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POLYNOMIAL TIME ALGORITHMS FOR MAXIMIZING THE INTERSECTION VOLUME OF POLYTOPES

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Dedicated to Masakazu Kojima on the occasion of his 60th birthday.

Abstract: Suppose that we have several polytopes in $\mathbb{R}^d$ and we can translate them without rotation. Here we consider the intersection maximization problem, which asks the positions of the polytopes which maximizes the volume of their intersection. In this paper, we address this problem, and show that the problem can be solved in oracle polynomial time by an ellipsoid method, exploiting two important facts. Namely, the objective function is a continuous piecewise-polynomial function and its dth root is concave. We further study the structure of the problem in depth for the two dimensional case, and propose an algorithm which solves it in $O(n^4)$ time for two non-convex polygons, where n is the total number of vertices.

Key words: *polytope, volume, ellipsoid method, polynomial time, algorithm arrangement, face lattice*

Mathematics Subject Classification: *52B05, 65K05*

1 Introduction

Let A and B be two polygons in the Euclidean plane where the position of A is fixed and the polytope B can be translated freely without rotation. Then, the intersection of A and B changes as the move. Here we consider the *intersection maximization problem*, that is, to maximize the volume of the intersection by translation.

Let us consider a simple example of two convex polygons in Figure 1. By shifting horizontally the triangle from left to right, the volume of the intersection of the triangle and the square initially increases as a convex function in the translation variable. After the whole triangle is contained in the square, the volume is constant, and then decreases as a concave function. Thus, the volume of the intersection has both features of convex and concave functions. One can also see that the function is continuous, but not differentiable. Consequently, maximization of such functions may not be done in a straightforward manner.

In 1998, de Berg et al. studied this problem and proposed an algorithm running in $O(n \log n)$ time for two convex polygons in plane [6], where n is the total number of vertices. There is also a study of sublinear time randomized algorithm for two convex polygons in plane [1]. However, to the best of our knowledge, neither the non-convex case nor higher dimensional cases has been studied.

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In higher dimensions, the computation of the volume itself is known to be #P-hard [12] when the input is a V-polytope (a convex polytope given as the convex hull of points) or an H-polytope (a convex polytope given as the solution set to a system of linear inequalities). If the input two polytopes are identical (i.e. $A = B$), the maximum volume of their intersection equals the volume of the input polytope. Thus, the associated problem to decide whether the maximum intersection volume is a given value is #P-hard for both H-polytopes and V-polytopes. This fact itself does not make the intersection maximization problem worthless to investigate, since one can compute the volume of a V-polytope and H-polytope quickly for considerably complex polytopes in modest (say up to 10) dimensions, see [8]. A natural question then is whether there exists an oracle polynomial time algorithm for the intersection maximization problem where the oracle returns the volume of a polytope.

In this paper, we address this problem from the computational point of view. First, we show that the dth root of the objective function is concave for any finite number of d-dimensional convex polytopes. We also show that the hyperplanes spanned by the facets of the polytopes define an arrangement where the objective function is a polynomial function in each of its regions. The problem has this structure even in non-convex cases. It follows that one can solve the problem in oracle polynomial time by an ellipsoid method where the oracle returns the volume, which is the objective function of the problem, and its subgradient. By using these properties, we propose a strongly polynomial time algorithm for the non-convex case, where the input is given by the union of several polytopes. In the two dimensional case of two non-convex polytopes, we propose an enumeration based algorithm which runs in $O(n^4)$ time and $O(n^2)$ space.

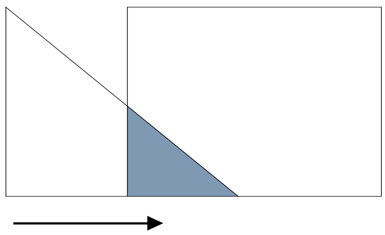

Figure 1: The volume of the intersection of two polygons

The organization of this paper is as follows. Section 2 sets basic definitions and notations. We show the concavity of the dth root of the objective function in Section 3, and that it is a piecewise-polynomial function in Section 4. Section 5 describes an ellipsoid method running in oracle polynomial time. In Section 6, we explore the structure of the problem in the two dimensional case, and present an algorithm running in $O(n^4)$ time for the non-convex case.

2 Preliminaries

We begin with the definitions related to polytopes. See textbooks, such as [14] to see more details.

