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GENERALIZED TYPE I INVEXITY AND DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE VARIATIONAL PROBLEMS*

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Abstract: We consider a new class of generalized V-type I invex functions for a class of nondifferentiable multiobjective variational problems. A number of sufficient optimality results are established under generalized V-type I and related invex functions. Duality theorems are obtained for Mond-Weir type duals under the aforesaid assumptions.

Key words: *sufficient optimality conditions, duality; nondifferentiable variational problems, generalized convexity*

Mathematics Subject Classification: *90C29; 49N15, 90C46*

1 Introduction

Hanson [7] observed that variational and control problems are continuous-time analogue of finite dimensional nonlinear programming problems. Since then the fields of nonlinear programming problems and the calculus of variations have to some extent merged together within optimization theory, hence enhancing the potential for continued research in both fields. Optimality conditions and duality results are obtained for scalar valued variational problems by Mond and Hanson [19] under convexity. Mond *et al.* [21] extended the concept of invexity (see, Hanson [8]) to the continuous case and used it to generalize earlier duality results for a class of variational problems. Mond and Smart [20] extended the duality theorems for a class of static nondifferentiable problems with Wolfe type and Mond-Weir type duals, and further extended these for the continuous analogues. Mishra and Mukherjee [16] extended the work of Mond *et al.* [21] for multiobjective variational problems which in particular extended an earlier work of Bector and Husain [3]. Jeyakumar and Mond [10] introduced the class of V-invex functions, which preserves the sufficient optimality and duality results in the scalar case and avoids the major difficulty of verifying that the inequality holds for the same kernel function. Mukherjee and Mishra [22] extended the work of Jeyakumar and Mond [10] to variational problems with the concept of weak minima. Mishra [15] established a close relationship between variational problems and nonlinear multiobjective

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programming problems. Kim and Kim [12] obtained duality results for a nondifferentiable multiobjective variational problem under generalized $(F,\ \rho)$-convexity, following Mishra and Mukherjee [17]. For other works on variational problems one can see, Bhatia and Mehra [2], Chankong and Haimes [4], Chen [5], Kim and Lee [11], Liu [14], Mishra and Mukherjee [18], Mukherjee and Mishra [23], Nahak and Nanda [24, 25] and Yang and Zang [26].

Recently, Kim and Kim [13] extended the concepts of generalized $V-$type I invex vector-valued functions introduced by Hanson *et al.* [9], to continuous case and established Mond-Weir type duality results under the aforesaid assumptions.

In this paper, we extend the work of Kim and Kim [13] to nondifferentiable case by adding a square root of a certain positive semidefinite quadratic form in every component of the objective function. In Section 2, we recall the necessary concepts and give the model of the problem. We establish several sufficient optimality conditions in Section 3. In Section 4, we present duality theorems for Mond-Weir type of dual under the aforesaid assumptions.

2 Notations and Preliminaries

Let $I = [a,\ b]$ be a real interval and $f : I \times R^n \times R^n \to R$ be a continuously differentiable function. In order to consider $f(t, x, \dot{x})$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, we denote the partial derivatives of f by f_t,

$$f_x = \left[\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right], \quad f_{\dot{x}} = \left[\frac{\partial f}{\partial \dot{x}^1}, \ldots, \frac{\partial f}{\partial \dot{x}^n}\right].$$

The partial derivatives of the other functions used will be written similarly. Let $C(I,\ R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$u^i = Dx^i \Leftrightarrow x^i(t) = \alpha + \int_a^t u^i(s)\ ds,$$

in which α is a given boundary value. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

For a multiobjective continuous programming:

$$\text{(MP)} \qquad \text{minimize} \int_a^b f(t, x, \dot{x})\, dt = \left(\int_a^b f_1(t, x, \dot{x})\, dt, \ldots, \int_a^b f_p(t, x, \dot{x})\, dt\right)$$

subject to

$$x(a) = \alpha,\ x(b) = \beta$$

$$g(t, x, \dot{x}) \ \leq\ 0,\ t \in I,$$

$$x \in C(I,\ R^n),$$

where $f_i : I \times R^n \times R^n \to R,\ i \in P = \{1, \ldots, p\}, g : I \times R^n \times R^n \to R^m$ are assumed to be continuously differentiable functions.

