Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 3, No. 2, pp. 227 – 234, May 2007

AN OPTIMAL UTILITY VALUE METHOD WITH MAJORITY EQUILIBRIUM FOR PUBLIC FACILITY ALLOCATION

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Abstract: We consider the public facility allocation problem decided through an optimal utility value under the majority rule in public facility management. A location of the public facility is a majority rule winner with optimal utility value if no other location in the network is with better utility value than the winner. We define a weight function and establish the network model for the cases with one or more than one public facilities to be located. We show that there exists a modified weak quasi-Condorcet winner if the public facility allocation graph model is a tree. Based on above discussion we proposed a practical majority equilibrium method for more general public facility allocation problems.

Key words: *public facility allocation problems, optimal utility value method, majority equilibrium*

Mathematics Subject Classification: *49L20, 91A10, 91A25, 91A65*

1 Introduction

In many management problems, majority vote is often the ultimate decision making tool. The concept of majority equilibrium captures such a democratic spirit in requiring that no other solutions would please more than half of the participants (or more than half of the total voting weight for participants with weighted voting powers). A majority equilibrium solution is a stable point solution under a democratic (sometimes weighted) decision making mechanism, which is employed not only in public management but also in business management decision making processes. Such a perfectly defined solution, however, may not always exist. As in the famous Condorcet paradox, three agents have three different orders of preferences, $A > B > C$, $B > C > A$, $C > A > B$ among three alternatives A, B and C, would not yield a majority equilibrium solution. In reality, the paradox phenomena would have to be dealt with and a solution should be settled.

The public facility allocation problem is a case which would fit into one such decision problem. In this model, a group of collaborating retailing agents would have to decide on locations to set up public facilities that would benefit the majority of the agents. A closely related setting is considered by Demange for continuous and discrete spatial models of collective choice, aiming at characterization of the location problem of public services as a result of public voting process [4]. To facilitate a rigorous study of the related problem, Demange proposed four types of majority equilibrium solutions (call Condorcet Winners) and discussed corresponding results concerning conditions for their existences.

A weighted version of the discrete model of Demange for public facility allocation problem is of particular interests to us. The environment is represented by a network $G =$

$((V,w),(E,l))$ that link voters together. In the model, for each $i \in V$, $w(i)$ represents the voting power of voters reside at i, which can be decided by the voting system or by the decision power at a vertex. For each $e \in E$, $l(e)$ represents the distance between two voters i and j. We will consider a special type of utility function: the sum distance between the location of the public facility to all voters, which each voter want to minimize. While each desires to have the public facility to be close to itself, the decision has to be agreed upon by a majority of the votes.

Following Demange [4], a location $x \in V$ is a strong (respectively weak) Condorcet winner if, for any $y \in V$, the total weight of vertices that is closer to x than to y is more (respectively no less) than the total weight of vertices that is closer to y than to x. Similarly, it is a quasi-Condorcet winner if we change "closer to x than" to "closer to x than y or of the same distance to x as y". Of the four types of majority winner, strong Condorcet winner is the most restrictive of all, and weak quasi-Condorcet winner is the lest restrictive one and the other two are between them. For discrete models considered by Romero [13], Hansen and Thisse [10], it was known that, the order induced by strict majority relation (the weak Condorcet order) in a tree is transitive. Therefore, a weak Condorcet winner in any tree always exists. In addition, Demange extended the existence condition of a weak Condorcet winner to all single peaked orders on trees [5].

Motivated by the above results, we structured the weighted function based on the majority rule and proposed an optimal utility function for the public facility allocation problem. Our study distinguishes from previous work in our focus in weighted function and optimal utility value issues with the imperfect information from majority voting. The weighted function will depend on the majority voting process and the utility function is defined as the sum value of the distance between the location of the public facility to voters. In Section 2, we establish our majority voting process, introduce some denotations , define formal formulation of the single public facility location problem and modify the definition issue of Condorcet winners. in Section 3. We present a linear algorithm for finding a modified weak quasi-Condorcet winners of a tree with the proposed vertex-weight function and edge-length functions; and prove that in the case, the modified weak quasi-Condorcet points are the points which minimize the total weight-distance to the individuals' locations. Based on above discussion we will propose a practical majority equilibrium method for general cases in Section 4. In Section 5 we conclude the paper.

