at a given point x_0 which generalizes the notion of approximate convexity studied in [1, 21, 31, 37]. Starshaped sets and starshaped functions play an important role in various fields of mathematics (see [14, 20, 26, 45, 55] for a sample). The class of functions on X which are starshaped at some point $x_0 \in X$ is much larger than the class of convex functions; moreover it enjoys some properties (such as stability under infima for any subfamily of functions taking a given value at x_0) which are not shared by the class of convex functions. The class we study is still larger since we focus our attention on a property which is just approximate starshapedness. This property is deduced from starshapedness in a way which is similar to the passage from convexity to approximate convexity in the sense of [31]. The example of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x| - x^2$ with $x_0 := 0$ shows that there are approximately starshaped functions which are not starshaped. In Theorem 6.4 we show the striking result that a semismooth function is approximately starshaped.

Because our study is essentially limited to properties bearing on a specific point, the questions linked with integration are left apart although they are related with submonotonicity; see [22] and its references for such questions. A related study of regularity of sets and approximate convexity of sets is undertaken in [1, 13, 38]: among our aims is the wish to exhibit regularity criteria, i.e. conditions ensuring that some subdifferentials of a given function f coincide at some point. Nonsmooth analysis needs such unificating results.

$\boxed{2}$ Preliminaries

In the sequel, X is a Banach space and $\mathcal{F}(X)$ is the set of functions $f : X \to \mathbb{R} \cup \{+\infty\}$; $\mathcal{S}(X)$ denotes the set of lower semicontinuous functions $f \in \mathcal{F}(X)$ and $\mathcal{L}(X)$ stands for the class of locally Lipschitzian real-valued functions. If f is defined on a subset W of X, we extend it by $+\infty$ on $X \backslash W$. The open ball with center $x_0 \in X$ and radius $\rho > 0$ is denoted by $B(x_0, \rho)$; B_X and S_X stand for the closed unit ball and the unit sphere of X respectively. We recall that X is an *Asplund space* if every separable linear subspace of X has a separable dual. This class of spaces is important in nonsmooth analysis and convex analysis.

A *subdifferential* on a Banach space X will be here just a correspondence $\partial : \mathcal{F}(X) \times X \rightrightarrows X^*$ which assigns a subset $\partial f(x)$ of the dual space X^* of X to any $(f, x) \in \mathcal{F}(X) \times X$ in such a way that $\partial f(x)$ coincides with the subdifferential of convex analysis for a convex function f. However, we impose some limitations to this correspondence, so that in fact many usual properties are satisfied; moreover, for several statements, we require a mean value property. The simplest one is the Lebourg mean value theorem (see for instance [10, Thm 2.3.7], [2, Thm 19], valid for locally Lipschitzian functions.

For lower semicontinuous functions, a (more sophisticated) version can be given for a *reliable subdifferential* (for X), i.e. a subdifferential ∂ satisfying the following fuzzy property close to the basic fuzzy rule of [28] (see also [42, 65]):

(F) for any $f \in \mathcal{S}(X)$, $g \in \mathcal{L}(X)$ with g convex such that $f + g$ attains at x a local minimum and for any $\varepsilon > 0$, there exist $u, v \in B(x, \varepsilon)$, $u^* \in \partial f(u)$, $v^* \in \partial g(v)$ such that $|f(u) - f(x)| < \varepsilon$, $\|u^* + v^*\| < \varepsilon$.

We say that ∂ is *multi-reliable* (for X) if, for each $m \in \mathbb{N} \backslash \{0\}$, ∂ is reliable on X^m. That is the case for the Clarke–Rockafellar ([10, Thm 2.3.7], [53]), the Ioffe subdifferential ([28]) and the moderate subdifferential of Michel–Penot ([2, 32]) for any Banach space X and, when X is an Asplund space, for the Fréchet subdifferential and the limiting Fréchet subdifferential of Mordukhovich ([35, 65]).

Theorem 2.1 ([44]). *Let ∂ be a reliable subdifferential (for X). Then, ∂ is valuable (for X) in the sense that it satisfies the following property: given $f \in \mathcal{S}(X)$ finite at $\overline{x} \in X$, $\overline{y} \in X\backslash\{\overline{x}\}$ and $r \in \mathbb{R}$ such that $f(\overline{y}) \geq r$, there exists $u \in [\overline{x}, \overline{y})$ and sequences $(u_n) \to u$, (u_n^*) such that $u_n^* \in \partial f(u_n)$, $(f(u_n)) \to f(u)$,*

$$\liminf_n \langle u_n^*, \overline{y} - \overline{x} \rangle \geq r - f(\overline{x}), \tag{2.1}$$

$$\liminf_n \langle u_n^*, \frac{x - u_n}{\|x - u\|} \rangle \geq \frac{r - f(\overline{x})}{\|\overline{y} - \overline{x}\|} \qquad \forall x \in u + (0, \infty)(\overline{y} - \overline{x}). \tag{2.2}$$

In particular, when X is an Asplund space, any subdifferential larger than the Fréchet subdifferential is valuable. The following multidirectional mean value theorem is more powerful: in view of the compactness of the segment $[\overline{x}, \overline{y}]$, taking a sequence $(\varepsilon_n) \to 0_+$, it implies the preceding one in the case $Y := \{\overline{y}\}$ since in that case $\liminf_{s \to 0_+} f(Y + sB_X) \geq f(\overline{y})$. Thus a multi-valuable subdifferential in the sense of the following statement is valuable.

Theorem 2.2 ([11, 12, 28, 65]). *Let ∂ be a multi-reliable subdifferential (for X). Then, ∂ is multi-valuable (for X) in the sense that it satisfies the following property: given a bounded closed convex subset Y of X, $\overline{x} \in X\backslash Y$, $D := [\overline{x}, Y] := \{(1-t)\overline{x} + ty : t \in [0,1], y \in Y\}$ and $f \in \mathcal{S}(X)$ finite at $\overline{x}$, bounded below on $D + \sigma B_X$ for some $\sigma > 0$, then for any $\varepsilon > 0$ and $r \in \mathbb{R}$ such that*

$$r \leq \liminf_{s \to 0_+} f(Y + sB_X),$$

there exist $z \in [\overline{x}, Y] + \varepsilon B_X$ and $z^* \in \partial f(z)$ such that

$$r - f(\overline{x}) < \langle z^*, y - \overline{x} \rangle + \varepsilon \|y - \overline{x}\| \ \text{ for all } \ y \in Y.$$

In several cases of interest, the subdifferential we use is *tangentially determined* in the sense that it is defined with the help of some generalized directional derivative f^∂ of $f \in \mathcal{F}(X)$ via the formula

$$\partial f(x) := \{x^* : \langle x^*, \cdot \rangle \leq f^\partial(x, \cdot)\}. \tag{2.3}$$

This not always the case. For example, the Ioffe approximate subdifferential and the *firm (or Fréchet or regular) subdifferential* of $f \in \mathcal{F}(X)$ at x are not tangentially determined. The latter (studied at length in [4]) is the set $\partial^- f(x)$ of $x^* \in X^*$ such that for any $\varepsilon > 0$ there exists some $\rho > 0$ for which

$$f(w) - f(x) - \langle x^*, w - x \rangle \geq -\varepsilon \|w - x\| \qquad \forall w \in B(x, \rho).$$

Let us just mention the directional derivatives we will use. The *lower directional derivative* (or contingent derivative or lower epiderivative or lower Hadamard derivative) of f is given by

$$f^!(x, u) := \liminf_{(t,v) \to (0_+, u)} \frac{1}{t}(f(x + tv) - f(x)).$$

It can also be denoted by $f'(x, u)$ in view of its importance, but here we keep this notation for the case the *directional derivative* of f at x in the direction u exists in the sense that $f^!(x, u) = f^\sharp(x, u)$, where

$$f^\sharp(x, u) := \limsup_{(t,v) \to (0_+, u)} \frac{1}{t}(f(x + tv) - f(x)) = -(-f)^!(x, u).$$

The *Clarke-Rockafellar* derivative [10], [53] or *circa-derivative* of $f \in \mathcal{S}(X)$ is given by

$$f^{\uparrow}(x,u) := \sup_{r>0} \limsup_{\substack{(t,y)\to(0_+,x) \\ f(y)\to f(x)}} \inf_{v\in B(u,r)} \frac{1}{t}(f(y+tv)-f(y)).$$

When $f \in \mathcal{L}(X)$, $f^{\uparrow}$ coincides with the *Clarke's derivative* f^0 [10] which, for $f \in \mathcal{S}(X)$, and with the notation $B_f(x,\delta) := \{y \in B(x,\delta) : |f(y)-f(x)| < r\}$, is defined by

$$f^0(x,u) := \inf_{r>0} \sup_{(t,y,v)\in(0,r)\times B_f(x,r)\times B(u,r)} \frac{1}{t}(f(y+tv)-f(y)),$$

For f continuous at x this expression can be simplified into

$$f^0(x,u) := \limsup_{(t,y,v)\to(0_+,x,u)} \frac{1}{t}(f(y+tv)-f(y)).$$

The subdifferentials associated with $f^!$ and $f^{\uparrow}$ will be denoted by $\partial^!$ and $\partial^{\uparrow}$ respectively.