Let K denote the set of all feasible solutions for (MP), that is

$$K = \{x \in C(I,\ R^n):\ x(a) = \alpha,\ x(b) = \beta,\ g(t,\ x(t),\ \dot{x}(t)) \ \leq\ 0, t \in I\}.$$

Craven [6] obtained Kuhn-Tucker type necessary conditions for the above problem and proved that the necessary conditions are also sufficient if the objective functions are pseudoconvex and the constraints are quasiconvex.

Definition 2.1. A point $x^* \in K$ is said to be an *efficient solution* of (MP) if for all $x \in K$

$$\int_a^b f_i(t, x^*(t), \dot{x}^*(t))\ dt \geq \int_a^b f_i(t, x(t), \dot{x}(t))\ dt,\ \forall\ i \in P$$

$$\Rightarrow \int_a^b f_i(t, x^*(t), \dot{x}^*(t))\ dt = \int_a^b f_i(t, x(t), \dot{x}(t))\ dt,\ \forall\ i \in P.$$

We now recall the definitions of generalized V-type I invex functions from Kim and Kim [13].

Definition 2.2. We say the problem (PM) to be V-type I invex at $u \in C(I,\ R^n)$ with respect to η, α_i and β_j if there exist vector function $\eta : I \times R^n \times R^n \to R^n$ with $\eta(t, x, x) = 0$ and real valued functions $\alpha_i \in R_+ \setminus \{0\}$ and $\beta_j \in R_+ \setminus \{0\}$ such that

$$\int_a^b f_i(t, x, \dot{x})\ dt\ -\ \int_a^b f_i(t, u, \dot{u})\ dt$$
$$\geq\ \int_a^b \left[\alpha_i(x,\ u,\ \dot{x},\ \dot{u})\,\eta\ (t, x, u)\left\{f_x^i(t, u, \dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u})\right\}\right] dt$$

and

$$-\int_a^b g(t, u, \dot{u})\ dt\ \geq \int_a^b \left[\beta_j(x, u, \dot{x}, \dot{u})\,\eta(t, x, u)\left\{g_x^j(t, u, \dot{u}) - \frac{d}{dt} g_{\dot{x}}^j(t, u, \dot{u})\right\}\right] dt,$$

$\forall\ x \in K$, and for all $i \in P = \{1, 2, \ldots, p\}$, $j \in M = \{1, 2, \ldots, m\}$.

If the first inequality is strict (whenever $x \neq x^*$) we say that (MP) is *semi strictly* V-type I invex at x^*.

Definition 2.3. We say the problem (MP) is *semi strictly quasi* V-type I at $u \in C(I,\ R^n)$ with respect to η, α_i and β_j if there exist vector function $\eta : I \times R^n \times R^n \to R^n$ with $\eta(t, x, x) = 0$ and real valued functions $\alpha_i \in R_+ \setminus \{0\}$ and $\beta_j \in R_+ \setminus \{0\}$ such that for some vector $\tau \in R^p$, $\tau \geq 0$, and piecewise smooth function $\lambda : I \to R^m$, $\lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, u, \dot{x}, \dot{u})\, f_i(t, x, \dot{x})\ dt\ \leq\ \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, u, \dot{x}, \dot{u})\, f_i(t, u, \dot{u})\ dt$$

$$\Rightarrow\ \int_a^b \sum_{i=1}^p \tau_i\, \eta\ (t, x, u)\left\{f_x^i(t, u, \dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u})\right\} dt\ <\ 0,$$

and

$$-\int_a^b \sum_{j=1}^m \lambda_j(t)\,\beta_j(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})\,g_j(t,u,\dot{u})\ dt\ \le\ 0$$

$$\Rightarrow \int_a^b \sum_{j=1}^m \lambda_j(t)\eta(t,x,u)\left\{g_x^j(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}^j(t,u,\dot{u})\right\}\ dt\ \le\ 0\,,$$

whenever $x \neq x^*$, $\forall\, x \in K$, for all $i \in P = \{1,2,\ldots,p\}$ and $j \in M = \{1,2,\ldots m\}$.

