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TOWARDS A PIVOTING PROCEDURE FOR A CLASS OF SECOND-ORDER CONE PROGRAMMING PROBLEMS HAVING MULTIPLE CONE CONSTRAINTS

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Dedicated to Professor Masakazu Kojima on the occasion of his 60th birthday.

Abstract: An implementable pivoting procedure for a class of second-order cone programming having one second-order cone was for the first time proposed by [8]. In this paper, we consider a wider class of problems having multiple second-order cones. We derive some fundamental properties necessary for establishing a pivoting algorithm for the class.

Key words: *second-order cone programming, pivot, the simplex method*

Mathematics Subject Classification: *90C49*

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1 Introduction

To date, second-order cone programming (SOCP) problems are solved solely by the interior-point method ([5, 7]) and, to the best of the author's knowledge, no public implementation of pivoting method for SOCP is available. Muramatsu [8] proposed an implementable pivoting method for a subclass of SOCP, and Kurita and Muramatsu [3] reported its numerical efficiency. However, their algorithm can deal with only SOCP problems having single second-order cone.

The purpose of this paper is to establish the theoretical basis necessary to develop a pivoting procedure for SOCP problems having multiple second-order cones by extending the results of [8]. Specifically, we extend the dictionary defined in [8] to the multiple second-order cone case, though the class of the problems is still a proper subclass of the general SOCP. Using the dictionary, we consider subproblems which are solved in the pivoting procedure to determine entering and leaving variables. The main theorem of this paper is that if all the subproblems corresponding to the dictionary have trivial optimal solutions (see Section 3 for the definition of trival optimal solutions), then the current basic solution is optimal.

Another contribution of this paper is to propose a dual solution corresponding to the dictionary, which is a tentative solution for dual, and an extension of dual solution in LP. With this dual solution, we can state the theorem for the multiple cone case in a clearer way, and, furthermore, tell which subproblem has a nontrivial optimal solution without actually solving them. The latter feature may enhance the computational efficiency of the pivoting procedure.

This paper is organized as follows. In the rest of this section, we introduce the SOCP problem we deal with, and define a dictionary for it. In Section 2, we introduce the nondegeneracy, and define a dual solution for a nondegenerate and feasible basic solution. In Section 3, assuming that we are given a feasible dictionary, we consider subproblems to decrease the objective function value of the corresponding basic solution, and describe the main theorem together with its proof. During the course, we show some properties of dual solutions and subproblems which are necessary for developing a pivoting procedure for the class of SOCP. Section 4 contains some concluding remarks.

Thoughout the paper, we use the following notation. For an $m \times n$ matrix A, and an index set $B \subseteq \{1,\ldots,n\}$, we denote by A_B the $m \times |B|$ matrix whose columns are those of A corresponding to B. For $i \in \{1,\ldots,m\}$, A_{iB} is the i-th row vector of A_B. For $N \subseteq \{1,\ldots,m\}$, A_{NB} is the $|N| \times |B|$ matrix whose rows are those of A_B corresponding to N. For $j \in B$, A_{Nj} is the j-th column vector of A_{NB}. Similarly, for a vector $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}_B$ is a sub-vector of $\boldsymbol{x}$ corresponding to B.

We consider a second-order cone programming (SOCP) problem having n nonnegative variables and p second-order cones:

$$\langle P\rangle \begin{cases} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{d}^1)^T\boldsymbol{u}^1 + \ldots + (\boldsymbol{d}^p)^T\boldsymbol{u}^p + d_0^1 u_0^1 + \ldots + d_0^p u_0^p \\ \text{subject to} & A\boldsymbol{x} + R^1\boldsymbol{u}^1 + \ldots + R^p\boldsymbol{u}^p = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0, (u_0^1, \boldsymbol{u}^1) \in \mathcal{K}_{r_1+1}, \ldots, (u_0^p, \boldsymbol{u}^p) \in \mathcal{K}_{r_p+1}, \end{cases}$$

where $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{d}^\mu \in \mathbb{R}^{r_\mu}$ $(\mu = 1,\ldots,p)$, $A \in \mathbb{R}^{m\times n}$, and $R^\mu \in \mathbb{R}^{m\times r_\mu}$ $(\mu = 1,\ldots,p)$, and

$$\mathcal{K}_{r+1} = \left\{ (u_0, \boldsymbol{u}) \in \mathbb{R}^{r+1} \mid u_0 \geq \|\boldsymbol{u}\| \right\}$$

denotes the $r+1$-dimensional second-order cone. A pivoting procedure for the single second-order cone case where $p = 1$ in $\langle P\rangle$ was studied by Muramatsu [8]. In this paper, we consider the case where $p \geq 1$.

We note that u_0^μ $(\mu = 1,\ldots,p)$ does not appear in the equality condition. This is essential to the development of the current pivoting algorithm in SOCP. However, $\langle P\rangle$ still includes important problems such as quadratic programming problems and problems of minimizing a sum of Euclidean norms ([9]). LP is a special case of $\langle P\rangle$ where no second-order cone constraints exist $(p = 0)$. The results in this paper are still valid in this case.

We assume that $d_0^\mu \geq 0$ for $\mu = 1,\ldots,p$, because otherwise $\langle P\rangle$ is obviously unbounded. We then assume that the matrix $(A\ R^1\ R^2\ \cdots R^p)$ is of full rank.

The dual of $\langle P\rangle$ is

$$\langle D\rangle \begin{cases} \text{maximize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{s} + A^T\boldsymbol{y} = \boldsymbol{c} \\ & \boldsymbol{z}^\mu + (R^\mu)^T\boldsymbol{y} = \boldsymbol{d}^\mu\ (\mu = 1,\ldots,p) \\ & \boldsymbol{s} \geq \boldsymbol{0}, (d_0^\mu, \boldsymbol{z}^\mu) \in \mathcal{K}_{r_\mu+1}\ (\mu = 1,\ldots,p). \end{cases}$$

Let us choose index sets $(B, B_1, \ldots, B_p)$ such that

$$G = (A_B, R^1_{B_1}, \ldots, R^p_{B_p}) \in \mathbb{R}^{m\times m}$$

is invertible. Necessarily, we have $|B| + \sum_{\mu=1}^p |B_\mu| = m$. Some of the index sets B and B_μ $(\mu = 1,\ldots,p)$ can be empty. Premultiplying G^{-1} to the equality condition of $\langle P\rangle$, we obtain

$$\begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{u}^1_{B_1} \\ \vdots \\ \boldsymbol{u}^p_{B_p} \end{pmatrix} = G^{-1}\boldsymbol{b} - G^{-1}A_N\boldsymbol{x}_N - \sum_{\mu=1}^{p} G^{-1}R^{\mu}_{N_\mu}\boldsymbol{u}_{N_\mu},$$

where $N = \{1, \dots, n\}\backslash B$ and $N_\mu = \{1, \dots, r_\mu\}\backslash B_\mu$ $(\mu = 1, \dots, p)$.