We will need the next result in which $\overline{\mathrm{co}}^*(S)$ denotes the weak* closed convex hull of a subset S of X^* and $w \to_f x$ means $w \to x$ with $f(w) \to f(x)$. Moreover, if $F : T \rightrightarrows X^*$ is a multifunction from a metric space to the dual of a Banach space, we denote by $\limsup_{t\to s} F(t)$ the set of weak* limit points of bounded nets $(x_i^*)_{i\in I}$ such that there exist a net $(t_i)_{i\in I} \to s$ with $x_i^* \in F(t_i)$ for each $i \in I$.

Theorem 2.3 ([3], [35, Thm 8.11]). *Let E be a closed subset of an Asplund space X, let $x \in E$ and let $f \in \mathcal{S}(X)$. Then, with $\overline{\partial^-} f(x) := \limsup_{w\to_f x} \partial^- f(w)$, $\partial^\infty f(x) := \limsup_{(t,w)\to_f(0^+,x)} t\partial^- f(w)$, one has*

$$\partial^{\uparrow} f(x) = \overline{\mathrm{co}}^*(\overline{\partial^-} f(x) + \partial^\infty f(x)). \tag{2.4}$$

In view of the abundance of concepts of subdifferentials, it is of interest to detect conditions ensuring some coincidence. The following definition will be convenient for such an aim; it is compatible with the terminology calling f regular or subdifferentially regular when f is $\partial^{\uparrow}$-regular ([10]).

Definition 2.4. A function $f \in \mathcal{F}(X)$ is said to be ∂-regular at x_0 for some subdifferential ∂ if $\partial f(x_0) = \partial^! f(x_0)$.

3 Semismoothness

The following result is valid for any lower semicontinuous function on X finite at x_0 and has an independent interest. Here (relative) *radial continuity* means continuity along segments whose extremities belong to the domain of the function. This mild continuity assumption is satisfied by all convex functions.

Lemma 3.1. *Let $f \in \mathcal{S}(X)$ be finite at $x_0 \in X$ and let ∂ be a subdifferential.*

(a) *If ∂ is valuable for X, then, for each $u \in S_X$, one has*

$$\liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in \partial f(x_0+tv)\} \le f^!(x_0,u) := \liminf_{(t,v)\to(0_+,u)} \frac{1}{t}\left(f(x_0+tv)-f(x_0)\right). \tag{3.1}$$

(b) *If ∂ is multi-valuable for X, and if f is radially continuous at x_0, then for each $u \in S_X$ one has*

$$\limsup_{(t,v)\to(0_+,u)} \sup\{\langle x^*, v\rangle : x^* \in \partial f(x_0+tv)\} \geq f^\sharp(x_0,u) := \limsup_{(t,v)\to(0_+,u)} \frac{1}{t}(f(x_0+tv) - f(x_0)). \tag{3.2}$$

Proof. (a) Relation (3.1) is obvious when $f^!(x_0,u) = \infty$. Given $r > r' > f^!(x_0,u)$ and $\delta > 0$ we can find $(t,v) \in (0,\delta)\times B(u,\delta)$ such that $t^{-1}(f(x_0+tv)-f(x_0)) < r'$. Then, relation (2.1) applied to the pair (x_0+tv, x_0) yields some $s \in [0,1)$, some sequences $(x_n) \to x_0+tv-stv$, (x_n^*), and $(\varepsilon_n) \to 0_+$ such that $x_n^* \in \partial f(x_n)$,

$$\langle x_n^*, \frac{x_0-x_n}{\|x_0-x_n\|}\rangle \geq \frac{f(x_0)-f(x_0+tv)}{\|x_0-(x_0+tv)\|} - \varepsilon_n.$$

Since (v_n) defined by $x_n := x_0 + t(1-s)v_n$ converges to v, we have $v_n \in B(u,\delta)$ for n large enough, $t' := (1-s)t \in (0,\delta)$, $x_n^* \in \partial f(x_0+t'v_n)$ and since $q_n := \|v\|^{-1}\|v_n\| \to 1$, we get for n large enough

$$\langle x_n^*, v_n\rangle \leq q_n r' + \varepsilon_n \|v_n\| < r.$$

Therefore

$$\inf\{\langle x^*, v'\rangle : (t',v') \in (0,\delta)\times B(u,\delta),\ x^* \in \partial f(x_0+t'v')\} \leq r.$$

Since $\delta > 0$ is arbitrarily small and r is arbitrarily close to $f^!(x_0,u)$, we get relation (3.1).

(b) Let us first consider the case $f^\sharp(x_0,u) := -(-f)^!(x_0,u) > 0$. Let $r' > r > 0$ with $r' < f^\sharp(x_0,u)$ and let $\delta \in (0,1)$ be given. Let us pick $\beta \in (0,\delta/2)$ such that $1+2\beta < r'/r$,

$$\frac{w}{\|w\|} \in B(u,\delta) \qquad \forall w \in B(u,2\beta) \tag{3.3}$$

and let us pick some $(t,v) \in (0,\beta)\times B(u,\beta)$ such that $t^{-1}(f(x_0+tv) - f(x_0)) > r'$. By the radial continuity of f at x_0, we can choose $p \in (0,\beta t)$ such that $t^{-1}(f(x_0+tv)-f(x_0+pv))) > r'$. Let us take $\gamma \in (0,\beta p)$ such that

$$\inf_{y\in B(x_0+tv,\gamma)} f(y) - f(x_0+pv) > r't.$$

We also impose that f is bounded below on $[x_0+pv, B(x_0+tv,\gamma)] + \gamma B_X$, which is possible in view of the lower semicontinuity of f and the compactness of the segment $[x_0+pv, x_0+tv]$. Let $\varepsilon > 0$ be given such that $\varepsilon \leq \gamma p(t-p)^{-1}$, $\varepsilon < \beta p - \gamma$, $\varepsilon < r'(1+2\beta)^{-1} - r$.

Applying the multidirectional mean value theorem to the pair $(x_0+pv, B(x_0+tv,\gamma))$, we get some $z \in [x_0+pv, B(x_0+tv,\gamma)] + \varepsilon B_X$ and $z^* \in \partial f(z)$ such that

$$r't < \langle z^*, y - x_0 - pv\rangle + \varepsilon\|y - x_0 - pv\| \qquad \forall y \in B(x_0+tv,\gamma) \tag{3.4}$$

Let $q \in [0,1]$, $b, b' \in B_X$ be such that $y := x_0 + tv + \gamma b \in B(x_0+tv,\gamma)$, $z := (1-q)(x_0 + pv) + qy + \varepsilon b'$. Then, for $m := p + q(t-p)$, $w := v + m^{-1}(q\gamma b + \varepsilon b')$, we have

$$z - x_0 = pv + q(t-p)v + q\gamma b + \varepsilon b' = mw.$$

Since $(t-p)(q\gamma+\varepsilon) \leq (t-p)q\gamma + p\gamma = m\gamma$, we have

$$w = v + \frac{\gamma}{t-p}b'' \qquad \text{with} \qquad b'' := \frac{t-p}{m\gamma}(q\gamma b + \varepsilon b') \in B_X.$$

Since $\gamma < \beta p < \beta t/2 < \beta(t-p)$, we get $w \in B(v,\beta) \subset B(u, 2\beta)$ and, by (3.3),

$$\frac{z-x_0}{\|z-x_0\|} = \frac{w}{\|w\|} \in B(u,\delta).$$

Let us set $y' := x_0 + tv + \gamma b''$, so that $y' \in B(x_0+tv, \gamma)$ and

$$z - x_0 = mv + q\gamma b + \varepsilon b' = m(t-p)^{-1}\left((t-p)v + \gamma b''\right) = m(t-p)^{-1}(y' - x_0 - pv)$$

We obtain from (3.4), after division by $s := \|y' - x_0 - pv\| \leq (t-p)\|v\| + \gamma \leq t(1+\beta)+\gamma < t(1+2\beta)$,

$$\langle z^*, \frac{z-x_0}{\|z-x_0\|}\rangle \geq r'(1+2\beta)^{-1} - \varepsilon > r. \tag{3.5}$$

For $u' := \|z-x_0\|^{-1}(z-x_0) \in B(u,\delta)$, $t' := \|z-x_0\| = \|mw\| \leq m(\|u\|+2\beta) \leq t(1+2\beta) \leq \beta(1+\delta) < \delta$ since $\beta < \delta/2$, $\delta < 1$, we have $z = x_0 + t'u'$ and $\langle z^*, u'\rangle > r$. Therefore

$$\sup\{\langle x^*, u'\rangle : t' \in (0,\delta),\ u' \in B(u,\delta),\ x^* \in \partial f(x_0 + t'u')\} > r \tag{3.6}$$

and $\limsup_{(t,v)\to(0_+,u)} \sup\{\langle x^*, v\rangle : x^* \in \partial f(x_0+tv)\} \geq f^\sharp(x_0,u)$.