$\boxed{2}$ Denotations and Definitions

In [4], Demange has surveyed and discussed some spatial models of collective choice, some results concerning the transitivity of the majority rule and the existence of a majority winner. Let $S = \{1, 2, \cdots, n\}$ be a society representing a set of n individuals, and X be a set of alternatives (or choice space). Each individual $i \in S$ has a preference order, denoted $\geq_i$, on X. The n-tuple $(\geq_i)_{i\in S}$ is called the *profile* of the society. Given a profile $(\geq_i)_{i\in S}$ of the society X, an alternative $x \in X$ is called:

(a) *Weak quasi-Condorcet winner* if for every $y \in X$ distinct of x,

$$|\{i \in S : y >_i x\}| \leq \frac{n}{2}; \text{ i.e. } |\{i \in S : x \geq_i y\}| \geq \frac{n}{2}.$$

(b) *Strong quasi-Condorcet winner* if for every $y \in X$ distinct of x,

$$|\{i \in S : y >_i x\}| < \frac{n}{2}; \text{ i.e. } |\{i \in S : x \geq_i y\}| > \frac{n}{2}.$$

(c) *Weak Condorcet winner* if for every $y \in X$ distinct of x,

$$|\{i \in S : x >_i y\}| \geq |\{i \in S : y >_i x\}|.$$

(d) *Strong Condorcet winner* if for every $y \in X$ distinct of x,

$$|\{i \in S : x >_i y\}| > |\{i \in S : y >_i x\}|.$$

In [3], motivated by Demange's results and based on the proposed formal formulation of the public facility location problem with a single facility in a network, Chen et al are interested in classify the types of networks for which a Condorcet winner can be found in linear time. As a warm-up example, they present the solution for trees and a linear algorithm for finding the weak quasi-Condorcet winners of a tree with vertex-weight and edge-length functions, and prove that in the case, the weak quasi-Condorcet points are the points which minimize the total weight-distance to the individuals' locations. Furthermore, they give a sufficient and necessary condition for a point to be a weak quasi-Condorcet point for cycles if the edge-length function is a constant, and present a more interesting linear time algorithm.

We note that by Demange's definitions of Condorcet winner, only the number of the society's members are countered, but the decision power difference among candidate locations or the voting asymmetric information are ignored. And in [3], although the definition of vertex-weight is introduced it is needed to establish the meaningful and exact vertex-weight function. In this paper, we modify Demange's definitions by incorporating weighted functions and utility functions and give the exact definition of vertex-weight in [3]. We will give some denotations and definition needed afterward.

In this paper we consider the following majority voting process:

The definition of majority voting process:

Step 1. Design the vote such that it includes the following three options: the name of the voter, all of the public facility location's candidates, and the distance of the voter's location to the public facility location voted.
Step 2. Hand out the votes and every voter can choose one or more location listed in the vote. Let k denote the number of locations voted by a voter. The voter's decision power for the voted locations is defined as just $\frac{1}{k}$.
Step 3. Collect the votes and make decision.

We first consider a single public facility location problem.

Denote $V = v_1, v_2, \cdots, v_n$ the set of n public facility candidate locations. In every location v_i,$i = 1, 2, \cdots, n$, there are u_i voters and denote them as $(u_1^i, u_2^i, \cdots, u_{u_i}^i)$.

Denote $d(u_k^i, v_j)$ the distance from k-th voter in location i to location j, where $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, u_i$.

Denote by $d_G(v, v')$ the length of a shortest chain joining two locations v and v' in G, and call it *the distance between the two locations v and v' in G.*

Define the weight function at vertex v_i as $\omega(v_i) = \sum_{j=1}^{u_i} f_i(u_j^i)$ where $f_i(u_j^i)$ is defined as follows:$f_i(u_j^i) = \frac{1}{k_{ij}}$, if the voter u_j^i marked k_{ij} candidates including the vertex v_i in V and $0 < k_{ij} \leq n$, where $i = 1, 2, \cdots, n$; otherwise, $f_i(u_j^i) = 0$, with the voter u_j^i does not vote the vertex v_i. We call the value $f_i(u_j^i)$ is the decision power of voter u_j^i at location v_i and the weight function $\omega(v_i)$ is the decision power of vertex v_i by the weighted cumulating votes.