Following [8], we will define *dictionary*. In LP, a basic solution is determined by a dictionary. In contrast in SOCP, we need some information on the basic solution to define a dictionary. Namely, we assume that N_1 to N_p parts of the *basic solution*, $\tilde{\boldsymbol{u}}^1_{N_1} \in \mathbb{R}^{|N_1|}, \dots, \tilde{\boldsymbol{u}}^p_{N_p} \in \mathbb{R}^{|N_p|}$, are given. These parts which can be chosen arbitrary define a dictionary, and the other parts of the basic solution will be defined by using dictionary. Precise definition of the dictionary will be given later.

Under the assumption, the above equality is rewritten as

$$\begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{u}^1_{B_1} \\ \vdots \\ \boldsymbol{u}^p_{B_p} \end{pmatrix} = G^{-1}(\boldsymbol{b} - \sum_{\mu=1}^{p} R^{\mu}_{N_\mu}\tilde{\boldsymbol{u}}^{\mu}_{N_\mu}) - G^{-1}A_N\boldsymbol{x}_N - \sum_{\mu=1}^{p} G^{-1}R^{\mu}_{N_\mu}(\boldsymbol{u}^{\mu}_{N_\mu} - \tilde{\boldsymbol{u}}^{\mu}_{N_\mu}).$$

Introducing new variables

$$\boldsymbol{v}^{\mu}_{N_\mu} = \boldsymbol{u}^{\mu}_{N_\mu} - \tilde{\boldsymbol{u}}^{\mu}_{N_\mu} \ (\mu = 1, \dots, p),$$

we have

$$\begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{u}^1_{B_1} \\ \vdots \\ \boldsymbol{u}^p_{B_p} \end{pmatrix} = \tilde{\boldsymbol{b}} - G^{-1}A_N\boldsymbol{x}_N - \sum_{\mu=1}^{p} G^{-1}R^{\mu}_{N_\mu}\boldsymbol{v}^{\mu}_{N_\mu}$$

where $\tilde{\boldsymbol{b}} = G^{-1}(\boldsymbol{b} - \sum_{\mu=1}^{p} R^{\mu}_{N_\mu}\tilde{\boldsymbol{u}}^{\mu}_{N_\mu})$. Putting

$$\begin{pmatrix} \tilde{\boldsymbol{x}}_B \\ \tilde{\boldsymbol{u}}^1_{B_1} \\ \vdots \\ \tilde{\boldsymbol{u}}^p_{B_p} \end{pmatrix} = \tilde{\boldsymbol{b}}, \qquad \begin{pmatrix} D_{BN} \\ D_{B_1N} \\ \vdots \\ D_{B_pN} \end{pmatrix} = G^{-1}A_N,$$

$$\begin{pmatrix} D_{BN_1} \\ D_{B_1N_1} \\ \vdots \\ D_{B_pN_1} \end{pmatrix} = G^{-1}R^1_{N_1}, \ \dots, \ \begin{pmatrix} D_{BN_p} \\ D_{B_1N_p} \\ \vdots \\ D_{B_pN_p} \end{pmatrix} = G^{-1}R^p_{N_p},$$

we can write the equality condition of $\langle P \rangle$:

$$\boldsymbol{x}_B = \tilde{\boldsymbol{x}}_B - D_{BN}\boldsymbol{x}_N - \sum_{\rho=1}^{p} D_{BN_\rho}\boldsymbol{v}^{\rho}_{N_\rho}$$

$$\boldsymbol{u}^{\mu}_{B_\mu} = \tilde{\boldsymbol{u}}^{\mu}_{B_\mu} - D_{B_\mu N}\boldsymbol{x}_N - \sum_{\rho=1}^{p} D_{B_\mu N_\rho}\boldsymbol{v}^{\rho}_{N_\rho} \ (\mu = 1, \dots, p).$$

Substituting these variables, we express the objective function in terms of $(\boldsymbol{x}_N, \boldsymbol{v}^1_{N_1}, \ldots, \boldsymbol{v}^p_{N_p})$:

$$\begin{aligned}\theta &= \boldsymbol{c}_B^T(\tilde{\boldsymbol{x}}_B - D_{BN}\boldsymbol{x}_N - \sum_{\mu=1}^{p} D_{BN_\mu}\boldsymbol{v}^\mu_{N_\mu}) + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ &\quad + \sum_{\mu=1}^{p}\left((\boldsymbol{d}^\mu_{B_\mu})^T(\tilde{\boldsymbol{u}}^\mu_{B_\mu} - D_{B_\mu N}\boldsymbol{x}_N - \sum_{\rho=1}^{p} D_{B_\mu N_\rho}\boldsymbol{v}^\rho_{N_\rho}) + (\boldsymbol{d}^\mu_{N_\mu})^T(\tilde{\boldsymbol{u}}^\mu_{N_\mu} + \boldsymbol{v}^\mu_{N_\mu}) + d^\mu_0 u^\mu_0\right) \\ &= \tilde{\theta} + \tilde{\boldsymbol{s}}_N^T\boldsymbol{x}_N + \sum_{\mu=1}^{p}(\tilde{\boldsymbol{z}}^\mu_{N_\mu})^T\boldsymbol{v}_{N_\mu} + \sum_{\mu=1}^{p} d^\mu_0 u^\mu_0\end{aligned}$$

where

$$\begin{aligned}\tilde{\theta} &= \boldsymbol{c}_B^T\tilde{\boldsymbol{x}}_B + \sum_{\rho=1}^{p}(\boldsymbol{d}^\rho_{B_\rho})^T\tilde{\boldsymbol{u}}^\rho_{B_\rho} + \sum_{\rho=1}^{p}(\boldsymbol{d}^\rho_{N_\rho})^T\tilde{\boldsymbol{u}}^\rho_{N_\rho} \\ \tilde{\boldsymbol{s}}_N &= \boldsymbol{c}_N - D_{BN}^T\boldsymbol{c}_B - \sum_{\rho=1}^{p} D_{B_\rho N}^T\boldsymbol{d}^\rho_{B_\rho} \\ \tilde{\boldsymbol{z}}_{N_\mu} &= \boldsymbol{d}^\mu_{N_\mu} - D_{BN_\mu}^T\boldsymbol{c}_B - \sum_{\rho=1}^{p} D_{B_\rho N_\mu}^T\boldsymbol{d}^\rho_{B_\rho} \ (\mu = 1, \ldots, p).\end{aligned}$$