Now let us consider the case $f^\sharp(x_0,u) \leq 0$. Then, let $r < r' < f^\sharp(x_0,u)$, hence $r < r' < 0$ and $r'/r < 1$. In that case, in order to secure the passage from (3.4) to the outer inequality of (3.5), we replace the requirement $1+2\beta < r'/r$ by the inequality $(1-\beta)^2 > r'/r$ which is satisfied if $\beta \in (0,\delta/2)$ is small enough. Again, we take $p \in (0,\beta t)$ such that $t^{-1}(f(x_0+tv) - f(x_0+pv))) > r'$. Then we have $(t-p)(1-\beta) > t(1-\beta)^2 > tr'/r$; and we can choose $\gamma > 0$ such that $(t-p)(1-\beta) - \gamma > tr'/r$, so that

$$s := \|y' - x_0 - pv\| \geq (t-p)\|v\| - \gamma\|a\| \geq (t-p)(1-\beta) - \gamma > tr'/r$$

or $tr'/s > r$; thus, if we take $\varepsilon > 0$ small enough (and depending on r, r', t, β, γ only), we have $tr'/s - \varepsilon > r$ and, again, by inequality (3.4) and the definition of y' and z we get

$$\langle z^*, \frac{z-x_0}{\|z-x_0\|}\rangle = \langle z^*, \frac{y'-x_0-pv}{s}\rangle \geq r'\frac{t}{s} - \varepsilon > r.$$

The same choices of u' and t' show that relation (3.6) is again satisfied. □

Remark 3.2. In the usual case in which $\partial(f+h)(x) = \partial f(x) + h$ for any $f \in \mathcal{S}(X)$, $h \in X^*$, the second part of the proof of assertion b) can be avoided by changing f into $f+h$ for some $h \in X^*$ with $h(u)$ large enough to ensure $(f+h)^\sharp(x_0,u) > 0$.

Example 3.3. Inequality (3.2) is not valid for any lower semicontinuous function, as the case of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ for $x \in \mathbb{R}\backslash\{0\}$, $f(0) = -1$: then $f^\sharp(0,1) = +\infty$, but $\partial^- f(x) = \{1\}$ for each $x \in (0,1)$.

The following definition, partially introduced in [43] is convenient to deal with the important concept of semismoothness defined in [33]. In the original paper [33], only the case of locally Lipschitzian functions, with $\partial = \partial^\uparrow$ was considered.

Definition 3.4. A multimapping $M : X \rightrightarrows X^*$ is extendedly thin at x_0 if for any $u \in S_X$, any sequences $(u_n) \to u$, $(t_n) \to 0_+$, and any $x_n^* \in M(x_0 + t_n u_n)$ the sequence $(\langle x_n^*, u_n\rangle)_n$ has a limit in the extended reals. It is thin at x_0 if that limit is always finite.

Given a subdifferential ∂, a function f on X is said to be ∂-semismooth (resp. extendedly ∂-semismooth) at $x_0 \in X$ if $f(x_0)$ is finite and ∂f is thin (resp. extendedly thin) at x_0.

For the sake of simplicity, we will essentially limit our study to semismooth functions.

Remark 3.5. If the requirement that $(\langle x_n^*, u_n \rangle)_n$ has a finite limit is changed into the requirement that $(\langle x_n^*, u \rangle)_n$ has a finite limit, we say that M is radially thin; that condition was the original definition in [43]. When M is bounded on a neighborhood of x_0, as assumed in [43], both notions coincide. In particular, for a locally Lipschitz function and a subdifferential contained in the Clarke subdifferential, the definition of semismoothness coincides with the one given in [33] and elsewhere. Since here ∂f is not necessarily locally bounded, we have to make that tuning.

One may wonder whether the definition really depends on the choice of a subdifferential. A partial answer is as follows.

Proposition 3.6. *Let ∂' be the subdifferential associated with a given subdifferential ∂ by setting for $f \in \mathcal{S}(X)$, $x \in X$, $\partial' f(x) := \overline{\mathrm{co}}^*(\limsup_{w \to_f x} \partial f(w) + \limsup_{(t,w) \to_f (0^+, x)} t\partial f(w))$. Then $f \in \mathcal{S}(X)$ is ∂'-semismooth at $x_0 \in X$ iff it is ∂-semismooth at x_0.*

If X is an Asplund space, if $f \in \mathcal{S}(X)$ and if ∂_1, ∂_2 are subdifferentials such that $\partial^- f \subset \partial_i f \subset \partial^\uparrow f$ for $i = 1, 2$, then f is ∂_1-semismooth at $x_0 \in X$ if, and only if, f is ∂_2-semismooth at $x_0 \in X$.

Proof. It suffices to show that f is ∂'-semismooth at $x_0 \in X$ when f is ∂-semismooth at $x_0 \in X$. By definition, for any $u \in S_X$ there exists some $\lambda \in \mathbb{R}$ such that, to any $\varepsilon > 0$ corresponds some $\delta > 0$ for which

$$|\langle x^*, v \rangle - \lambda| \le \varepsilon \quad \forall v \in B(u, \delta),\ \forall t \in (0, \delta),\ \forall x^* \in \partial f(x_0 + tv).$$

Let $x \in C_\delta := x_0 + (0, \delta) B(u, \delta)$, $x := x_0 + tv$ with $t \in (0, \delta)$, $v \in B(u, \delta)$, and let

$$x^* \in S(x) := \limsup_{w \to x} \partial f(w) + \limsup_{(t,w) \to (0^+, x)} t\partial f(w).$$

There exist $w^*, y^* \in X^*$, with $x^* = w^* + y^*$, nets $(w_i) \to x$, $(z_j) \to x$, $(r_j) \to 0_+$, and nets $(w_i^*) \to w^*$, (z_j^*) in X^* such that $(w_i^*) \to w^*$, $(r_j z_j^*) \to y^*$ for the weak* topology, with (w_i^*) and $(r_j z_j^*)$ bounded. Since C_δ is open, we may suppose $w_i, z_j \in C_\delta$ and write $w_i = x_0 + t_i v_i$ with $(v_i) \to v$. Then we have $|\langle w_i^*, v_i \rangle - \lambda| \le \varepsilon$, hence $|\langle w^*, v \rangle - \lambda| \le \varepsilon$. Similarly, writing $z_j = x_0 + t_j v_j$, with $(v_j) \to v$ and observing that $\left|\langle z_j^*, v_j \rangle - \lambda\right| \le \varepsilon$, we get $|\langle y^*, v \rangle| = \lim_j \left|\langle r_j z_j^*, v_j \rangle\right| \le \lim_j |r_j \lambda| + \lim_j r_j \varepsilon = 0$. Thus $|\langle x^*, v \rangle - \lambda| \le \varepsilon$. Then a convexity and closure argument shows that one also has $|\langle x^*, v \rangle - \lambda| \le \varepsilon$ for each $x^* \in \overline{\mathrm{co}}^*(S(x))$.

The second part follows immediately from the first part and Theorem 2.3. □

The following result has been proved in [33, Lemma 2] for the Clarke subdifferential in the case of a locally Lipschitzian function; see [49, Prop. 2.1] for an extension to the vector-valued case. It shows the compatibility of the terminologies in [16, 33, 43] with the preceding definition. Since the Clarke subdifferential is large, the assumption of ∂-semismoothness we make here is in general less stringent than $\partial^\uparrow$-semismoothness. Here we use *directional convergence* as in the preceding definition: we write $x \xrightarrow{u} x_0$ to mean that we take $x = x_0 + tv$ with $v \to u$, $t \to 0_+$.

Lemma 3.7. *Suppose ∂ is multi-valuable for X and $f \in \mathcal{S}(X)$ is radially continuous at x_0 and extendedly ∂-semismooth at $x_0 \in X$. Then for each $u \in S_X$ the directional derivative $f'(x_0, u)$ exists and*

$$f'(x_0, u) = \lim_{x \xrightarrow{u} x_0,\ x^* \in \partial f(x)} \langle x^*, \frac{x - x_0}{\|x - x_0\|} \rangle.$$

Proof. The existence of the directional derivative and the equality are immediate consequences of Lemma 3.1 and of the definition of extended ∂-semismoothness. $\square$

Example 3.8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 0$ for $(x, y) \in \mathbb{R}_- \times \mathbb{R}$, $f(x, y) = xg(y/x)$ for $(x, y) \in (0, +\infty) \times \mathbb{R}$, where $g : \mathbb{R} \to \mathbb{R}$ is a C^∞ function satisfying $g(r) = r$ for $r \in [-1, 1]$, $g(r) = 0$ for $|r| \geq 2$. Then f is ∂-semismooth at any point of $\mathbb{R}^2$ for any subdifferential ∂ such that $\partial^- f \subset \partial f \subset \partial^\uparrow f$.

Example 3.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 0$ for $(x, y) \in \mathbb{R}_- \times \mathbb{R}$, $f(x, y) = 2x^{3/2} \sin(y/x)$ for (x, y) with $x > 0$. Then f is Fréchet differentiable on $\mathbb{R}^2$ with $f'(x, y) = 0$ for $(x, y) \in \mathbb{R}_- \times \mathbb{R}$ and $f'(x, y) = (3x^{1/2} \sin(y/x) - 2yx^{-1/2} \cos(y/x), 2x^{1/2} \cos(y/x))$ for (x, y) with $x > 0$. Obviously, f is semismooth at $(0, 0)$ and non-Lipschitz around $(0, 0)$.