For any $R \subseteq V$, we set $\omega(R) = \sum_{i \; \| v_i \in R} \sum_{j=1}^{u_i} f_i(u_j^i)$. In particular, if $R = V$, we write $\omega(G)$ instead of $\omega(V)$, i.e. $\omega(G) = \sum_{i=1}^{n} \sum_{j=1}^{u_i} f_i(u_j^i)$. A vertex v of G is said to be *pendant* if v has exact one neighbor in G.

We can model our public facility allocation problem as follows:

The graph model of public facility allocation (GMPFA):

We consider the undirected graph model $G = (V, E)$ of order n with the weight function ω that assigns to each vertex v of G a non-negative weight $\omega(v)$ defined above, and a length function l that assigns to each edge e of G the distance between the two end locations of the edge e. If P is a chain of G, then we denote by $l(P)$ the sum of lengths of all edges of P.

Modified definitions of Condorcet winner in the proposed model(GMPFA):

Given a graph $G = (V, E)$ with $V = \{v_1, v_2, \cdots, v_n\}$, each $v_i \in V$ has a preference order $\geq_i$ on V induced by the distance on G. That is, we have $x \geq_i y$ for any two vertices x and y of G if and only if $d_G(v_i, x) \leq d_G(v_i, y)$. The following definition is an extension of that given in [4]

Definition 2.1. Given a graph $G = (V, E)$ and profile $(\geq_i)_{v_i \in V}$ on V, denote $\Phi = \{i \; \| v_i \in V, \; u >_i v_0\}$ and $\Psi = \{i \; \| v_i \in V : v_0 \geq_i u\}$. A vertex v_0 in V is called:

(1) *Modified weak quasi-Condorcet winner*, if for every $u \in V$ distinct of v_0,

$$\omega(\{v_i \in V : u >_i v_0\}) \leq \frac{\omega(G)}{2}; \text{ i.e. } \omega(\{v_i \in V : v_0 \geq_i u\}) \geq \frac{\omega(G)}{2}.$$

or in other words,

$$\sum_{i \in \Phi} \sum_{j=1}^{u_i} f_i(u_j^i) \leq \frac{\sum_{i=1}^{n} \sum_{j=1}^{u_i} f_i(u_j^i)}{2};$$

i.e.

$$\sum_{i \in \Psi} \sum_{j=1}^{u_i} f_i(u_j^i) \geq \frac{\sum_{i=1}^{n} \sum_{j=1}^{u_i} f_i(u_j^i)}{2}. \tag{2.1}$$

(2) *Modified strong quasi-Condorcet winner*, if for every $u \in V$ distinct of v_0,

$$\omega(\{v_i \in V : u >_i v_0\}) < \frac{\omega(G)}{2}; \text{ i.e. } \omega(\{v_i \in V : v_0 \geq_i u\}) > \frac{\omega(G)}{2}.$$

or or in other words,

$$\sum_{i \in \Phi} \sum_{j=1}^{u_i} f_i(u_j^i) < \frac{\sum_{i=1}^{n} \sum_{j=1}^{u_i} f_i(u_j^i)}{2};$$

i.e.

$$\sum_{i \in \Psi} \sum_{j=1}^{u_i} f_i(u_j^i) > \frac{\sum_{i=1}^{n} \sum_{j=1}^{u_i} f_i(u_j^i)}{2}. \tag{2.2}$$

(3) *Modified weak Condorcet winner*, if for every $u \in V$ distinct of v_0,

$$\omega(\{v_i \in V : v_0 >_i u\}) \geq \omega(\{v_i \in V : u >_i v_0\}).$$

or in other words,

$$\sum_{i \in \Psi} \sum_{j=1}^{u_i} f_i(u_j^i) \geq \sum_{i \in \Phi} \sum_{j=1}^{u_i} f_i(u_j^i). \tag{2.3}$$

(4) *Modified strong Condorcet winner*, if for every $u \in V$ distinct of v_0,

$$\omega(\{v_i \in V : v_0 >_i u\}) > \omega(\{v_i \in V : u >_i v_0\}).$$

or in other words,

$$\sum_{i \in \Psi} \sum_{j=1}^{u_i} f_i(u_j^i) > \sum_{i \in \Phi} \sum_{j=1}^{u_i} f_i(u_j^i). \tag{2.4}$$