Definition 2.4. We say the problem (MP) is *semi strictly pseudo $V-$type I* at $u \in C(I,\ R^n)$ with respect to η, α_i and β_j if there exist vector function $\eta : I \times R^n \times R^n \to R^n$ with $\eta(t,x,x) = 0$ and real valued functions $\alpha_i \in R_+\backslash\{0\}$ and $\beta_j \in R_+\backslash\{0\}$ such that for some vector $\tau \in R^p$, $\tau \geq 0$, and piecewise smooth function $\lambda : I \to R^m$, $\lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^p \tau_i\eta\ (t,x,u)\left\{f_x^i(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}^i(t,u,\dot{u})\right\}\ dt\ \ge\ 0$$

$$\Rightarrow \int_a^b \sum_{i=1}^p \tau_i\,\alpha_i(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})\,f_i(t,x,\dot{x})\ dt\ >\ \int_a^b \sum_{i=1}^p \tau_i\,\alpha_i(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})\,f_i(t,u,\dot{u})\ dt$$

and

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\,\eta\ (t,x,u)\left\{g_x^j(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}^j(t,u,\dot{u})\right\}\ dt\ \ge\ 0$$

$$\Rightarrow \int_a^b \sum_{j=1}^m \lambda_j(t)\,\beta_j(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})\,g(t,u,\dot{u})\ dt\ \le\ 0,$$

whenever $x \neq x^*$, $\forall\, x \in K$, for all $i \in P = \{1,2,\ldots,p\}$ and $j \in M = \{1,2,\ldots,m\}$.

Definition 2.5. We say the problem (MP) is *quasi strictly pseudo $V-$type I* at $u \in C(I,\ R^n)$ with respect to η, α_i and β_j if there exist vector function $\eta : I \times R^n \times R^n \to R^n$ with $\eta(t,x,x) = 0$ and real valued functions $\alpha_i \in R_+\backslash\{0\}$ and $\beta_j \in R_+\backslash\{0\}$ such that for some vector $\tau \in R^p$, $\tau \geq 0$, and piecewise smooth function $\lambda : I \to R^m$, $\lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^p \tau_i\alpha_i(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})\,f_i(t,x,\dot{x})\ dt\ \le\ \int_a^b \sum_{i=1}^p \tau_i\alpha_i(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})\,f_i(t,u,\dot{u})\ dt$$

$$\Rightarrow \int_a^b \sum_{i=1}^p \tau_i\,\eta\ (t,x,u)\left\{f_x^i(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}^i(t,u,\dot{u})\right\}\ dt\ \le\ 0,$$

and

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t)\,\eta(t,x,u)\left\{g_x^j(t,u,\dot u) - \frac{d}{dt} g_{\dot x}^j(t,u,\dot u)\right\} dt \ \geq\ 0$$

$$\Rightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t)\,\beta_j(x\,,\,u\,,\,\dot x\,,\,\dot u)\,g(t,u,\dot u)\ dt\ <\ 0,$$

$\forall\ x \in K$, for all $i \in P = \{1,2,\ldots,p\}$ and $j \in M = \{1,2,\ldots,m\}$.

Definition 2.6. We say the problem (MP) is *pseudo quasi V-type I* at $u \in C(I,\ R^n)$ with respect to η, α_i and β_j if there exist vector function $\eta : I \times R^n \times R^n \to R^n$ with $\eta(t,x,x) = 0$ and real valued functions $\alpha_i \in R_+\setminus\{0\}$ and $\beta_j \in R_+\setminus\{0\}$ such that for some vector $\tau \in R^p$, $\tau \geq 0$, and piecewise smooth function $\lambda : I \to R^m$, $\lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^{p} \tau_i \eta\ (t,x,u)\left\{f_x^i(t,u,\dot u) - \frac{d}{dt} f_{\dot x}^i(t,u,\dot u)\right\} dt\ \geq\ 0$$

$$\Rightarrow \int_a^b \sum_{i=1}^{p} \tau_i\,\alpha_i(x\,,\,u\,,\,\dot x\,,\,\dot u)\,f_i(t,x,\dot x)\ dt\ \geq\ \int_a^b \sum_{i=1}^{p} \tau_i\,\alpha_i(x\,,\,u\,,\,\dot x\,,\,\dot u)\,f_i(t,u,\dot u)\ dt$$