Example 3.10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(0) = 0$, $f(x) = x^2 \sin(1/x)$ for $x \in \mathbb{R}\backslash\{0\}$. Albeit f has a derivative at $x_0 = 0$, it is not semismooth at 0 for any subdifferential ∂ such that $\partial^- f \subset \partial f \subset \partial^\uparrow f$.

Example 3.11. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -|x|$ is semismooth, but not regular at 0.

4 Directional Submonotonicity at a Point

The following definition is a weakening of a notion introduced by Spingarn in [56] and used in [54, 57]; it is adapted to the comparison we have in view. In order to be precise, let us recall that a multimapping $M : X \rightrightarrows X^*$ is said to be *submonotone around* x_0, or, as in [37] approximately monotone around x_0 (or strictly submonotone around x_0, but we prefer not to keep the word "strict" which may be confusing, since strict monotonicity is a well established notion) if $M(x_0)$ is nonempty and if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in B(x_0, \delta)$ and any $x_1^* \in M(x_1)$, $x_2^* \in M(x_2)$ one has

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|.$$

The weakening we have in view does not consist in just giving a directional variant as in [43, Def. 2.2], [22, 25]; the main difference with the original concept is the fact that one of the two points is fixed. This fact justifies the terminology "at x_0" in contrast with "around x_0". A neater distinction would be obtained by using a distinct word such as "stellar", but we prefer to keep close to the usual terminology. We introduce another distinction which disappears when $M(x_0)$ is a singleton or a compact subset of X^*.

Definition 4.1. A multimapping $M : X \rightrightarrows X^*$ is submonotone at x_0 if $M(x_0)$ is nonempty and if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(x_0, \delta)$ and any $x_0^* \in M(x_0)$, $x^* \in M(x)$ one has

$$\langle x^* - x_0^*, x - x_0 \rangle \geq -\varepsilon \|x - x_0\|.$$

It is said to be extendedly directionally submonotone at x_0 if $M(x_0)$ is nonempty and if for any $x_0^* \in M(x_0)$, $u \in S_X$, $\varepsilon > 0$, there exists $\delta > 0$ such that for any $s \in (0, \delta)$, $v \in B(u, \delta)$ and any $x^* \in M(x)$ with $x := x_0 + sv$, one has

$$\langle x^* - x_0^*, x - x_0 \rangle \geq -\varepsilon \|x - x_0\|.$$

If $\delta > 0$ can be chosen independently of $x_0^* \in M(x_0)$, M is said to be directionally submonotone at x_0.

In other words, $M : X \rightrightarrows X^*$ is submonotone at x_0 if one has

$$\liminf_{x\to x_0,\, x\neq x_0} \inf_{x_0^*\in M(x_0)} \inf_{x^*\in M(x)} \langle x^* - x_0^*, \frac{x-x_0}{\|x-x_0\|}\rangle \geq 0;$$

It is directionally submonotone (resp. extendedly directionally submonotone) at x_0 if one has

$$\liminf_{x\xrightarrow{d}x_0} \inf_{x_0^*\in M(x_0)} \inf_{x^*\in M(x)} \langle x^* - x_0^*, \frac{x-x_0}{\|x-x_0\|}\rangle \geq 0,$$

$$\text{(resp.} \qquad \inf_{x_0^*\in M(x_0)} \liminf_{x\xrightarrow{d}x_0} \inf_{x^*\in M(x)} \langle x^* - x_0^*, \frac{x-x_0}{\|x-x_0\|}\rangle \geq 0),$$

where $x \xrightarrow{d} x_0$ means that x directionally converges to x_0 in a sense made precise in the preceding definition. The terminology adopted here slightly differs from the one in [43, Def. 2.2]. However, for M locally bounded, the case of study in [43, 56], the terminologies agree, as shown in the following lemma.

Lemma 4.2. *A multimapping $M : X \rightrightarrows X^*$ is extendedly directionally submonotone at $x_0 \in X$, if, and only if, $M(x_0)$ is nonempty and for each $u \in S_X$, one has*

$$\liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in M(x_0+tv)\} \geq \sup\{\langle x_0^*, u\rangle : x_0^* \in M(x_0)\}. \tag{4.1}$$

If moreover $M(x_0)$ is bounded, then M is directionally submonotone at x_0.

Proof. Let us assume M is extendedly directionally submonotone at x_0. Given $u \in S_X$, $r < \sigma_{M(x_0)}(u) := \sup\{\langle x_0^*, u\rangle : x_0^* \in M(x_0)\}$, and picking $\varepsilon \in (0,1)$, $x_0^* \in M(x_0)$ satisfying $\langle x_0^*, u\rangle > r + 3\varepsilon$, we can find some $\delta \in (0,1)$ such that $\langle x_0^*, v\rangle > r + 2\varepsilon$ for each $v \in B(u,\delta)$ and

$$\langle x^* - x_0^*, tv\rangle \geq -\varepsilon\,\|tv\| \geq -2\varepsilon t \qquad \forall t \in (0,\delta), \forall v \in B(u,\delta), \forall x^* \in M(x_0+tv).$$

Thus $\inf\{\inf\{\langle x^*, v\rangle : x^* \in M(x_0+tv)\} : t \in (0,\delta), v \in B(u,\delta)\} \geq r$ and (4.1) follows.

Now suppose (4.1) holds and $M(x_0)$ is nonempty. Given $x_0^* \in M(x_0)$, $\varepsilon \in (0,1)$, $u \in S_X$ and $c \geq \max(\|x_0^*\|, 1)$, relation (4.1) ensures that one can find $\delta \in (0, \varepsilon/3c)$ such that

$$\langle x^*, v\rangle \geq \sigma_{M(x_0)}(u) - \frac{1}{3}\varepsilon \geq \langle x_0^*, u\rangle - \frac{1}{3}\varepsilon \qquad \forall t \in (0,\delta),\ v \in B(u,\delta),\ x^* \in M(x_0+tv).$$

Since $|\langle x_0^*, u-v\rangle| \leq c\delta$ and $\|v\| \geq 2/3$ for all $v \in B(u,\delta)$, we get

$$\langle x^*-x_0^*, tv\rangle \geq -\frac{1}{3}\varepsilon t - c\delta t \geq -\frac{2}{3}\varepsilon t \geq -\varepsilon\,\|tv\| \qquad \forall t \in (0,\delta),\ v \in B(u,\delta),\ x^* \in M(x_0+tv),$$

so that M is extendedly directionally submonotone at x_0. When $M(x_0)$ is bounded, we can choose $c \geq 1$ and $\delta > 0$ independently of $x_0^* \in M(x_0)$. □

Corollary 4.3. *For a multimapping $M : X \rightrightarrows X^*$ and $x_0^* \in X$, the following assertions are equivalent:*

(a) *M and its opposite $-M$ are extendedly directionally submonotone at x_0;*

(b) *M is thin at x_0, $M(x_0)$ is a singleton $\{x_0^*\}$ and, for each $u \in S_X$, $\langle x^*, v\rangle \to \langle x_0^*, u\rangle$ as $(t,v) \to (0_+,u)$ and $x^* \in M(x_0+tv)$.*

Proof. (a)⇒(b) Given $u \in S_X$ we have

$$\begin{aligned}\sup\{\langle x_0^*, u\rangle : x_0^* \in M(x_0)\} &\leq \liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in M(x_0+tv)\}\\ &\leq \limsup_{(t,v)\to(0_+,u)} \sup\{\langle x^*, v\rangle : x^* \in M(x_0+tv)\}\\ &= -\liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in -M(x_0+tv)\}\\ &\leq -\sup\{\langle x_0^*, u\rangle : x_0^* \in -M(x_0)\} = \inf\{\langle x_0^*, u\rangle : x_0^* \in M(x_0)\},\end{aligned}$$

so that these inequalities are equalities. Thus, for each $u \in S_X$ the set $\{\langle x_0^*, u\rangle : x_0^* \in M(x_0)\}$ is a singleton. It follows that $M(x_0)$ is a singleton and (b) holds.

(b)⇒(a) Given $u \in S_X$ and $\varepsilon > 0$, our convergence assumption means that we can find $\delta > 0$ such that

$$|\langle x^*, v\rangle - \langle x_0^*, u\rangle| < \varepsilon/2 \qquad \forall (t,v) \in (0,\delta) \times B(u,\delta), \quad x^* \in M(x_0+tv).$$

We can take $\delta \in (0, 1/2)$ small enough to have $\delta \|x_0^*\| \leq \varepsilon/2$. Then, for $v \in B(u,\delta)$, we have $|\langle x^*, tv\rangle - \langle x_0^*, tv\rangle| < \varepsilon t \leq 2\varepsilon t \|v\| = 2\varepsilon \|x - x_0\|$ for all $(t,v) \in (0,\delta) \times B(u,\delta)$, $x^* \in M(x_0+tv)$, since $|\langle x_0^*, v\rangle - \langle x_0^*, u\rangle| \leq \varepsilon/2$ and $\|v\| \geq 1/2$, so that M and $-M$ are thin at x_0. □

Definition 4.4 ([43]). Given a subdifferential ∂, a function $f \in \mathcal{S}(X)$ is said to be (*extendedly*) *directionally ∂-subconvex at x_0* if ∂f is (*extendedly*) directionally submonotone at x_0.