Now a dictionary is defined as follows:

$$\begin{aligned}\theta &= \tilde{\theta} + \tilde{\boldsymbol{s}}_N^T\boldsymbol{x}_N + \sum_{\rho=1}^{p}\tilde{\boldsymbol{z}}_{N_\rho}^T\boldsymbol{v}^\rho_{N_\rho} + \sum_{\rho=1}^{p} d^\rho_0 u^\rho_0 \\ \boldsymbol{x}_B &= \tilde{\boldsymbol{x}}_B - D_{BN}\boldsymbol{x}_N - \sum_{\rho=1}^{p} D_{BN_\rho}\boldsymbol{v}^\rho_{N_\rho} \\ \boldsymbol{u}^\mu_{B_\mu} &= \tilde{\boldsymbol{u}}^\mu_{B_\mu} - D_{B_\mu N}\boldsymbol{x}_N - \sum_{\rho=1}^{p} D_{B_\mu N_\rho}\boldsymbol{v}^\rho_{N_\rho} \quad (\mu = 1, \ldots, p) \\ \boldsymbol{u}^\mu_{N_\mu} &= \tilde{\boldsymbol{u}}^\mu_{N_\mu} + \boldsymbol{v}^\mu_{N_\mu} \quad (\mu = 1, \ldots, p)\end{aligned} \tag{1}$$

In the following, an upper index of a vector is sometimes omitted when it is easily deduced from the lower index. For example, $\tilde{\boldsymbol{u}}_{N_\mu}$ is a simplified form of $\tilde{\boldsymbol{u}}^\mu_{N_\mu}$. This may enhance the readability.

We denote $\mathcal{B} = (B_1, \ldots, B_p)$ and $\mathcal{N} = (N_1, \ldots, N_p)$. An aggregation of vectors over $\mathcal{N}$ is denoted by its subscript; for example, $\tilde{\boldsymbol{u}}_\mathcal{N} = (\tilde{\boldsymbol{u}}^1_{N_1}, \ldots, \tilde{\boldsymbol{u}}^p_{N_p})$. Because the dictionary is determined by B, $\mathcal{B}$, and $\tilde{\boldsymbol{u}}_\mathcal{N}$, we denote (1) by $\mathcal{D}(B, \mathcal{B}; \tilde{\boldsymbol{u}}_\mathcal{N})$.

Now we define a *basic solution* corresponding to $\mathcal{D}(B, \mathcal{B}; \tilde{\boldsymbol{u}}_\mathcal{N})$, which is derived by putting $\boldsymbol{x}_N = \boldsymbol{0}$, $\boldsymbol{v}_\mathcal{N} = \boldsymbol{0}$, and

$$u^\mu_0 = \tilde{u}^\mu_0 = \|\tilde{\boldsymbol{u}}^\mu\| \ \ (\mu = 1, \ldots, p).$$

Specifically, the basic solution associated with (1) is

$$(\boldsymbol{x}_B, \boldsymbol{x}_N, u^1_0, \boldsymbol{u}_{B_1}, \boldsymbol{u}_{N_1}, \ldots, u^p_0, \boldsymbol{u}_{B_p}, \boldsymbol{u}_{N_p}) = (\tilde{\boldsymbol{x}}_B, \boldsymbol{0}, \tilde{u}^1_0, \tilde{\boldsymbol{u}}_{B_1}, \tilde{\boldsymbol{u}}_{N_1}, \ldots, \tilde{u}^p_0, \tilde{\boldsymbol{u}}_{B_p}, \tilde{\boldsymbol{u}}_{N_p}). \tag{2}$$

We denote $\bar{B} = \{\, \mu \in \{1, \ldots, p\} \mid \tilde{u}^\mu_0 > 0 \,\}$ and $\bar{N} = \{\, \mu \in \{1, \ldots, p\} \mid \tilde{u}^\mu_0 = 0 \,\}$.

Note that the basic solution (2) is feasible for $\langle P \rangle$ if and only if $\tilde{\boldsymbol{x}}_B \geq \boldsymbol{0}$. In this case, we say the dictionary $\mathcal{D}(B, \mathcal{B}; \tilde{\boldsymbol{u}}_\mathcal{N})$ is feasible. If the basic solution is optimal for $\langle P \rangle$, the dictionary is called optimal.

2 Nondegeneracy and a Dual Solution

The primal nondegeneracy at a feasible basic solution $(\tilde{\boldsymbol{x}}_B, \mathbf{0}, \tilde{u}_0^1, \tilde{\boldsymbol{u}}_{B_1}, \tilde{\boldsymbol{u}}_{N_1}, \ldots, \tilde{u}_0^p, \tilde{\boldsymbol{u}}_{B_p}, \tilde{\boldsymbol{u}}_{N_p})$ is defined as follows. Let $\mathcal{T}$ be the tangent space of the cone at the basic solution. We denote by $\bar{A}$ the coefficient matrix of $\langle P \rangle$ which corresponds to the equality condition. The basic solution is *primal nondegenerate* if $\mathcal{T} + \mathrm{ker}\bar{A} = \mathbb{R}^{\bar{n}}$, where $\bar{n} = n + \sum_{\mu=1}^{p}(1 + r_\mu)$.

This definition is the SOCP version of the degeneracy in symmetric cone programming derived in [2]. The same condition is also found for SOCP by [1]. See [6] for more on the symmetric cone programming.

On the other hand, in the rest of this paper, we sometimes assume the following conditions for a basic solution.

Condition A:

1. $\tilde{\boldsymbol{x}}_B > \mathbf{0}$
2. $B_\rho = \emptyset$ if $\rho \in \bar{N}$.

A feasible and nondegenerate basic solution associated with a dictionary does not necessarily satisfy Condition A. However, we will show that if we are given a nondegenerate basic solution, then we can adjust B and B_μ so that Condition A is satisfied; we can change the dictionary without changing the basic solution to satisfy Condition A.