and

$$-\int_a^b \sum_{j=1}^{m} \lambda_j(t)\,\beta_j(x\,,\,u\,,\,\dot x\,,\,\dot u)\,g_j(t,u,\dot u)\ dt\ \leq\ 0$$

$$\Rightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t)\eta(t,x,u)\left\{g_x^j(t,u,\dot u) - \frac{d}{dt} g_{\dot x}^j(t,u,\dot u)\right\} dt\ \leq\ 0\,,$$

$\forall\ x \in K$, for all $i \in P = \{1,2,\ldots,p\}andj \in M = \{1,2,\ldots,m\}$.

We consider the following nondifferentiable multiobjective continuous programming problem (CNP):

$$\text{minimize}\ \left(\int_a^b \left\{f_1(t,x,\dot x) + \left(x^T(t)\,B_1\,x(t)\right)^{1/2}\right\} dt, \ldots,\right.$$

$$\left.\int_a^b \left\{f_p(t,x,\dot x) + \left(x^T(t)\,B_p\,x(t)\right)^{1/2}\right\} dt\right)$$

subject to

$$x(a) = \alpha,\ x(b) = \beta$$

$$g(t,x,\dot x)\ \leq\ 0,\ t \in I,$$

$$x \in C(I,\ R^n)\,,$$

where $f_i : I \times R^n \times R^n \to R,\ i \in P = \{1, \ldots, p\}, g : I \times R^n \times R^n \to R^m$ are assumed to be continuously differentiable functions and each B_i, $\forall\, i \in P$, is an $n \times n$ positive semidefinite matrix.

Let K_{CNP} denote the set of all feasible solutions for (CNP), that is

$$K_{CNP} = \{x \in C(I,\ R^n) :\ x(a) = \alpha,\ x(b) = \beta,\ g(t,\ x(t),\ \dot{x}(t))\ \leq\ 0, t \in I\}.$$

In the subsequent analysis, we shall frequently use the following generalized Schwarz inequality

$$x^T Bz \leq \left(x^T Bx\right)^{1/2} \left(z^T Bz\right)^{1/2},$$

where B is an $n \times n$ positive semidefinite matrix.

3 Optimality Conditions

In this section we establish some sufficient conditions for an $x^* \in K_{CNP}$ to be efficient solution of problem (CNP) under various generalized V-type I invexity assumptions.

Theorem 3.1 (Sufficiency). *Let x^* be feasible for (CNP) and suppose that there exist $\tau^* \in R^p$, $\tau^* \geq 0$ and a piecewise smooth function $\lambda^* : I \to R^m$, $\lambda^*(t) \geq 0$, such that* $\forall\, t \in I$,

$$\sum_{i=1}^{p} \left(\tau_i^* f_x^i(t, x^*, \dot{x}^*) + B_i(t)\ z_i(t)\right) + \sum_{j=1}^{m} \lambda_j^*(t)\ g_x^j(t, x^*, \dot{x}^*)$$

$$= \frac{d}{dt}\left(\sum_{i=1}^{p} \tau_i^* f_{\dot{x}}^i(t, x^*, \dot{x}^*) + \sum_{j=1}^{m} \lambda_j^* g_{\dot{x}}^j(t, x^*, \dot{x}^*)\right), \tag{3.1}$$

$$z_i^T B_i z_i \leq 1, \quad i = 1, 2, \ldots, p \tag{3.2}$$

$$\int_a^b \sum_{j=1}^{m} \lambda_j^* g_j(t, x^*, \dot{x}^*)\ dt = 0. \tag{3.3}$$

If the problem (CNP) is quasi strictly pseudo $V-$type I invex with respect to η, α_i and β_j, $\forall\, i \in P,\ j \in M$. Then x^* is an efficient solution for (CNP).