We denote the d-dimensional Euclidean space by $\mathbb{R}^d$. A *convex body* A is a convex compact set with nonempty interior, i.e., A is closed, bounded, and the affine hull of A is $\mathbb{R}^d$, and for any points x and y in A, the line segment connecting x and y is included in A. A *convex polytope* is the intersection of a finite number of closed halfspaces. A convex polytope is simply called a *polytope*. A polytope in $\mathbb{R}^d$ is called *full dimensional* if its affine hull is $\mathbb{R}^d$, and *c-dimensional* if the dimension of its affine hull is c. If a polytope is full dimensional, it is a convex body. A c-dimensional polytope is called a *c-polytope.*

For $c \in \mathbb{R}^d, b \in \mathbb{R}$, a linear inequality $c^T x \leq b$ is said to be *valid* for a polytope P if it is satisfied by all points x in P. A *face* of a polytope P is a set of form $\{x \in P \mid c^T x = b\}$ for some valid inequality $c^T x \leq b$ for P. The *dimension* of a face is the dimension of its affine hull. A face of dimension c is called a *c-face.* The 0-faces of P are the *vertices* of P, and the $(d-1)$-faces of P are the *facets* of P. The empty set is a unique (-1)-face and P a unique d-face. The set of all faces of P ordered by inclusion is the *face lattice* of P, whose least element is the empty set and whose largest element is P. For a facet F of P, the hyperplane spanned by F is called the *affine hyperplane* of F, and denoted by $H(F)$. We denote by $H^+(F)$ the closed halfspace defined by $H(F)$ containing P, and by $H^-(F)$ the other closed halfspace defined by $H(F)$. The *vertex-facet incidence* of P is the bipartite graph $G(P) = (V_1, V_2, E)$ where V_1 is the set of vertices, V_2 the set of facets and $(v, F) \in E$ ($v \in V_1$ and $F \in V_2$ are adjacent in G) if and only if $v \in F$. The following property is basic, see e.g. [14].

Property 2.1 *The face lattices of polytopes P_1 and P_2 are isomorphic if and only if there is an isomorphism between $G(P_1)$ and $G(P_2)$ preserving the given bipartitions (i.e. the vertices in $V_1(P_1)$ are mapped to those in $V_1(P_2)$).*

A *d-simplex* is a d-polytope with exactly $d+1$ vertices. In particular, a d-simplex is often called simplex, simply. A *triangulation* Δ of a d-polytope is a set of d-simplices and their faces such that their union is P and the intersection of any two members of Δ is their common face.

For a convex body P in $\mathbb{R}^d$, we denote its d-dimensional volume by $vol(P)$. For a vector h in $\mathbb{R}^d$, $P+h$ is the convex body obtained by translating P by h, i.e., $P+h = \{x+h | x \in P\}$. Whenever there is no confusion, for simplicity, we call a face of $P+h$ a face of P, instead of correctly saying "a translated face of P." For a sequence $\mathcal{P}$ of polytopes $P_1, \ldots, P_k$, their intersection forms a polytope. We call the polytope the *intersection polytope*, and denote it by $\cap(\mathcal{P})$, i.e., $\cap(\mathcal{P}) = \cap_{i=1}^k P_i$. For a (long) vector‡ $h = (h_2, \ldots, h_k) \in \mathbb{R}^{d\times(k-1)}$, we denote $\cap(\{P_1, P_2 + h_2, \ldots, P_k + h_k\})$ by $\cap(\mathcal{P} + h)$. For a given set of polytopes $\mathcal{P} = \{P_1, \ldots, P_k\}$ in $\mathbb{R}^d$, the intersection maximization problem is to maximize $vol(\cap(\mathcal{P} + h))$ subject to $h \in \mathbb{R}^{d\times(k-1)}$. We call the objective function of the problem the *volume function.* For simplicity, we may denote $vol(\cap(\mathcal{P} + h))$ by $vol(h)$.

Hereafter, unless otherwise specified, $\mathcal{P}$ denotes the set of k polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^d$, and $h = (h_2, \ldots, h_k) \in \mathbb{R}^{d\times(k-1)}$. Without loss of generality, we assume no two input polytopes P_i and P_j have the faces whose affine hulls can be the same by translation, thus $F \neq F'$ for any face F of P_i and any face F' of P_j, as we may translate their initial positions. We denote by $\mathcal{F}(\mathcal{P})$ the set of all facets of all polytopes in $\mathcal{P}$. By the assumption above, every facet in $\mathcal{F}(\mathcal{P})$ is a facet of a unique polytope P_i. Here we present some properties of the intersection of the polytopes.