Again, in nice spaces, such a definition is somewhat independent of the choice of the subdifferential in a reasonable range. Before proving that, we establish a regularity result. This regularity result is given in [43, Prop. 4.5] in the case of a locally Lipschitzian function; here we extend it to the case f is lower semicontinuous. Again, we denote by $\sigma(S, \cdot)$ the support function of a subset S of X^* : for $u \in X$,

$$\sigma_S(u) := \sigma(S, u) := \sup\{\langle x^*, u\rangle : x^* \in S\}.$$

Proposition 4.5. *Let $f \in \mathcal{S}(X)$ be extendedly directionally ∂-subconvex at $x_0 \in X$ for some valuable subdifferential ∂. Then one has $\sigma(\partial f(x_0), \cdot) \leq f^!(x_0, \cdot)$ and $\partial f(x_0) \subset \partial^! f(x_0)$. Thus, if $\partial^! f(x_0) \subset \partial f(x_0)$, then f is ∂-regular at x_0 in the sense that $\partial f(x_0) = \partial^! f(x_0)$.*

Proof. Lemmas 3.1 and 4.2 show that for any $u \in S_X$ one has

$$\begin{aligned}f^!(x_0, u) &\geq \liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in \partial f(x_0+tv)\} \geq \sup\{\langle x_0^*, u\rangle : x_0^* \in \partial f(x_0)\}\\ &= \sigma(\partial f(x_0), u)\end{aligned}$$

hence, by definition of $\partial^! f(x_0)$, one has $\partial f(x_0) \subset \partial^! f(x_0)$ and equality holds when $\partial^! f(x_0) \subset \partial f(x_0)$. □

The preceding result has interesting consequences in the case the subdifferential is tangentially determined by a closed convex positively homogeneous function; this is the case for the Clarke subdifferential $\partial^\uparrow$ and for the moderate subdifferential of Michel-Penot ([32]).

Corollary 4.6. *Let $f \in \mathcal{S}(X)$ be extendedly directionally ∂-subconvex at $x_0 \in X$ for some valuable subdifferential ∂. Suppose that $\partial f(x_0) = \{x_0^* \in X^* : x_0^* \leq f^\partial(x_0, \cdot)\}$, where $f^\partial(x_0, \cdot)$*

is a lower semicontinuous and sublinear function such that $f^{\partial}(x_0,0)=0$ and $f^{\partial}(x_0,\cdot)\geq f^{!}(x_0,\cdot)$. Then f is ∂-tangentially regular at x_0 in the sense that $f^{\partial}(x_0,\cdot)=f^{!}(x_0,\cdot)$. Moreover, in that case, for each $u\in X$, one has

$$f^{!}(x_0,u)=\liminf_{(t,v)\to(0_+,u)}\inf\{\langle x^*,v\rangle : x^*\in\partial f(x_0+tv)\}=f^{\partial}(x_0,u)=\sigma(\partial f(x_0),u).$$

Proof. Let $f^{\partial}(x_0,\cdot)$ be a lower semicontinuous and sublinear function defining $\partial f(x_0)$. For each $u\in X$ a form of the Hahn-Banach theorem yields $\sigma(\partial f(x_0),u)=f^{\partial}(x_0,u)$. Thus, when $f^{\partial}(x_0,\cdot)\geq f^{!}(x_0,\cdot)$, the preceding proposition yields $f^{\partial}(x_0,u)=\sigma(\partial f(x_0),u)=f^{!}(x_0,u)$. □

Proposition 4.7. *Let X be an Asplund space and let $f\in\mathcal{S}(X)$ be (extendedly) directionally ∂-subconvex at $x_0\in X$ for some subdifferential ∂ such that $\partial^- f\subset\partial f\subset\partial^{\uparrow}f$. Then f is (extendedly) directionally ∂'-subconvex at x_0 for any other subdifferential ∂' satisfying $\partial' f\subset\partial^{\uparrow}f$, $\partial' f(x_0)\subset\partial f(x_0)$.*

Proof. Again, the result stems from the fact that for any $\delta>0$, $u\in S_X$, $v\in B(u,\delta)$, $s\in(0,\delta)$ and $x=x_0+sv$, the set $\partial^{\uparrow}f(x)$ is the weak* closed convex hull of

$$\overline{\partial}f(x)+\partial^{\infty}f(x):=\limsup_{w\to_f x}\partial f(w)+\limsup_{(t,w)\to_f(0_+,x)}t\partial f(w);$$

since $\partial' f(x_0)\subset\partial f(x_0)$, the definition of (extended) directional submonotonicity allows to pass from ∂f to $\partial' f$. □

Thus, in an Asplund space X, the notion of ∂-subconvexity at $x_0\in X$ for functions which are ∂-regular at x_0 does not depend on the choice of ∂ in the class of subdifferentials satisfying $\partial^- f\subset\partial f\subset\partial^{\uparrow}f$. In fact, in view of Proposition 4.5, if f is directionally $\partial^{!}$-subconvex at x_0 and if ∂ is a subdifferential such that $\partial^- f\subset\partial f\subset\partial^{\uparrow}f$, then f is directionally ∂-subconvex at x_0 if, and only if, f is ∂-regular at x_0. In particular there exist functions f which are directionally $\partial^{!}$-subconvex at x_0 but not directionally $\partial^{\uparrow}$-subconvex at x_0.

Example 4.8. For $c\in(0,1)$ let $g_c:[0,1]\to[-1,0]$ be the restriction of a function of class C^1 on $\mathbb{R}$ such that $g_c(0)=0=g_c(1)$, $g_c'(r)\in[-c,0]$ for $r\in[0,1-c]$, $g_c'(r)\in[0,2]$ for $r\in[1-c,1]$ with $g_c'(1)=2$. Let us set $h(c,r):=g_c(r)$ and let us define an even function $f:\mathbb{R}\to\mathbb{R}$ by $f(0):=0$, $f(x):=2^{-n}h(2^{-n},2^n x-1)$ for $x\in[2^{-n},2^{-n+1}]$, $f(x)=0$ for $x\geq 1$. Then $\partial^{!}f(0)=\{0\}$, $\partial^{\uparrow}f(0)=[-2,2]$ and it is easy to check that f is directionally $\partial^{!}$-subconvex at x_0 but not $\partial^{\uparrow}$-subconvex at 0.

Example 4.9. Let $(r_n)_{n=1}^{n=\infty}$ be an enumeration of the set of all rational numbers. Take the function h from [58, p. 216, Ex. 6] defined on $\mathbb{R}$ by

$$h(t)=t+\sum_{n=1}^{\infty}\frac{(t-r_n)^{1/3}}{n^2(1+|r_n|)^{1/3}}.$$

Then h is an increasing function from $\mathbb{R}$ onto $\mathbb{R}$, which has a derivative (possibly infinite) at each $t\in\mathbb{R}$ with

$$h'(t)=1+\sum_{n=1}^{\infty}\frac{1}{3n^2(t-r_n)^{2/3}(1+|r_n|)^{1/3}}.$$

Therefore, $h'(t)>1$ for all $t\in\mathbb{R}$ and $h'(r_n)=\infty$ for all $n\in\mathbb{N}$; moreover $h'(\cdot)$ is lower semicontinuous. Thus, there exists $t_0\in(0,1)$ such that

$$h'(t_0)=\min\{h'(t)\ :\ t\in[0,1]\}<\infty.$$

Let $k = h^{-1}$. Then $0 \leq k'(x) < 1$ for all $x \in \mathbb{R}$ and $k'(h(r_n)) = 0$ for all $n \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = k(x)$ for $x \leq x_0 := h(t_0)$, $f(x) = t_0 + k'(x_0)(x - x_0)$ otherwise. Then f is Lipschitzian and $f'(x_0) = k'(x_0) > 0$; on the other hand, $\partial^{\uparrow} f(x_0) = [0, k'(x_0)]$. Obviously, f is $\partial^!$-subconvex at x_0 but not $\partial^{\uparrow}$-subconvex at this point.

5 Relationships between Semismoothness and Subconvexity

Let us start our study of the links between semismoothness, regularity and subconvexity with the following simple result. Note that the assumption $\partial f(x_0) \subset \partial^! f(x_0)$ is natural since it is a necessary condition for ∂-directional subconvexity at x_0 by Proposition 4.5.

Proposition 5.1. *Let $f \in \mathcal{S}(X)$ be radially continuous, finite at x_0 and (extendedly) ∂-semismooth at x_0 for a multi-valuable (for X) subdifferential ∂ such that $\partial f(x_0) \subset \partial^! f(x_0)$. Then f is (extendedly) ∂-directionally subconvex at x_0.*

Proof. Since f is extendedly ∂-semismooth at x_0, for each $u \in S_X$ one has

$$\liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in \partial f(x_0 + tv)\} \geq f'(x_0, u) \geq \sup\{\langle x_0^*, u\rangle : x_0^* \in \partial f(x_0)\}.$$

Thus, Lemma 4.2 applies. When f is ∂-semismooth at x_0, the uniform boundedness theorem and the inclusion $\partial f(x_0) \subset \{x^* \in X^* : x^* \leq f'(x_0, \cdot)\}$ of Proposition 4.5 ensure that $\partial f(x_0)$ is bounded. □

Combining that result with Proposition 4.5, we get the following consequence.