Proof. Suppose that x^* is not an efficient solution for (CNP), then there exists an $x \in K_{CNP}$ such that

$$\int_a^b \left\{f_i(t, x, \dot{x}) + \left(x^T(t) B_i(t) x(t)\right)^{1/2}\right\} dt$$

$$\leq \int_a^b \left\{f_i(t, x^*, \dot{x}^*) + \left(x^{*T}(t) B_i(t) x^*(t)\right)^{1/2}\right\} dt, \quad \forall\, i \in P,$$

and

$$\int_a^b \left\{f_{i_0}(t, x, \dot{x}) + \left(x^T(t) B_{i_0}(t) x(t)\right)^{1/2}\right\} dt$$

$$< \int_a^b \left\{f_{i_0}(t, x^*, \dot{x}^*) + \left(x^{*T}(t) B_{i_0}(t) x^*(t)\right)^{1/2}\right\} dt, \quad \text{for some } i_0 \in P,$$

which implies that

$$\int_a^b \sum_{i=1}^{p} \tau_i^* \alpha_i (x , x^* , \dot{x} , \dot{x}^*) \left\{ f_i (t, x, \dot{x}) + \left(x^T (t) B_i (t) x (t)\right)^{1/2} \right\} dt$$

$$\leq \int_a^b \sum_{i=1}^{p} \tau_i^* \alpha_i (x , x^* , \dot{x} , \dot{x}^*) \left\{ f_i (t, x^*, \dot{x}^*) + \left(x^{*T} (t) B_i (t) x^* (t)\right)^{1/2} \right\} dt.$$

From the above inequality, the inequality (3.2), the generalized Schwarz inequality and the invexity hypothesis on the problem, we have

$$\int_a^b \sum_{i=1}^{p} \tau_i^* \eta (t , x , x^*) \left\{ f_x^i (t , x^*, \dot{x}^*) + B_i (t) z_i (t) - \frac{d}{dt} f_{\dot{x}}^i (t , x^*, \dot{x}^*) \right\} dt \leq 0 . \qquad (3.4)$$

From (3.1) and (3.4), we have

$$\int_a^b \sum_{j=1}^{m} \lambda_j^* (t) \eta (t, x, x^*) \left\{ g_x^j (t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j (t, x^*, \dot{x}^*) \right\} dt \geq 0. \qquad (3.5)$$

From the invexity hypothesis on the problem (CNP) and (3.5), we have

$$\int_a^b \sum_{j=1}^{m} \lambda_j^* (t) \beta_j (x , x^*, \dot{x} , \dot{x}^*) \; g_j (t , x^*, \dot{x}^*) \; dt < 0 . \qquad (3.6)$$

Since $\beta_j \in R_+ \setminus \{0\}$, $\forall j \in M$, from (3.3), we have

$$\int_a^b \sum_{j=1}^{m} \lambda_j^* (t) \beta_j (x , x^*, \dot{x} , \dot{x}^*) g_j (t , x^*, \dot{x}^*) \; dt = 0,$$

which contradicts the inequality (3.6) and hence x^* is an efficient solution for (CNP). □

Theorem 3.2 (Sufficiency). *Let x^* be feasible for (CNP) and suppose that there exist $\tau^* \in R^p$, $\tau^* > 0$ and a piecewise smooth function $\lambda^* : I \to R^m$, $\lambda^* (t) \geq 0$, such that $\forall\, t \in I$, the inequalities (3.1)-(3.3) from Theorem 3.1 hold. If the problem (CNP) is pseudo quasi $V-$type I invex with respect to η, α_i and β_j, $\forall\, i \in P$, $j \in M$. Then x^* is an efficient solution for (CNP).*

Proof. Suppose that x^* is not an efficient solution for (CNP), then there exists an $x \in K_{CNP}$ such that

$$\int_a^b \left\{ f_i (t, x, \dot{x}) + \left(x^T (t) B_i (t) x (t)\right)^{1/2} \right\} dt$$

$$\leq \int_a^b \left\{ f_i (t, x^*, \dot{x}^*) + \left(x^{*T} (t) B_i (t) x^* (t)\right)^{1/2} \right\} dt , \quad \forall\, i \in P ,$$