Example 2.2. Denote by K_2 and K_3 the complete graphs of orders 2 and 3, respectively. Suppose that the length functions on edge set and the weight functions on vertex set in K_2 and K_3 are constant which means that the decision power at any location is same. Then K_2 has modified weak Condorcet winners, and hence has also modified weak quasi-Condorcet winners, but has no modified strong Condorcet winners, and hence has no modified strong quasi-Condorcet winners; K_3 has modified strong quasi-Condorcet winner, modified weak Condorcet winner and modified weak quasi-Condorcet winner, but has no modified strong Condorcet winner.

In the sequel, we will only consider the algorithm for finding modified weak quasi-Condorcet winner of a tree. The properties and algorithms for other three types of Condorcet winners can be discussed in a similar way.

3 Weak Quasi-Condorcet Winner of a Tree

Romero In [13], Hansen and Thisse In [10] pointed out that the family of orders induced by a distance on a tree guarantees the existence of a weak Condorcet winner. Furthermore, the weak Condorcet points are the points which minimize the total distance to the individuals' locations [4]. In this section we propose a linear algorithm for finding the modified weak quasi-Condorcet winners on a tree with vertex-weight and edge-length functions; and prove that in this case, the modified weak quasi-Condorcet winners are the same as points which minimize the total weight-distance to the individuals' locations. In fact, the same conclusions hold for modified weak Condorcet winners.

Given two vertices $v, x \in V$, the set of *quasi-friend* vertices of v relative to x is defined as

$$F_G(v, x) = \{u : \ d_G(u, v) \leq \ d_G(u, x)\};$$

and the set of *hostile* vertices of v relative to x is defined as

$$H_G(v, x) = \{u : \ d_G(u, v) > \ d_G(u, x)\}.$$

By the definition of modified weak quasi-Condorcet winner, a vertex $v_0 \in V$ is a modified weak quasi-Condorcet winner of G, if for any vertex $x \neq v_0$,

$$\omega(F_G(v_0, x)) \geq \ \frac{1}{2}\omega(G), \quad \text{or equivalently,} \ \ \omega(F_G(v_0, x)) \geq \ \omega(H_G(v_0, x)),$$

or

$$\sum_{i\in\Phi}\sum_{j=1}^{u_i} f_i(u_j^i) \le \frac{\sum_{i=1}^{n}\sum_{j=1}^{u_i} f_i(u_j^i)}{2};$$

i.e.

$$\sum_{i\in\Psi}\sum_{j=1}^{u_i} f_i(u_j^i) \ge \frac{\sum_{i=1}^{n}\sum_{j=1}^{u_i} f_i(u_j^i)}{2}. \tag{3.1}$$

Similar to the proofs in [3], we can show that

Theorem 3.1. *Every tree has one modified weak quasi-Condorcet winner , or two adjacent modified weak quasi-Condorcet winners. We can find it or them in linear time.*

Theorem 3.2. *Let T be a tree. Then v_0 is a modified weak quasi-Condorcet winner of T if and only if v_0 is a barycenter of T.*

Theorem 3.3. *Let $T = (V, E)$ be a tree, $N = \{1, 2, \cdots, n\}$ be the set of voters with positive weight $\omega : N \to R^+$. The majority rule $\pi : V^n \to V$ of choosing modified weak quasi-Condorcet winners satisfies the property of (group)strategy-proofness.*

4 A Practical Algorithm to Get Modified Weak Quasi-Condorcet Winner for Connected Public Facility Allocation

We consider the undirected graph model $G = (V, E)$ of order n with a weight function ω that assigns to each vertex v_i of G with $\omega(v_i) = \sum_{j=1}^{u_i} f_i(u_j^i)$ where $i = 1, \cdots, n$ and $f_i(u_j^i)$ is defined as above, and a length function l that assigns to each edge e of G a positive length $l(e)$. Notice that for any $R \subseteq V$, we set $\omega(R) = \sum_{i,\|v_i\in R}\sum_{j=1}^{u_i} f_i(u_j^i)$. In particular, $\omega(V) = \omega(G) = \sum_{i,\|v_i\in V}\sum_{j=1}^{u_i} f_i(u_j^i)$ and notice that a vertex v of G is said to be *pendant* if v has exact one neighbor in G. In section 3, we have shown the existence of the majority equilibrium when the connected undirected graph model $G = (V, E)$ of order n of public facility allocation problem is a tree. From the proofs in Section 3 and with the minimum spanning tree technique we can establish a practicable algorithm to get the modified weak Quasi-Condorcet winner for the connected graph cases.