To show this, let $B = P \cup P'$ where $\tilde{\boldsymbol{x}}_P > \mathbf{0}$ and $\tilde{\boldsymbol{x}}_{P'} = \mathbf{0}$. Then the tangent space at the basic solution is

$$\mathcal{T} = \mathbb{R}^{|P|} \otimes \{\mathbf{0}_{P'}\} \otimes_{\mu \in \bar{B}} \mathcal{T}_\mu \otimes_{\mu \in \bar{N}} \{\mathbf{0}_{r_\mu+1}\},$$

and its orthogonal compelment

$$\mathcal{T}^\perp = \{\mathbf{0}_P\} \otimes \mathbb{R}^{|P'|} \otimes_{\mu \in \bar{B}} \mathcal{T}_\mu^\perp \otimes_{\mu \in \bar{N}} \mathbb{R}^{r_\mu+1},$$

where

$$\mathcal{T}_\mu = \left\{ (f_0, \boldsymbol{f}) \in \mathbb{R}^{1+r_\mu} \;\middle|\; \tilde{u}_0^\mu f_0 - (\tilde{\boldsymbol{u}}^\mu)^T \boldsymbol{f} = 0 \right\}.$$

Taking the complement of the nondegeneracy condition, we have $\mathcal{T}^\perp \cap \mathrm{Im}\bar{A}^T = \{\mathbf{0}\}$, which implies that the system

$$\begin{array}{rcl} \mathbf{0} & = & A_P^T \boldsymbol{y}, \\ \boldsymbol{f}^\mu & = & (R^\mu)^T \boldsymbol{y} \ (\mu \in \bar{B}), \\ (0, \boldsymbol{f}^\mu) & \in & \mathcal{T}_\mu^\perp \ (\mu \in \bar{B}) \end{array}$$

has the unique zero solution. From the last relation, it follows that $\boldsymbol{f}^\mu = \mathbf{0}$, thus the system is equivalent with

$$A_P^T \boldsymbol{y} = \mathbf{0}, \ (R^\mu)^T \boldsymbol{y} = \mathbf{0} \ (\mu \in \bar{B}). \tag{3}$$

If we denote the coefficient matrix of (3) by $\tilde{A}^T$, then (3) is equivalent with $\mathrm{Im}\tilde{A} = \mathbb{R}^m$, in other words, the matrix

$$G^{-1}\tilde{A} = (G^{-1}A_P | G^{-1}R^\mu(\mu \in \bar{B})) = \left(\begin{array}{c|c} \begin{matrix} I_P \\ O_{P'P} \\ O_{B_1P} \\ \vdots \\ O_{B_\mu P} \\ \vdots \\ O_{B_pP} \end{matrix} & \begin{pmatrix} O_{PB_\mu} & D_{PN_\mu} \\ O_{P'B_\mu} & D_{P'N_\mu} \\ O_{B_1B_\mu} & D_{B_1N_\mu} \\ \vdots & \vdots \\ I_{B_\mu} & D_{B_\mu N_\mu} \\ \vdots & \vdots \\ O_{B_pB_\mu} & D_{B_pN_\mu} \end{pmatrix} (\mu \in \bar{B}) \end{array} \right)$$

has rank m. Looking at the structure of this matrix, we see that the above matrix has rank m if and only if

$$\mathrm{span}(D_{P'N_\mu} : \mu \in \bar{B}) = \mathbb{R}^{|P'|} \text{ and } \mathrm{span}(D_{B_\rho N_\mu} : \mu \in \bar{B}) = \mathbb{R}^{|B_\rho|} \ (\rho \in \bar{N}). \tag{4}$$

Now assume that P' is nonempty. Then (4) shows that we can express $\boldsymbol{x}_{P'}$ in terms of variables in N_μ ($\mu \in \bar{B}$) which are located on the right-hand side of the dictionary. Expressing $\boldsymbol{x}_{P'}$ by such variables means that $\boldsymbol{x}_{P'}$ are now nonbasic variables, and fortunately the basic solution is not changed because $\tilde{\boldsymbol{x}}_{P'} = \boldsymbol{0}$. Therefore, without loss of generality, we can assume that P' is empty in a feasible and nondegenerate dictionary.

Next suppose that a variable, say u_j^ρ where $\rho \in \bar{N}$ is located on the left-hand side of the dictionary without being substituted by $\tilde{u}_j^\rho + v_j^\rho$. Then (4) shows that u_j^ρ can be expressed by a variable in N_μ where $\mu \in \bar{B}$ on the right-hand side; we can exchange u_j^ρ and that variable to obtain a new dictionary without changing the basic solution and having u_j^ρ on the right-hand side. Continuing this process, we can eliminate all the variables in the cone $\rho \in \bar{N}$ from the left-hand side. Therefore, without loss of generality, we can assume Condition A for a feasible basic solution under the primal nondegeneracy assumption.

Assuming that the current basic solution satisfies Condition A, we will define a *dual solution*

$$(\hat{\boldsymbol{y}}, \hat{\boldsymbol{s}}_B, \hat{\boldsymbol{s}}_N, \hat{z}_0^1, \hat{\boldsymbol{z}}_{B_1}, \hat{\boldsymbol{z}}_{N_1}, \ldots, \hat{z}_0^p, \hat{\boldsymbol{z}}_{B_p}, \hat{\boldsymbol{z}}_{N_p}),$$

in the following. To do this, we first put

$$\hat{\boldsymbol{s}}_B = \boldsymbol{0} \text{ and } \hat{\boldsymbol{z}}_{B_\mu} = -\frac{d_0^\mu}{\tilde{u}_0^\mu}\tilde{\boldsymbol{u}}_{B_\mu} \quad (\mu = 1, \ldots, p).$$