and

$$\int_a^b \left\{ f_{i_0}(t,x,\dot{x}) + \left(x^T(t)\,B_{i_0}(t)\,x(t)\right)^{1/2} \right\} dt$$
$$< \int_a^b \left\{ f_{i_0}(t,x^*,\dot{x}^*) + \left(x^{*T}(t)\,B_{i_0}(t)\,x^*(t)\right)^{1/2} \right\} dt, \quad \text{for some } i_0 \in P,$$

which implies that

$$\int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) \left\{ f_i(t,x,\dot{x}) + \left(x^T(t)\,B_i(t)\,x(t)\right)^{1/2} \right\} dt$$
$$< \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) \left\{ f_i(t,x^*,\dot{x}^*) + \left(x^{*T}(t)\,B_i(t)\,x^*(t)\right)^{1/2} \right\} dt. \quad (3.7)$$

Since $\beta_j \in R_+ \setminus \{0\}$, $\forall j \in M$, from (3.3), we have

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\,\beta_j(x, x^*, \dot{x}, \dot{x}^*)\,g_j(t, x^*, \dot{x}^*)\,dt = 0.$$

From the above inequality, the invexity hypothesis on the problem, we have

$$\int_a^b \sum_{j=1}^m \lambda_j^* \eta(t, x, x^*) \left\{ g_x^j(t,x^*,\dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t,x^*,\dot{x}^*) \right\} dt \leq 0. \quad (3.8)$$

From (3.1) and (3.8), we have

$$\int_a^b \sum_{i=1}^p \tau_i^* \eta(t, x, x^*) \left\{ f_x^i(t, x^*, \dot{x}^*) + B_i(t)\,z_i(t) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*) \right\} dt \geq 0. \quad (3.9)$$

From the invexity hypothesis on the problem (CNP) and (3.9), we have

$$\int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) \left\{ f_i(t,x,\dot{x}) + \left(x^T(t)\,B_i(t)\,x(t)\right)^{1/2} \right\} dt$$
$$\geq \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) \left\{ f_i(t,x^*,\dot{x}^*) + \left(x^{*T}(t)\,B_i(t)\,x^*(t)\right)^{1/2} \right\} dt,$$

which contradicts the inequality (3.7) and hence x^* is an efficient solution for (CNP).

The sufficiency conditions can be established under semi strictly quasi V-type I and semi strictly pseudo V-type I invexity assumptions as well. More precisely, we have following Theorems 3.3 and 3.4. However, as the proofs will be similar to the above Theorems 3.1 and 3.2, we omit the proofs. □

Theorem 3.3 (Sufficiency). *Let x^* be feasible for (CNP) and suppose that there exist $\tau^* \in R^p$, $\tau^* \geq 0$ and a piecewise smooth function $\lambda^* : I \to R^m$, $\lambda^*(t) \geq 0$, such that $\forall\, t \in I$, the inequalities (3.1)-(3.3) from Theorem 3.1 hold. If the problem (CNP) is semi strictly quasi V-type I invex with respect to η, α_i and β_j, $\forall\, i \in P$, $j \in M$. Then x^* is an efficient solution for (CNP).*

Theorem 3.4 (Sufficiency). *Let x^* be feasible for (CNP) and suppose that there exist $\tau^* \in R^p$, $\tau^* \geq 0$ and a piecewise smooth function $\lambda^* : I \to R^m$, $\lambda^*(t) \geq 0$, such that $\forall\, t \in I$, the inequalities (3.1)-(3.3) from Theorem 3.1 hold. If the problem (CNP) is semi strictly pseudo V-type I invex with respect to η, α_i and β_j, $\forall\, i \in P$, $j \in M$. Then x^* is an efficient solution for (CNP).*

4 Mond-Weir Duality

We consider the following dual problem to (CNP):