Corollary 5.2. *Let $f \in \mathcal{S}(X)$ be radially continuous, finite at x_0, and ∂-semismooth at x_0 for a multi-valuable (for X) subdifferential ∂ such that $\partial^! f \subseteq \partial f$. Then f is ∂-directionally subconvex at x_0 if, and only if, f is ∂-regular at x_0.*

We deduce from Lemma 4.2 a criterion for ∂-semismoothness which generalizes [43, Cor. 4.7] from the class $\mathcal{L}(X)$ to the class $\mathcal{S}(X)$. Note that its assumption $\sigma(\partial f(x_0), \cdot) \geq f^!(x_0, \cdot)$ is satisfied when $\partial f(x_0) = \partial^{\uparrow} f(x_0) \neq \varnothing$ and that $\partial^! f(x_0)$ is bounded whenever f is quiet at x_0, in the sense that there exist $c, r > 0$ such that $f(x) - f(x_0) \leq c\,\|x - x_0\|$ for $x \in B(x_0, r)$, i.e. $-f$ is calm at x_0.

Proposition 5.3. *Let $f \in \mathcal{S}(X)$. If f is directionally ∂-subconvex at $x_0 \in X$ then assertion (a) below implies assertion (b') and moreover, for each $u \in S_X$, one has $\sigma(\partial f(x_0 + tv), v) \to \sigma(\partial f(x_0), u)$ as $(t, v) \to (0_+, u)$.*

If f is radially continuous at x_0, if ∂ is multi-valuable for X and if $\sigma(\partial f(x_0), \cdot) \geq f^!(x_0, \cdot)$, the reverse implication holds, hence also (b)⇒(a).

When f is directionally ∂-subconvex at x_0 and $\partial f(x_0)$ is bounded and nonempty (a)⇒(b).

(a) *for each $u \in S_X$ one has $\limsup_{(t,v)\to(0_+,u)} \sigma(\partial f(x_0 + tv), v) \leq \sigma(\partial f(x_0), u)$;*

(b) *f is ∂-semismooth at x_0;*

(b') *f is extendedly ∂-semismooth at x_0.*

Proof. (a)⇒(b') Let $u \in S_X$. By our upper semicontinuity assumption, using (4.1) with $M := \partial f$, we get

$$\begin{aligned}\sigma(\partial f(x_0), u) &\geq \limsup_{(t,v)\to(0_+,u)} \sup\{\langle x^*, v\rangle : x^* \in \partial f(x_0 + tv)\} \\ &\geq \liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in \partial f(x_0 + tv)\} \geq \sigma(\partial f(x_0), u),\end{aligned}$$

and ∂f is extendedly thin at x_0.

When $\partial f(x_0)$ is bounded, nonempty, the above limit $\sigma(\partial f(x_0), \cdot)$ is finite and ∂f is thin at x_0.

(b')$\Rightarrow$(a) Suppose on the contrary that there exist $u \in S_X$, $\alpha \in \mathbb{R}$ and a sequence $((t_n, u_n)) \to (0_+, u)$ such that $\lim_n \sigma(\partial f(x_0 + t_n u_n), u_n) > \alpha > \sigma(\partial f(x_0), u)$. Then, for each $n \in \mathbb{N}$ large enough, there exists some $x_n^* \in \partial f(x_0 + t_n u_n)$ such that $\langle x_n^*, u_n \rangle > \alpha > \sigma(\partial f(x_0), u) \geq f^!(x_0, u)$ in view of our assumption. This is a contradiction with Lemma 3.7 when f is extendedly ∂-semismooth at x_0, since in that case $(\langle x_n^*, u_n \rangle) \to f'(x_0, u) = f^!(x_0, u)$. □

Let us give specializations of the preceding criterion; they extend the ones in [5, 16]. The first one covers Example 3.

Corollary 5.4. *Let X be an Asplund space and let $f \in \mathcal{S}(X)$ be extendedly directionally ∂-subconvex at $x_0 \in X$ and such that $f^!(x_0, \cdot)$ is convex, finite at 0 and such that for each $u \in S_X$ the function $(t, v) \mapsto f^!(x_0 + tv, v)$ is upper semicontinuous (u.s.c.) at $(0_+, u)$. Then f is extendedly $\partial^!$-semismooth at x_0. If moreover $f^!(x_0, \cdot)$ is finite, f is semismooth at x_0.*

Proof. Our assumptions ensure that $\sigma(\partial^! f(x_0), \cdot) = f^!(x_0, \cdot)$; moreover condition (a) of the preceding proposition is satisfied since $\sigma(\partial^! f(x_0 + tv), v) \leq f^!(x_0 + tv, v)$ for any $t > 0$, $v \in X$. □

Corollary 5.5. *Let $f \in \mathcal{S}(X)$ be extendedly directionally $\partial^\uparrow$-subconvex at $x_0 \in X$ and such that for each $u \in S_X$ the function $(t, v) \mapsto f^\uparrow(x_0 + tv, v)$ is u.s.c. at $(0_+, u)$, with $\partial^\uparrow f(x_0)$ nonempty. Then f is extendedly $\partial^\uparrow$-semismooth at x_0 and $\partial^\uparrow$-regular at x_0.*

Proof. The $\partial^\uparrow$-semismoothness of f at x_0 is a special case of Proposition 5.3: for $\partial = \partial^\uparrow$ and for any $(t, v) \in (0, +\infty) \times X$ one has $\sigma(\partial f(x_0 + tv), v) \leq f^\uparrow(x_0 + tv, v)$ and $\sigma(\partial^\uparrow f(x_0), \cdot) = f^\uparrow(x_0, \cdot)$ since $\partial^\uparrow f(x_0) \neq \varnothing$, so that condition a) of Proposition 5.3 is satisfied. Since $\partial^! f(x_0) \subset \partial^\uparrow f(x_0)$, Clarke regularity stems from Proposition 4.5. □

Corollary 5.6. *Let $f \in \mathcal{S}(X)$ be continuous at x_0 and such that $f^0(x_0, 0) = 0$. Then f is directionally $\partial^\uparrow$-subconvex at $x_0 \in X$, if and only if f is $\partial^\uparrow$-semismooth at x_0 and $\partial^\uparrow$-regular at x_0.*

Proof. The sufficient condition is a consequence of Proposition 5.1 since $\partial^\uparrow f(x_0) \subset \{x^* \in X^* : x^* \leq f^0(x_0, \cdot)\}$ is bounded when $f^0(x_0, 0) = 0$ as $f^0(x_0, \cdot)$ is sublinear and continuous at each point its domain ([53, Thm 3]). The necessary condition is a special case of Proposition 5.3 and of the following lemma. □

Lemma 5.7. *For any $f \in \mathcal{S}(X)$ and $x_0 \in \operatorname{dom} f$ the function $(x, v) \mapsto f^0(x, v)$ is upper semicontinuous at (x_0, u) for each $u \in X$ and, on its domain, the function $f^0(x_0, \cdot)$ coincides with $f^\uparrow(x_0, \cdot)$. Moreover, $\partial^0 f(x_0) = \partial^\uparrow f(x_0)$ when f is directionally Lipschitzian around x_0 in the sense that there exists some $u \in X$ such that $f^0(x, u) < +\infty$.*

Proof. When $f \in \mathcal{L}(X)$ the result is given in [10, Prop. 2.1.1]. In the general case one uses the fact that the strict epigraph $H(f, x)$ of $f^0(x, \cdot)$ is the hypertangent cone at $(x, f(x))$ to the epigraph E of f, where the *hypertangent cone* $H(E, e)$ at $e \in E$ to a subset E of some normed vector space Z is the set of $z \in Z$ such that $E \cap B(e, \delta) + (0, \delta)B(z, \delta) \subset E$ for some $\delta > 0$ (see [10, p. 57], [53, p. 267]). Clearly, given $z \in H(E, e)$, one has $z' \in H(E, e')$ for $e' \in E$ close enough to e and z' close enough to z. This fact proves the upper semicontinuity of

$(x, v) \mapsto f^0(x, v)$ at (x_0, u) for each $u \in X$. The equalities $H(E, e) = \mathrm{int}T^{\uparrow}(E, e)$, $T^{\uparrow}(E, e) = \mathrm{cl}H(E, e)$ when $H(E, e)$ is nonempty ([10, Thm 2.4.8]) entail the string of equivalences

$$x^* \in \partial^0 f(x_0) \Leftrightarrow (x^*, -1) \in H(E, e)^0 \Leftrightarrow (x^*, -1) \in T^{\uparrow}(E, e)^0 \Leftrightarrow x^* \in \partial^{\uparrow} f(x_0).$$

The fact that $f^0(x_0, \cdot)$ coincides with $f^{\uparrow}(x_0, \cdot)$ on its domain is contained in [53, Thm 3]. $\square$

Another criterion is obtained by means of a variant of a notion used in [40]. A function f is said to be *directionally stable at* $x_0 \in X$ if $f(x_0) < +\infty$ and if for any $u \in S_X$ there exist $\delta > 0$ and $c > 0$ such that

$$|f(x_0 + tv) - f(x_0 + tu)| \leq ct \qquad \forall v \in B(u, \delta),\ t \in (0, \delta). \tag{5.1}$$

Note that this condition is satisfied when $f^!(x_0, \cdot)$ and $(-f)^!(x_0, \cdot)$ are finite.