In the definition of $\hat{\boldsymbol{z}}_{B_\mu}$, when $\tilde{u}_0^\mu = 0$, we just ignore this subvector because in this case B_μ is empty due to Condition A. The other parts of the dual solution are defined to satisfy the equality condition of $\langle D \rangle$. Namely, we have

$$\hat{\boldsymbol{y}} = G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B_1} + \frac{d_0^1}{\tilde{u}_0^1}\tilde{\boldsymbol{u}}_{B_1} \\ \vdots \\ \boldsymbol{d}_{B_p} + \frac{d_0^p}{\tilde{u}_0^p}\tilde{\boldsymbol{u}}_{B_p} \end{pmatrix},$$

$$\begin{aligned}
\hat{\boldsymbol{s}}_N = \boldsymbol{c}_N - A_N^T\hat{\boldsymbol{y}} &= \boldsymbol{c}_N - A_N^T G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B_1} + \frac{d_0^1}{\tilde{u}_0^1}\tilde{\boldsymbol{u}}_{B_1} \\ \vdots \\ \boldsymbol{d}_{B_p} + \frac{d_0^p}{\tilde{u}_0^p}\tilde{\boldsymbol{u}}_{B_p} \end{pmatrix} \\
&= \boldsymbol{c}_N - \begin{pmatrix} D_{BN}^T & D_{B_1N}^T & \ldots & D_{B_pN}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B_1} + \frac{d_0^1}{\tilde{u}_0^1}\tilde{\boldsymbol{u}}_{B_1} \\ \vdots \\ \boldsymbol{d}_{B_p} + \frac{d_0^p}{\tilde{u}_0^p}\tilde{\boldsymbol{u}}_{B_p} \end{pmatrix} \\
&= \boldsymbol{c}_N - \left(D_{BN}^T\boldsymbol{c}_B + \sum_{\rho=1}^p D_{B_\rho N}^T(\boldsymbol{d}_{B_\rho} + \frac{d_0^\rho}{\tilde{u}_0^\rho}\tilde{\boldsymbol{u}}_{B_\rho}) \right) = \tilde{\boldsymbol{s}}_N - \sum_{\rho=1}^p \frac{d_0^\rho}{\tilde{u}_0^\rho} D_{B_\rho N}^T\tilde{\boldsymbol{u}}_{B_\rho},
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{z}}_{N_\mu} &= \boldsymbol{d}^\mu_{N_\mu} - (R^\mu)^T \hat{\boldsymbol{y}} = \boldsymbol{d}^\mu_{N_\mu} - (R^\mu)^T G^{-T} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}^1_{B_1} + \frac{d^1_0}{\tilde{u}^1_0}\tilde{\boldsymbol{u}}^1_{B_1} \\ \vdots \\ \boldsymbol{d}^p_{B_p} + \frac{d^p_0}{\tilde{u}^p_0}\tilde{\boldsymbol{u}}^p_{B_p} \end{pmatrix} \\
&= \boldsymbol{d}^\mu_{N_\mu} - \begin{pmatrix} D^T_{BN_\mu} & D^T_{B_1N_\mu} & \cdots & D^T_{B_pN_\mu} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{d}_{B_1} + \frac{d^1_0}{\tilde{u}^1_0}\tilde{\boldsymbol{u}}_{B_1} \\ \vdots \\ \boldsymbol{d}_{B_p} + \frac{d^p_0}{\tilde{u}^p_0}\tilde{\boldsymbol{u}}_{B_p} \end{pmatrix} \\
&= \boldsymbol{d}^\mu_{N_\mu} - \left(D^T_{BN_\mu}\boldsymbol{c}_B + \sum_{\rho=1}^{p} D^T_{B_\rho N_\mu}(\boldsymbol{d}_{B_\rho} + \frac{d^\rho_0}{\tilde{u}^\rho_0}\tilde{\boldsymbol{u}}_{B_\rho}) \right) = \tilde{\boldsymbol{z}}^\mu_{N_\mu} - \sum_{\rho=1}^{p} \frac{d^\rho_0}{\tilde{u}^\rho_0} D^T_{B_\rho N_\mu}\tilde{\boldsymbol{u}}_{B_\rho}.
\end{aligned} \tag{5}$$

Notice that if $\hat{\boldsymbol{s}}_N \geq \boldsymbol{0}$ and $\|(\hat{\boldsymbol{z}}_{B_\mu}, \hat{\boldsymbol{z}}_{N_\mu})\| \leq 1$ $(\mu = 1, \ldots, p)$, then the dual solution is feasible for $\langle D \rangle$.

3 Subproblems

Suppose that we are given a feasbile dictionary together with its basic solution satisfying Condition A. Extending the way proposed by Muramatsu [8], we consider three types of subproblems to perform a pivot.

First, we consider to decrease the objective value by increasing x_i for $i \in N$ from 0. Because the next solution should be feasible for $\langle P \rangle$, we obtain a subproblem:

$$\langle S_i \rangle \left\{ \begin{array}{ll} \text{minimize}_{\,(x_i, u^1_0, \ldots, u^p_0)} & \tilde{s}_i x_i + \sum_{\rho=1}^p d^\rho_0 u^\rho_0 \\ \text{subject to} & \tilde{\boldsymbol{x}}_B - x_i D_{Bi} \geq \boldsymbol{0}, \quad x_i \geq 0 \\ & \begin{pmatrix} u^\rho_0 \\ \tilde{\boldsymbol{u}}_{B_\rho} - x_i D_{B_\rho i} \\ \tilde{\boldsymbol{u}}_{N_\rho} \end{pmatrix} \in \mathcal{K}_{r_\rho+1} \quad (\rho = 1, \ldots, p). \end{array} \right.$$

Here, D_{Bi} is the i-th column vector of D_{BN}.

Similarly for $\mu \in \bar{B}$ and $j \in B_\mu$, we consider

$$\langle Z^\mu_j \rangle \left\{ \begin{array}{ll} \text{minimize}_{\,(v_j, u^1_0, \ldots, u^p_0)} & \tilde{z}_j v_j + \sum_{\rho=1}^p d^\rho_0 u^\rho_0 \\ \text{subject to} & \tilde{\boldsymbol{x}}_B - v_j D_{Bj} \geq \boldsymbol{0} \\ & \begin{pmatrix} u^\rho_0 \\ \tilde{\boldsymbol{u}}_{B_\rho} - v_j D_{B_\rho j} \\ \tilde{\boldsymbol{u}}_{N_\rho} \end{pmatrix} \in \mathcal{K}_{r_\rho+1} \quad (\rho = 1, \ldots, p, \rho \neq \mu) \\ & \begin{pmatrix} u^\mu_0 \\ \tilde{\boldsymbol{u}}_{B_\mu} - v_j D_{B_\mu j} \\ \tilde{\boldsymbol{u}}_{N_\mu} + v_j \boldsymbol{e}_j \end{pmatrix} \in \mathcal{K}_{r_\mu+1}. \end{array} \right.$$

Notice that v_j is a free variable, while x_i in $\langle S_i \rangle$ should be nonnegative.