(MWD)

$$\text{maximize} \left(\int_a^b \{f_1(t, u(t), \dot{u}(t)) + u(t)^T B_1(t) z_1(t)\} dt, \ldots, \int_a^b \{f_p(t, u(t), \dot{u}(t)) + u(t)^T B_p(t) z_p(t)\} dt \right)$$

subject to

$$u(a) = \alpha, \quad u(b) = \beta$$

$$\sum_{i=1}^{p} \tau_i f_x^i(t, u, \dot{u}) + B_i(t)\, z_i(t) + \sum_{j=1}^{m} \lambda_j(t)\, g_x^j(t, u, \dot{u}) = \frac{d}{dt}\left(\sum_{i=1}^{p} \tau_i f_{\dot{x}}^i(t, u, \dot{u}) + \sum_{j=1}^{m} \lambda_j g_{\dot{x}}^j(t, u, \dot{u}) \right), \tag{4.1}$$

$$z_i^T B_i z_i \leq 1, \quad i = 1, 2, \ldots, p \tag{4.2}$$

$$\int_a^b \lambda_j(t) g_j(t, u, \dot{u})\, dt \geq 0, \ \forall\, j \in M, \tag{4.3}$$

$$\lambda(t) \geq 0, \ t \in I. \tag{4.4}$$

We let Y_0 be the set of feasible solutions of the problem (MWD). Now we establish some duality theorems for the pair of problems (CNP) and (MWD).

Theorem 4.1 (Weak Duality). *Let x be feasible for (CNP) and $(u, \tau, \lambda) \in Y_0$. If the problem (CNP) is pseudo quasi V-type I invex with respect to η, α_i and β_j, $\forall\ i \in P,\ \ j \in M$. Then the following cannot hold:*

$$\int_a^b \left\{ f_i(t, x, \dot{x}) + \left(x^T(t) B_i(t) x(t)\right)^{1/2} \right\} dt \leq \int_a^b \left\{ f_i(t, u, \dot{u}) + \left(u^T(t) B_i(t) u(t)\right)^{1/2} \right\} dt, \quad \forall\ i \in P$$

and

$$\int_a^b \left\{ f_{i_0}(t, x, \dot{x}) + \left(x^T(t) B_{i_0}(t) x(t)\right)^{1/2} \right\} dt < \int_a^b \left\{ f_{i_0}(t, u, \dot{u}) + \left(u^T(t) B_{i_0}(t) u(t)\right)^{1/2} \right\} dt, \quad \textit{for some } i_0 \in P.$$

Proof. Suppose contrary to the result, since $\alpha_i \in R_+ \backslash \{0\}$, $\forall i \in P$, and $\tau > 0$, we get

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, u, \dot{x}, \dot{u}) \left\{ f_i(t, x, \dot{x}) + \left(x^T(t) B_i(t) x(t)\right)^{1/2} \right\} dt < \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, u, \dot{x}, \dot{u}) \left\{ f_i(t, u, \dot{u}) + \left(u^T(t) B_i(t) u(t)\right)^{1/2} \right\} dt. \quad (4.5)$$

Since $\beta_j \in R_+ \backslash \{0\}$, $\forall j \in M$, by duality constraint (4.3), we have

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\ \beta_j(x, u, \dot{x}, \dot{u})\ g_j(t, u, \dot{u})\ dt \geq 0. \quad (4.6)$$

By the generalized invexity hypothesis and (4.6), we get

$$\int_a^b \sum_{j=1}^m \lambda_j \eta(t, x, u) \left\{ g_x^j(t, u, \dot{u}) - \frac{d}{dt} g_{\dot{x}}^j(t, u, \dot{u}) \right\} dt \leq 0. \quad (4.7)$$

From (4.1) and (4.7), we have

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t, x, u) \left\{ f_x^i(t, u, \dot{u}) + B_i(t) z_i(t) - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u}) \right\} \geq 0. \quad (4.8)$$

By the generalized invexity hypothesis, generalized Schwarz inequality and (4.8), we get

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i (x , u , \dot{x} , \dot{u}) \left\{ f_i (t, x, \dot{x}) + \left(x^T (t) B_i (t) x (t) \right)^{1/2} \right\} dt$$
$$\geq \int_a^b \sum_{i=1}^p \tau_i \alpha_i (x , u , \dot{x} , \dot{u}) \left\{ f_i (t, u , \dot{u}) + \left(u^T (t) B_i (t) u (t) \right)^{1/2} \right\} dt,$$