Lemma 5.8. *If $f \in \mathcal{S}(X)$ is directionally stable at $x_0 \in \mathrm{dom}\, f$, directionally Lipschitzian around x_0 and continuous at x_0, and if $\partial^! f(x_0) = \partial^{\uparrow} f(x_0)$ is nonempty, then, for each $u \in S_X$, the function $(x, v) \mapsto f^{\uparrow}(x, v)$ is u.s.c. at (x_0, u). In fact, $f^!(x_0, \cdot) = f^{\uparrow}(x_0, \cdot) = f^0(x_0, \cdot)$.*

Proof. In view of Lemma 5.7, it suffices to prove that for each $u \in S_X$ we have $f^{\uparrow}(x_0, u) = f^0(x_0, u)$. We first observe that since f is directionally Lipschitzian around x_0, by [52, Cor. 1], from the relation $\partial^! f(x_0) = \partial^{\uparrow} f(x_0)$ we deduce that $f^{\uparrow}(x_0, \cdot) = f^!(x_0, \cdot)$.

Now let $u \in \mathrm{dom}\, f^{\uparrow}(x_0, \cdot)$ and let $\delta > 0$ and $c > 0$ be as in (5.1). Then,

$$|f(x_0 + tv) - f(x_0 + tw)| \leq 2ct \qquad \forall v, w \in B(u, \delta),\ t \in (0, \delta).$$

It follows that for any $v \in B(u, \delta)$ we have

$$\liminf_{(t,v') \to (0_+, v)} \frac{1}{t} \left(f(x_0 + tv') - f(x_0)\right) \leq \liminf_{(t,u') \to (0_+, u)} \frac{1}{t} \left(f(x_0 + tu') - f(x_0)\right) + 2c$$

hence $v \in \mathrm{dom}\, f^!(x_0, \cdot) = \mathrm{dom}\, f^{\uparrow}(x_0, \cdot)$. Thus $\mathrm{dom}\, f^{\uparrow}(x_0, \cdot)$ is open. Since f is directionally Lipschitzian around x_0, [53, Thm 3] or [10, Thm 2.9.5] ensure that $f^{\uparrow}(x_0, \cdot) = f^0(x_0, \cdot)$ on $\mathrm{intdom}\, f^{\uparrow}(x_0, \cdot)$, hence on $\mathrm{dom}\, f^{\uparrow}(x_0, \cdot)$. Since $f^{\uparrow}(x_0, \cdot) \leq f^0(x_0, \cdot)$, these functions coincide. $\square$

Using Propositions 4.5 and 5.1 we get the following consequence.

Corollary 5.9. *Let $f \in \mathcal{S}(X)$ be finite, continuous at x_0, directionally stable at x_0 and directionally Lipschitzian around x_0 with $\partial^{\uparrow} f(x_0)$ nonempty. Then f is extendedly directionally $\partial^{\uparrow}$-subconvex at x_0 if, and only if, f is extendedly $\partial^{\uparrow}$-semismooth at x_0 and $\partial^{\uparrow}$-regular at x_0. If moreover $\partial^{\uparrow} f(x_0)$ is bounded, the equivalence holds without "extendedly".*

6 Approximate Starshapedness

The following notions are pointwise versions of the concepts of approximate convexity and directional approximate convexity introduced in [36, 21] respectively and studied in [1, 37].

Definition 6.1. A function $f \in \mathcal{F}(X)$, finite at x_0, is said to be approximately starshaped at x_0 if for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(x_0, \rho)$, $t \in [0, 1]$, one has

$$f((1 - t)x_0 + tx) \leq (1 - t)f(x_0) + tf(x) + \varepsilon t(1 - t) \|x - x_0\|. \tag{6.1}$$

A function $f \in \mathcal{F}(X)$, finite at x_0, is said to be directionally approximately starshaped at x_0 if for any $\varepsilon > 0$, $u \in S_X$ there exists $\delta > 0$ such that for any $s \in (0, \delta)$, $v \in B(u, \delta)$, $t \in [0, 1]$, relation (6.1) holds for $x := x_0 + sv$.

Clearly, if f is approximately starshaped at x_0 then it is directionally approximately starshaped at x_0 : given $\varepsilon > 0$ and $u \in S_X$ and taking $\rho > 0$ associated with ε as in Definition 6.1 we see that (6.1) is satisfied when $x = x_0 + sv$ with $s \in (0, \delta)$, $v \in B(u, \delta)$ provided $\delta \leq \min(1, \rho/2)$. Moreover, Definition 6.1 is a uniform version of the notion of directional approximate starshapedness. In finite dimensions, the two concepts coincide. They have interesting consequences we now delineate.

Lemma 6.2. *Let $f \in \mathcal{F}(X)$ be finite at x_0. If f is directionally approximately starshaped at x_0, then, for any $u \in S_X$, $\varepsilon > 0$, there exists $\delta > 0$ such that for any $s \in (0, \delta)$, $v \in B(u, \delta)$ one has, for $x := x_0 + sv$,*

$$f^!(x_0, sv) \leq f(x_0 + sv) - f(x_0) + \varepsilon \|sv\|, \tag{6.2}$$

$$f^!(x, -sv) \leq f(x - sv) - f(x) + \varepsilon \|sv\|. \tag{6.3}$$

Moreover, f is directionally $\partial^!$-subconvex at x_0, i.e. $\partial^! f$ is directionally submonotone at x_0.

Proof. Let $u \in S_X$. Given $\varepsilon > 0$, let $\delta > 0$ be such that relation (6.1) is satisfied for any $s \in (0, \delta)$, $v \in B(u, \delta)$, $t \in (0, 1]$, with $x := x_0 + sv$. Let $(s, v) \in (0, \delta) \times B(u, \delta)$ be fixed. Then, we have

$$f(x_0 + tsv) - f(x_0) \leq tf(x_0 + sv) - tf(x_0) + \varepsilon t(1 - t) \|sv\|,$$

hence, dividing by t and taking the limit inferior as $t \to 0_+$, we get

$$f^!(x_0, sv) \leq f(x_0 + sv) - f(x_0) + \varepsilon \|sv\|.$$

Setting $x := x_0 + sv$, $t' := 1 - t$ and writing (6.1) under the form

$$f(x - t'sv) - f(x) \leq t'f(x - sv) - t'f(x) + \varepsilon t'(1 - t') \|sv\|,$$

dividing by t' and taking the limit inferior as $t' \to 0_+$, we get

$$f^!(x, -sv) \leq f(x - sv) - f(x) + \varepsilon \|sv\|.$$

Given $x_0^* \in \partial^! f(x_0)$, $x^* \in \partial^! f(x)$, with $x := x_0 + sv$, using the relations $\langle x_0^*, sv\rangle \leq f^!(x_0, sv)$, $\langle x^*, -sv\rangle \leq f^!(x, -sv)$, adding sides by sides relations (6.2), (6.3), we get

$$\langle x_0^* - x^*, x - x_0\rangle \leq 2\varepsilon \|x - x_0\|.$$

Since $\varepsilon > 0$ is arbitrary, we get that $\partial^! f$ is directionally submonotone at x_0. □

Similarly, it can be shown that if f is approximately starshaped at x_0 then f is $\partial^!$-subconvex at x_0; moreover, in such a case, f is ∂^--regular at x_0 : $\partial^- f(x_0) = \partial^! f(x_0)$.

Proposition 6.3. *Let X be a Banach space such that $\partial^!$ is valuable for X and let $f \in \mathcal{S}(X)$ be finite at x_0 and directionally approximately starshaped at x_0. Then for each $u \in X \backslash \{0\}$ one has*

$$f^!(x_0, u) = \liminf_{(t,v) \to (0_+, u)} \inf\{\langle x^*, v\rangle : x^* \in \partial^! f(x_0 + tv)\}.$$

Moreover, if X is an Asplund space, and if f is ∂-regular at x_0 for a subdifferential ∂ such that $\partial^- f \subset \partial f \subset \partial^\uparrow f$, then f is directionally ∂-subconvex at x_0.

Proof. When $\partial^!$ is valuable for X, Lemma 3.1 ensures that

$$\ell := \liminf_{(t,v)\to(0_+,u)} \inf\{\langle x^*, v\rangle : x^* \in \partial^! f(x_0+tv)\} \le f^!(x_0,u\rangle.$$

Let $((t_i,v_i))_{i\in I} \to (0_+,u)$ and $x_i^* \in \partial^! f(x_0+t_iv_i)$ be such that $(\langle x_i^*, v_i\rangle)_{i\in I} \to \ell$. Then, by (6.3), for some net $(\varepsilon_i)_{i\in I} \to 0_+$, we have

$$\langle x_i^*, -v_i\rangle \le f^!(x_0+t_iv_i,-v_i) \le t_i^{-1}\left(f(x_0)-f(x_0+t_iv_i)\right)+\varepsilon_i\left\|v_i\right\|.$$

Thus

$$\ell \ge \liminf_{i\in I} t_i^{-1}\left(f(x_0+t_iv_i)-f(x_0)\right) \ge f^!(x_0,u).$$

The second assertion is a combination of the preceding lemma with Proposition 4.7. □

The following result shows a striking consequence of semismoothness.