At last, for $\mu \in \bar{N}$, we consider to move in the direction of $-\hat{\boldsymbol{z}}_{N_\mu}$:

$$\langle Z^\mu_{N_\mu} \rangle \left\{ \begin{array}{ll} \text{minimize}_{\,(\lambda, u_0^1, \ldots, u_0^p)} & -\lambda \tilde{\boldsymbol{z}}^T_{N_\mu} \hat{\boldsymbol{z}}_{N_\mu} + \sum_{\rho=1}^p d_0^\rho u_0^\rho \\ \text{subject to} & \tilde{\boldsymbol{x}}_B + \lambda D_{BN_\mu} \hat{\boldsymbol{z}}_{N_\mu} \geq \mathbf{0}, \ \lambda \geq 0, \\ & \begin{pmatrix} u_0^\rho \\ \tilde{\boldsymbol{u}}_{B_\rho} + \lambda D_{B_\rho N_\mu} \hat{\boldsymbol{z}}_{N_\mu} \\ \tilde{\boldsymbol{u}}_{N_\rho} \end{pmatrix} \in \mathcal{K}_{r_\rho+1} \quad (\rho = 1, \ldots, p, \rho \neq \mu) \\ & \begin{pmatrix} u_0^\mu \\ -\lambda \hat{\boldsymbol{z}}_{N_\mu} \end{pmatrix} \in \mathcal{K}_{r_\mu+1}. \end{array} \right.$$

Condition A implies that $N_\mu = \{1, \ldots, r_\mu\}$ in this case.

In those subproblems, each second-order cone constraint is satisfied on its boundary at optimality. This means that we can eliminate u_0^l $(l = 1, \ldots, p)$; these subproblems are essentially one-dimensional convex optimization problems. Because the current dictionary is feasible, 0 is always a feasible solution for these one-dimantional problems. Therefore, these subproblems can be solved easily by line search. We say that a one-dimensional problem has a *trivial* optimal solution if 0 is optimal.

The following is the main theorem of this paper.

Theorem 1 *Assume that we are given a feasible dictionary $\mathcal{D}(B, \mathcal{B}, \tilde{\boldsymbol{u}}_{\mathcal{N}})$ together with associated basic solution satisfying Condition A. If all the problems $\langle S_i \rangle$ $(i \in N)$, $\langle Z^\mu_j \rangle$ for $j \in N_\mu$ and $\mu \in \bar{B}$, and $\langle Z^\mu_{N_\mu} \rangle$ for $\mu \in \bar{N}$ have trivial optimal solutions, then the dual solution corresponding to $\mathcal{D}(B, \mathcal{B}, \tilde{\boldsymbol{u}}_{\mathcal{N}})$ is feasible for $\langle D \rangle$, and the basic solution and the dual solution are optimal for $\langle P \rangle$ and $\langle D \rangle$, respectively.*

Before proving the theorem, we observe several properties of the subproblems.

Lemma 2 *Assume that we are given a feasible dictionary $\mathcal{D}(B, \mathcal{B}, \tilde{\boldsymbol{u}}_{\mathcal{N}})$ together with associated basic solution satisfying Condition A. If one of $\langle S_i \rangle$, $\langle Z^\mu_j \rangle$, or $\langle Z^\mu_{N_\mu} \rangle$ is unbounded, then $\langle P \rangle$ is unbounded.*

We omit the proof because it is quite similar to that of Lemma 3.1 of [8].

Lemma 3 *Assume that we are given a feasible dictionary $\mathcal{D}(B, \mathcal{B}, \tilde{\boldsymbol{u}}_{\mathcal{N}})$ together with associated basic solution satisfying Condition A.*

1. *$\langle S_i \rangle$ has a trivial optimal solution if and only if $\hat{s}_i \geq 0$.*
2. *Assume that $x_i = 0$ is not optimal for $\langle S_i \rangle$, and at the optimal solution $x_i^* > 0$, $\tilde{\boldsymbol{x}}_B - D_{Bi} x_i^* > \mathbf{0}$. Then there exists $\mu \in \{1, \ldots, p\}$ such that $D_{B_\mu i} \neq \mathbf{0}$.*

Proof . Since the minimum of $\langle S_i \rangle$ is taken at

$$u_0^\rho = \sqrt{\|\tilde{\boldsymbol{u}}^\rho_{B_\rho} - x_i D_{B_\rho i}\|^2 + \|\tilde{\boldsymbol{u}}_{N_\rho}\|^2} \text{ for every } \rho,$$

we can rewrite $\langle S_i \rangle$ as

$$\left\{ \begin{array}{ll} \text{minimize} & \tilde{s}_i x_i + \sum_{\rho=1}^p d_0^\rho \sqrt{x_i^2 \|D_{B_\rho i}\|^2 - 2 x_i D^T_{B_\rho i} \tilde{\boldsymbol{u}}_{B_\rho} + (\tilde{u}_0^\rho)^2} \\ \text{subject to} & 0 \leq x_i, \ x_i D_{Bi} \leq \tilde{\boldsymbol{x}}_B. \end{array} \right.$$

Let us denote the objective function by $f(x_i)$. By Condition A, if $\tilde{u}_0^\rho = 0$, then B_ρ is empty, thus the corresponding term is not summed. Therefore, we obtain

$$f'(x_i) = \tilde{s}_i + \sum_{\rho=1}^{p} \frac{\|D_{B_\rho i}\|^2 x_i - D_{B_\rho i}^T \boldsymbol{u}_{B_\rho}}{\sqrt{x_i^2 \|D_{B_\rho i}\|^2 - 2x_i D_{B_\rho i}^T \tilde{\boldsymbol{u}}_{B_\rho} + (\tilde{u}_0^\rho)^2}}$$

and

$$f'(0) = \tilde{s}_i - \sum_{\rho=1}^{p} D_{B_\rho i}^T \boldsymbol{u}_{B_\rho} / \tilde{u}_0^\rho = \hat{s}_i.$$

It can be easily verified that $f''(x_i) \geq 0$ for every $x_i \geq 0$, thus f is convex. Therefore, 0 is optimal if and only if $f'(0) \geq 0$. This proves the first statement.