which contradicts (4.5), this completes the proof. □

Theorem 4.2 (Weak Duality). *Let x be feasible for (CNP) and $(u , \tau , \lambda) \in Y_0$. If the problem (CNP) is semi strictly V-type I invex with respect to η, α_i and β_j, $\forall\, i \in P,\ \ j \in M$. Then the following cannot hold:*

$$\int_a^b \left\{ f_i (t, x, \dot{x}) + \left(x^T (t) B_i (t) x (t) \right)^{1/2} \right\} dt$$
$$\leq \int_a^b \left\{ f_i (t, u , \dot{u}) + \left(u^T (t) B_i (t) u (t) \right)^{1/2} \right\} dt , \quad \forall\, i \in P,$$

and

$$\int_a^b \left\{ f_{i_0} (t, x, \dot{x}) + \left(x^T (t) B_{i_0} (t) x (t) \right)^{1/2} \right\} dt$$
$$< \int_a^b \left\{ f_{i_0} (t, u , \dot{u}) + \left(u^T (t) B_{i_0} (t) u (t) \right)^{1/2} \right\} dt , \quad \textit{for some } i_0 \in P.$$

Proof. Suppose contrary to the result, since $\alpha_i \in R_+ \setminus \{0\}$, $\forall\, i \in P$, and $\tau \geq 0$, we get

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i (x , u , \dot{x} , \dot{u})^{-1} \left\{ f_i (t, x, \dot{x}) + \left(x^T (t) B_i (t) x (t) \right)^{1/2} \right\} dt$$
$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i (x , u , \dot{x} , \dot{u})^{-1} \left\{ f_i (t, u , \dot{u}) + \left(u^T (t) B_i (t) u (t) \right)^{1/2} \right\} dt. \quad (4.9)$$

Since $\beta_j \in R_+ \setminus \{0\}$, $\forall\, j \in M$, by duality constraint (4.3), we have

$$\int_a^b \sum_{j=1}^m \lambda_j (t)\ \beta_j (x , u , \dot{x} , \dot{u})\ g_j (t , u , \dot{u})\ dt\ \geq 0 . \quad (4.10)$$

By the generalized invexity hypothesis and (4.10), we get

$$\int_a^b \sum_{j=1}^m \lambda_j \eta (t, x, u) \left\{ g_x^j (t, u , \dot{u}) - \frac{d}{dt} g_{\dot{x}}^j (t, u , \dot{u}) \right\} dt\ \leq\ 0. \quad (4.11)$$

From (4.1) and (4.11), we have

$$\int_a^b \sum_{i=1}^p \tau_i \eta\,(t\,,\,x\,,\,u)\,\left\{f_x^i\,(t\,,\,u,\,\dot{u}) + B_i\,(t)\,z_i\,(t) - \frac{d}{dt} f_{\dot{x}}^i\,(t\,,\,u,\,\dot{u})\right\} \geq 0\,. \tag{4.12}$$

By the generalized invexity hypothesis, generalized Schwarz inequality and (4.12), we get

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i\,(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})^{-1}\left\{f_i\,(t, x, \dot{x}) + \left(x^T\,(t)\,B_i\,(t)\,x\,(t)\right)^{1/2}\right\} dt$$
$$\geq \int_a^b \sum_{i=1}^p \tau_i \alpha_i\,(x\,,\,u\,,\,\dot{x}\,,\,\dot{u})^{-1}\left\{f_i\,(t, u\,, \dot{u}) + \left(u^T\,(t)\,B_i\,(t)\,u\,(t)\right)^{1/2}\right\}\,dt,$$

which contradicts (4.9), this completes the proof. □

Strong and converse duality theorems can be established on the lines of Mishra [15] and Kim and Kim [13], in the light of the discussions given in the present paper.

5 Conclusion

In the present paper, we have extended a recent work of Kim and Kim [13] to nondifferentiable case, as a by product our results extend an earlier work of Mishra [15] to more general class of generalized invexity and an earlier work of Hanson *et al.* [9] to nondifferentiable and continuous-time case, as well.

Moreover, it will be interesting to see if the results of the present paper can be extended to the class of functions given by Aghezaaf and Hachimi [1]. The results of Mishra and Mukherjee [18] can be extended to the classes of functions used in the present paper. Some of these problems will be topic of research of forth coming papers of the authors.

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