A Practical Algorithm for the connected graph cases:

Step 1. Find the minimum spanning tree of the graph G, and denote it as $T = (V, l'(e)$ where $l'(e) \subseteq l(e)$;

Step 2. Take the pendant vertex v of T such that $w(v) < \frac{1}{2}w(T)$;

Step 3. $T - v \Rightarrow T$, $w(v) + w(u) \Rightarrow w(u)$, where u is the (unique) neighbor of v;

Step 4. $n - 1 \Rightarrow n$, If $n = 1$, or, $n = 2$ and the two vertices have the same weights, then stop; otherwise go to Step 2.

According to the proof of Section 4 in [3], we can also claim that there exists a modified weak quasi-Condorcet winner for the proposed Practical Algorithm. The proof is omitted here.

$\boxed{5}$ Conclusions

In this work, we consider the public facility location problem decided via a voting process under the majority rule. Our study follows the network model that has been applied to the study of similar problems in economics [3, 10, 13, 14]. Our mathematical results depend on understanding of combinatorial structures of underlying networks.

Many problems open up from our study. The complexity study for other rules for public facility location is very interesting and deserves further study. It would also be interesting to extend our study to other areas and problems of public decision making process.

References

[1] S. Barbera, J. Masso and S. Serizawa, Strategy-proof voting on compact ranges, *Games Econom. Behav.* 25 (1998) 272–291.

[2] K. Border and J. Jordan, Straightforward elections, unanimity and phantom retailers, *Rev. Econ. Stud.* 50 (1983) 153–170.

[3] L. Chen, X. Deng, Q. Fang and F. Tian, Majority equilibrium for public facility allocation, in *Computing and combinatorics*, Lecture Notes in Comput. Sci. Vol. 2697, Springer, Berlin, 2003, pp. 435–445.

[4] G. Demange, Spatial models of collective choice, in *Locational Analysis of Public Facilities*, J.F. Thisse and H.G. Zoller (eds.), North-Holland Publishing Company, 1983.

[5] G. Demange, Single peaked orders on a tree, *Math. Soc. Sci.* 3 (1982) 389–396.

[6] X. Deng, Z. Li and S. Wang, On Computation of Arbitrage for Markets with Friction, in *Computing and combinatorics (Sydney, 2000)*, Lecture Notes in Comput. Sci. Vol. 1858, Springer, Berlin, 2000, pp. 310–319.

[7] X. Deng and C. Papadimitriou, On the complexity of cooperative game solution concepts, *Math. Oper. Res.* 19 (1994) 257–266.

[8] X. Deng and C. Papadimitriou, Decision-making by hierarchies of discordant agents, *Math. Program.* 86 (1999) 417–431.

[9] X. Deng, C. Papadimitriou and S. Safra, On the complexity of equilibria, in *Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing*, ACM, New York, 2002, pp. 67–71.

[10] P. Hansen and J.-F. Thisse, Outcomes of voting and planning: condorcet, weber and rawls locations, *J. Public Econom.* 16 (1981) 1–15.

[11] N. Megiddo, Computational complexity and the game theory approach to cost allocation for a tree, *Math. Oper. Res.* 3 (1978) 189–196.

[12] H. Moulin, On strategy proofness and single peakedness, *Pub. Choice* 35 (1980) 437–455.

[13] D. Romero, Variations sur l'effet Condorcet, Thèse de 3ème cycle, Université de Grenoble, 1978.

[14] J. Schummer and R.V. Vohra, Strategy-proof location on a network, *J. Econom. Theory* 104 (2002) 405–428.

Manuscript received 14 November 2005
revised 20 June 2006
accepted for publication 27 June 2006

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