To prove the second, suppose to the contrary that $D_{B_\rho i} = \mathbf{0}$ for all $\rho \in \{1, \ldots, p\}$. Then the second-order cone constraints do not include x_i, and at the optimum, $u_0^\rho = \tilde{u}_0^\rho$ ($\rho = 1, \ldots, p$). Since 0 is not optimal, $f'(0) = \tilde{s}_i < 0$. In fact, at any feasible point, $f'(x_i) = \tilde{s}_i$. Therefore, at least one of the inequality of $\tilde{\boldsymbol{x}}_B - x_i^* D_{Bi} \geq \mathbf{0}$ holds at equality, which contradicts the assumption. □

Lemma 4 *Assume that we are given a feasible dictionary $\mathcal{D}(B, \mathcal{B}, \tilde{\boldsymbol{u}}_\mathcal{N})$ together with associated basic solution satisfying Condition A. For $\mu \in \bar{B}$ and $j \in N_\mu$, $v_j^* = 0$ is an optimal solution of $\langle Z_j^\mu \rangle$ if and only if*

$$\hat{z}_j = -\frac{d_0^\mu}{\tilde{u}_0^\mu} \tilde{u}_j^\mu.$$

Proof . Since the minimum of $\langle Z_j^\mu \rangle$ is taken at

$$u_0^\rho = \sqrt{\|\tilde{\boldsymbol{u}}_{B_\rho} - v_j D_{B_\rho j}\|^2 + \|\tilde{\boldsymbol{u}}_{N_\rho}\|} \ (\rho \neq \mu) \text{ and } u_0^\mu = \sqrt{\|\tilde{\boldsymbol{u}}_{B_\mu}^\mu - v_j D_{B_\mu j}\|^2 + \|\tilde{\boldsymbol{u}}_{N_\mu} + v_j \boldsymbol{e}_j\|^2},$$

we can rewrite $\langle Z_j^\mu \rangle$ as

$$\begin{cases} \text{minimize} & \tilde{z}_j^\mu v_j + d_0^\mu \sqrt{(v_j)^2(\|D_{B_\mu j}\|^2 + 1) - 2v_j(D_{B_\mu j}^T \tilde{\boldsymbol{u}}_{B_\mu} - \boldsymbol{e}_j^T \tilde{\boldsymbol{u}}_{N_\mu}) + (\tilde{u}_0^\mu)^2} \\ & \quad + \sum_{\rho \neq \mu} d_0^\rho \sqrt{(v_j)^2 \|D_{B_\rho j}\|^2 - 2v_j D_{B_\rho j}^T \tilde{\boldsymbol{u}}_{B_\rho} + (\tilde{u}_0^\rho)^2} \\ \text{subject to} & \tilde{\boldsymbol{x}}_B - v_j D_{Bj} \geq \mathbf{0}. \end{cases}$$

Here, D_{Bj} is the j-th column of D_{BN_μ}.

Let us denote the objective function of the above by $f(v_j)$. In view of $\tilde{u}_0^\rho > 0$, we have

$$\begin{aligned} f'(v_j) = \tilde{z}^\mu + d_j^\mu & \frac{v_j(\|D_{B_\mu j}\|^2 + 1) - (D_{B_\mu j}^T \tilde{\boldsymbol{u}}_{B_\mu} - \boldsymbol{e}_j^T \tilde{\boldsymbol{u}}_{N_\mu})}{\sqrt{(v_j)^2(\|D_{B_\mu j}\|^2 + 1) - 2v_j(D_{B_\mu j}^T \tilde{\boldsymbol{u}}_{B_\mu} - \boldsymbol{e}_j^T \tilde{\boldsymbol{u}}_{N_\mu}) + (\tilde{u}_0^\mu)^2}} \\ & + \sum_{\rho \neq \mu} d_0^\rho \frac{v_j \|D_{B_\rho j}\|^2 - D_{B_\rho j}^T \tilde{\boldsymbol{u}}_{B_\rho}}{\sqrt{(v_j)^2 \|D_{B_\rho j}\|^2 - 2v_j D_{B_\rho j}^T \tilde{\boldsymbol{u}}_{B_\rho} + (\tilde{u}_0^\rho)^2}} \end{aligned}$$

and

$$f'(0) = \tilde{z}_j^\mu - d_0^\mu \frac{D_{B_\mu j}^T \boldsymbol{u}_{B_\mu} - \tilde{u}_j^\mu}{\tilde{u}_0^\mu} - \sum_{\rho \neq \mu} d_0^\rho \frac{D_{B_\rho j}^T \boldsymbol{u}_{B_\rho}}{\tilde{u}_0^\rho}.$$

Because $\tilde{\boldsymbol{x}}_B > \mathbf{0}$, v_j could be both positive and negative on the feasbile region. Hence $f'(0) = 0$ if and only if 0 is optimal. Therefore, we obtain

$$\hat{z}_j^\mu = \tilde{z}_j^\mu - \sum_{\rho=1}^{p} d_0^\rho D_{B_\rho j} \boldsymbol{u}_{B_\rho} / \tilde{u}_0^\rho = -d_0^\mu \tilde{u}_j^\mu / \tilde{u}_0^\mu .$$

□

Lemma 5 *Assume that we are given a feasible dictionary $\mathcal{D}(B, \mathcal{B}, \tilde{\boldsymbol{u}}_{\mathcal{N}})$ together with associated basic solution satisfying Condition A. For $\mu \in \bar{N}$, $\lambda = 0$ is an optimal solution of $\langle Z_{N_\mu}^\mu \rangle$ if and only if $\|\hat{\boldsymbol{z}}_{N_\mu}\| \leq d_0^\mu$.*

Proof . Following the idea of the proof of Lemma 4, we rewrite $\langle Z_{N_\mu}^\mu \rangle$ as a one-dimensional problem:

$$\left\{ \begin{array}{ll} \text{minimize} & -\lambda \left(\tilde{\boldsymbol{z}}_{N_\mu}^T \hat{\boldsymbol{z}}_{N_\mu} - d_0^\mu \|\hat{\boldsymbol{z}}_{N_\mu}\| \right) \\ & \quad + \sum_{\rho \neq \mu} d_0^\rho \sqrt{\lambda^2 \|D_{B_\rho N_\mu} \hat{\boldsymbol{z}}_{N_\mu}\|^2 + 2\lambda \hat{\boldsymbol{z}}_{N_\mu}^T D_{B_\rho N_\mu}^T \tilde{\boldsymbol{u}}_{B_\rho}^\rho + (\tilde{u}_0^\rho)^2} \\ \text{subject to} & \tilde{\boldsymbol{x}}_B + \lambda D_{B N_\mu} \hat{\boldsymbol{z}}_{N_\mu} \geq \mathbf{0}, \ \lambda \geq 0. \end{array} \right.$$