Theorem 6.4. *Let $f \in \mathcal{S}(X)$ be finite, radially continuous at x_0 and ∂-semismooth at x_0 for a multi-valuable subdifferential ∂ for X. Then f is directionally approximately starshaped at x_0.*

Proof. Assume f is ∂-semismooth at x_0 and radially continuous at x_0. In view of Lemma 3.7, given $u \in S_X$, $\varepsilon \in (0,1)$ there exists $\delta \in (0,1/2)$ such that for any $s \in (0,\delta)$, $v \in B(u,\delta)$ and any $x^* \in \partial f(x)$ with $x := x_0+sv$, one has

$$|\langle x^*, v\rangle - f'(x_0,u)| \le \varepsilon, \tag{6.4}$$

$$\left|\frac{f(x_0+sv)-f(x_0)}{s} - f'(x_0,u)\right| \le \varepsilon \tag{6.5}$$

Let $s \in (0,\delta)$, $v \in B(u,\delta)$, $x := x_0+sv$, $t \in (0,1)$, and let $y := tx+(1-t)x_0$. Without loss of generality, to prove inequality (6.1) we may assume that $x \ne x_0$ and $f(x) < +\infty$. Applying the mean value theorem (Theorem 2.1) to f on $[x,y]$, since by (6.5) $r := f(y) < +\infty$, we can find some $z \in [x,y)$, some sequences $(z_n) \to z$, (z_n^*) and $(\varepsilon_n) \to 0_+$ such that $\varepsilon_n \in (0,\varepsilon)$ and $z_n^* \in \partial^\uparrow f(z_n)$ for each n and

$$\frac{f(y)-f(x)}{\|y-x\|} \le \langle z_n^*, \frac{x_0-z_n}{\|x_0-z_n\|}\rangle + \varepsilon_n. \tag{6.6}$$

Let $r_n := \|z_n-x_0\|/\|v\|$, $v_n := (z_n-x_0)/r_n$, so that $(r_n) \to \|z-x_0\|/\|v\| \in (0,\delta)$ and $(v_n) \to v$. We have $v_n \in B(u,\delta)$ for n large enough. In view of (6.4), (6.6), by letting $n \to \infty$, we get, since $\|y-x\| = s(1-t)\|v\|$,

$$\frac{f(y)-f(x)}{s(1-t)} = \frac{f(y)-f(x)}{\|y-x\|}\|v\| \le \left(\langle z_n^*, \frac{r_nv_n}{r_n\|v\|}\rangle + \varepsilon_n\right)\|v\| \le -f'(x_0,u)+\varepsilon+\varepsilon_n\|v\|.$$

On the other hand, since $y = x_0+stv$, $\|y-x_0\| = st\|v\|$, (6.5) ensures that

$$\frac{f(y)-f(x_0)}{st} = \frac{f(y)-f(x_0)}{\|y-x_0\|}\|v\| \le f'(x_0,u)+\varepsilon.$$

Combining these two relations, we obtain, since $\|v\| \ge 1-\delta \ge 1/2$,

$$f(y)-tf(x)-(1-t)f(x_0) \le st(1-t)(2\varepsilon+\varepsilon_n) \le 2\|v\|st(1-t)(2\varepsilon+\varepsilon_n) = 6\varepsilon t(1-t)\|x-x_0\|.$$

Since ε is arbitrarily small, f is directionally approximately starshaped at x_0. □

For a directionally Lipschitzian function, semismoothness entails a stronger property we introduce now.

Definition 6.5. A function $f \in \mathcal{F}(X)$, finite at x_0, is said to be approximately straight at x_0 if f and $-f$ are approximately starshaped at x_0.

A function $f \in \mathcal{F}(X)$, finite at x_0, is said to be directionally approximately straight at x_0 if f and $-f$ are directionally approximately starshaped at x_0, i.e. if, for any $\varepsilon > 0$ and any $u \in S_X$, there exists $\delta > 0$ such that for any $s \in (0, \delta)$, $v \in B(u, \delta)$, $t \in [0, 1]$, one has, with $x := x_0 + sv$

$$|(1-t)f(x_0) + tf(x) - f((1-t)x_0 + tx)| \leq \varepsilon t(1-t) \|x - x_0\| . \tag{6.7}$$

This property requires a symmetry result which is well known when $f \in \mathcal{L}(X)$ or when $\dim X < +\infty$.

Lemma 6.6. *If the function $f : X \to \overline{\mathbb{R}}$ is finite and continuous at $x \in X$ and is directionally Lipschitzian around x, then $-f$ is directionally Lipschitzian around x and*

$$\partial^{\uparrow}(-f)(x) = -\partial^{\uparrow} f(x). \tag{6.8}$$

Proof. The first assertion is contained in [53, Thm 3]; it is also shown there that

$$(-f)^{\uparrow}(x, v) = -f^{\uparrow}(x, -v) \qquad\qquad \forall v \in X.$$

To prove (6.8), using [10, Thm 2.4.8, Cor. 2.9.3], let us first note that when f is directionally Lipschitzian around x, then

$$x^* \in \partial^{\uparrow}(-f)(x) \Leftrightarrow x^* \leq (-f)^{\uparrow}(x, \cdot) \Leftrightarrow -x^* \leq f^{\uparrow}(x, \cdot) \Leftrightarrow -x^* \in \partial^{\uparrow} f(x). \tag{6.9}$$

□

Corollary 6.7. *Let $f \in \mathcal{F}(X)$ be finitely valued, continuous, directionally Lipschitzian around x_0, and $\partial^{\uparrow}$-semismooth at x_0. Then f is directionally approximately straight at x_0.*

Proof. Since f is continuous and directionally Lipschitzian around x_0, it is directionally Lipschitzian around each point of a neighborhood of x_0; thus the result follows from Theorem 6.4 and the fact that, by the preceding lemma, $-f$ is also semismooth at x_0. □

Let us subsume several of our results in the following statement which shows strong relationships.

Theorem 6.8. *Let ∂ be a multi-valuable subdifferential and let $f \in \mathcal{S}(X)$ be radially continuous, finite at x_0 and such that $\partial^! f \subset \partial f \subset \partial^{\uparrow} f$. Among the following assertions, one has the implications*

$$(a) \Rightarrow (b) \Rightarrow (c), \quad (e) \Rightarrow (b), \quad (e) \Rightarrow (d) \Rightarrow (c).$$

If $\partial^{\uparrow} f(x_0)$ is nonempty, if f is continuous, directionally stable at x_0, directionally Lipschitzian around x_0 and $\partial^{\uparrow}$-regular around x_0, then (a)⇔(b)⇔(c')⇔(d')⇔(e).

(a) *f is ∂-semismooth at x_0;*
(b) *f is directionally approximately starshaped at x_0;*
(c) *f is directionally $\partial^!$-subconvex at x_0;*
(c') *f is directionally $\partial^!$-subconvex at x_0 and $\partial^! f(x_0)$ is bounded;*
(d) *$\partial^! f$ and $\partial^! (-f)$ are directionally submonotone at x_0;*
(d') *$\partial^! f$ and $\partial^! (-f)$ are directionally submonotone at x_0 and $\partial^! f(x_0)$ is bounded;*
(e) *f is directionally approximately straight at x_0.*

Proof. (e)⇒(b) and (d)⇒(c) are obvious; (b)⇒(c) and (e)⇒(d) are proved in Lemma 6.2, while (a)⇒(b) is given in Theorem 6.4. Under the additional assumption, Corollary 5.9 ensures (c')⇒(a). Lemma 6.6 shows that $-f$ is also directionally Lipschitzian around x_0 and $\partial^{\uparrow}(-f)(x_0)$ is bounded and nonempty, so that f and $-f$ are semismooth when (d') and the additional assumption hold, hence (d')⇒(e) and (c')⇒(d'). □

Remark 6.9. a) When X is Asplund, Proposition 4.7 enables to replace the assumption that f is $\partial^{\uparrow}$-regular around x_0, by the assumption that f is $\partial^{\uparrow}$-regular at x_0.

b) As shown in previous statements several implications can be obtained for extended semismoothness and subconvexity and without the boundedness assumption for $\partial^! f(x_0)$.

An example of starshaped function which is not semismooth can be given as follows.

Example 6.10. Given decreasing sequences (a_n), (c_n) of $(0, +\infty)$ with limit 0, let $f : \mathbb{R} \to \mathbb{R}$ be even, continuous and given by $f(x) = c_{2n}x$ for $x \in [a_{2n}, a_{2n-1}]$, $f(x) = (c_{2n+1} + 1)(x - a_{2n}) + c_{2n}a_{2n}$ for $x \in [a_{2n+1}, a_{2n}]$. Then f is starshaped at 0, differentiable at 0, but not semismooth at 0 nor approximately convex at 0.

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