Putting the objective function by $f(\lambda)$, we calculate

$$f'(\lambda) = -\left(\tilde{\boldsymbol{z}}_{N_\mu}^T \hat{\boldsymbol{z}}_{N_\mu} - d_0^\mu \|\hat{\boldsymbol{z}}_{N_\mu}\| \right) + \sum_{\rho \neq \mu} d_0^\rho \frac{\lambda \|D_{B_\rho N_\mu} \hat{\boldsymbol{z}}_{N_\mu}\|^2 + \hat{\boldsymbol{z}}_{N_\mu}^T D_{B_\rho N_\mu}^T \tilde{\boldsymbol{u}}_{B_\rho}}{\sqrt{\lambda^2 \|D_{B_\rho N_\mu} \hat{\boldsymbol{z}}_{N_\mu}\|^2 + 2\lambda \hat{\boldsymbol{z}}_{N_\mu}^T D_{B_\rho N_\mu}^T \tilde{\boldsymbol{u}}_{B_\rho} + (\tilde{u}_0^\rho)^2}}$$

and

$$f'(0) = -\left(\tilde{\boldsymbol{z}}_{N_\mu}^T \hat{\boldsymbol{z}}_{N_\mu} - d_0^\mu \|\hat{\boldsymbol{z}}_{N_\mu}\| \right) + \sum_{\rho \neq \mu} d_0^\rho \frac{\hat{\boldsymbol{z}}_{N_\mu}^T D_{B_\rho N_\mu}^T \tilde{\boldsymbol{u}}_{B_\rho}}{\tilde{u}_0^\rho}.$$

Since f is convex and $\lambda \geq 0$, $\lambda = 0$ is optimal if and only if $f'(0) \geq 0$. Recalling the definition of $\hat{\boldsymbol{z}}_{N_\mu}$, we have

$$\begin{aligned} f'(0) \geq 0 \quad & \Leftrightarrow \quad -\left(\tilde{\boldsymbol{z}}_{N_\mu}^T \hat{\boldsymbol{z}}_{N_\mu} - d_0^\mu \|\hat{\boldsymbol{z}}_{N_\mu}\| \right) + \sum_{\rho \neq \mu} d_0^\rho \frac{\hat{\boldsymbol{z}}_{N_\mu}^T D_{B_\rho N_\mu}^T \tilde{\boldsymbol{u}}_{B_\rho}}{\tilde{u}_0^\rho} \geq 0 \\ & \Leftrightarrow \quad d_0^\mu \|\hat{\boldsymbol{z}}_{N_\mu}\| - \hat{\boldsymbol{z}}_{N_\mu}^T \left(\tilde{\boldsymbol{z}}_{N_\mu} - \sum_{\rho \neq \mu} \frac{d_0^\rho D_{B_\rho N_\mu}^T \tilde{\boldsymbol{u}}_{B_\rho}}{\tilde{u}_0^\rho} \right) \geq 0 \\ & \Leftrightarrow \quad d_0^\mu \|\hat{\boldsymbol{z}}_{N_\mu}\| - \|\hat{\boldsymbol{z}}_{N_\mu}\|^2 \geq 0 \\ & \Leftrightarrow \quad d_0^\mu \geq \|\hat{\boldsymbol{z}}_{N_\mu}\|. \end{aligned}$$

□

Proof of Theorem 1 . Because the dual solution satisfies the equality condition of $\langle D \rangle$ by definition, it suffices to show that each component satisfies the inequality or second-order cone constraints to prove the feasiblity of the dual solution. In the following, we also check that the basic solution and the dual solution satisfy complementarity condition to show the optimality.

(1) of Lemma 3 shows that $\hat{\boldsymbol{s}}_N \geq \mathbf{0}$, if all $\langle S_i \rangle$ have a trivial optimal solution. By definition, $\hat{\boldsymbol{s}}_B = \mathbf{0}$, thus $\hat{\boldsymbol{s}} \geq \mathbf{0}$. Because $\tilde{\boldsymbol{x}}_N = \mathbf{0}$, we have $\tilde{\boldsymbol{x}}^T \hat{\boldsymbol{s}} = \tilde{\boldsymbol{x}}_B^T \hat{\boldsymbol{s}}_B + \tilde{\boldsymbol{x}}_N^T \hat{\boldsymbol{x}}_N = 0$.

For $\mu \in \bar{N}$, because $N_\mu = \{1, \ldots, r_\mu\}$, we have $\|\hat{\boldsymbol{z}}^\mu\| \leq d_0^\mu$ due to Lemma 5. In this case, $\tilde{u}_0^\mu = 0$ and $\tilde{\boldsymbol{u}} = \mathbf{0}$, thus we have $d_0^\mu \tilde{u}_0^\mu + (\hat{\boldsymbol{z}}^\mu)^T \tilde{\boldsymbol{u}} = 0$.

Finally, we consider the case where $\mu \in \bar{B}$. Combining (5) and Lemma 4, we see that

$$\hat{\boldsymbol{z}}^\mu = -\frac{d_0^\mu}{\tilde{u}_0^\mu} \tilde{\boldsymbol{u}}^\mu.$$

Therefore, we have

$$\|\hat{\boldsymbol{z}}^\mu\| = d_0^\mu, \text{ and } d_0^\mu \tilde{u}_0^\mu + (\hat{\boldsymbol{z}}^\mu)^T \tilde{\boldsymbol{u}}^\mu = 0.$$

This completes the proof. □

4 Concluding Remarks

Now we describe the outline of a pivoting algorithm for $\langle P \rangle$. Assume first that we are given a feasible and nondegenerate dictionary. If one of the subproblems is unbounded, then we conclude that $\langle P \rangle$ is unbounded due to Lemma 2. If all the subproblems have trivial optimal solutions, then Theorem 1 tells us that the current basic and dual solutions are optimal for $\langle P \rangle$ and $\langle D \rangle$, respectively. If at least one of the subproblems has a nontrivial optimal solution, then we can move to the next basic solution by exchanging basic and nonbasic variables or by sliding its displacement vector. In particular, when $\langle S_i \rangle$ has a nontrivial optimal solution and we cannot choose a leaving variable from the nonnegative variables, (2) of Lemma 3 ensures that we can exchange a basic variable in a second-order cone block and a nonbasic variable in the nonnegative block. Describing the pivoting procedure itself is straightforward but cumbersome, so we omit the details. See [8] for the details of the pivoting procedure.

In this paper, we have shown the basic theorem to develop a pivoting algorithm for $\langle P \rangle$. However, there are several problems in front of us to establish the pivoting algorithm for $\langle P \rangle$. Describing the details of the pivoting algorithm rigorously, and implementing it may be the next step. Proving or disproving global convergence of the pivoting algorithm is an interesting subject. Finally, developing a pivoting procedure for general SOCP is a challenging theme.

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