Property 2.2 *Any c-face of $\cap(\mathcal{P} + h)$ with $c < d$ is included in at least one facet of $\mathcal{F}(\mathcal{P})$.*

For a face f of P_i and $h = (h_2, h_3, \ldots, h_k) \in \mathbb{R}^{d\times(k-1)}$, we define $f + h$ by $f + h_i$ if $i \geq 2$ and f otherwise. For a set of facets $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq \mathcal{F}(\mathcal{P})$, we define $\cap(\mathcal{F} + h)$ by $\cap(\{F_1 + h, \ldots, F_m + h\})$. For an intersection polytope $P = \cap(\mathcal{P} + h)$ and a face f of P, we define the *topological representation* of f, denoted by $\mathcal{T}(f)$, by the set of facets F in $\mathcal{F}(\mathcal{P})$ such that $F + h$ includes f. The following properties immediately follow from the definition.

‡It might be more natural to call $h \in \mathbb{R}^{d\times(k-1)}$ a *matrix* but we regard this as a $d \times (k-1)$ dimensional vector.

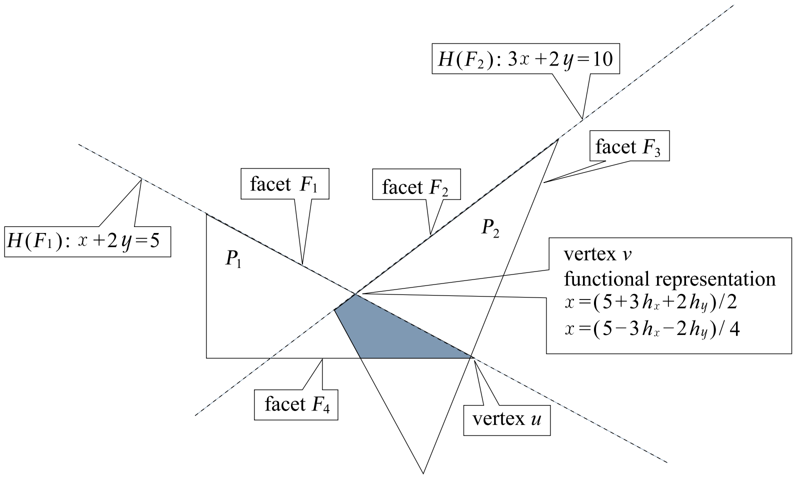

Figure 2: Affine hyperplanes of facets, and functional representation of vertex

Property 2.3 *Let $\cap(\mathcal{P}+h)$ be any intersection polytope. Then the following statements hold.*

(1) *No two distinct faces of $\cap(\mathcal{P}+h)$ have the same topological representation.*

(2) *For any faces f and f' of $\cap(\mathcal{P}+h)$, $f \subseteq f'$ if and only if $\mathcal{T}(f') \subseteq \mathcal{T}(f)$.*

For an intersection polytope $\cap(\mathcal{P}+h)$, we define its *topological vertex set* by the set of topological representations of all vertices in $\cap(\mathcal{P}+h)$.

For any vertex v of $\cap(\mathcal{P}+h)$, the position of v is given by the intersection of the facets in $\mathcal{T}(v)$, i.e., $\{v\} = \cap(\mathcal{T}(v)+h)$. Thus, v is represented as a unique solution to the linear equation system induced by $H(F_1+h), \ldots, H(F_m+h)$, where $\mathcal{T}(v) = \{F_1, \ldots, F_m\}$. By solving the system without assigning values to h, we obtain a function in h. We call this function the *functional representation* of a vertex v. For any vertex of any intersection polytope $\cap(\mathcal{P}+h)$, its functional representation is a linear function in h.

Example 2.4 *In Figure 2, the affine hyperplanes of facets F_1 and F_2 are $H(F_1) = \{(x,y) | x+2y=5\}$, and $H(F_2) = \{(x,y) | 3x+2y=10\}$, respectively. The topological representation of vertex v is $\{F_1, F_2\}$, and the topological representation of vertex u is $\{F_1, F_3, F_4\}$. The functional representation of v is the solution to the linear system;*

$$\begin{aligned} &x + 2y = 5, \\ &3(x - h_x) + 2(y - h_y) = 10. \end{aligned}$$

Thus, $(x,y) = \left(\frac{5+3h_x+2h_y}{2}, \frac{5-3h_x-2h_y}{4}\right)$.

For a given vector $h \in \mathbb{R}^{d\times(k-1)}$, a facet F in $\mathcal{F}(\mathcal{P})$ is *irredundant* if $F+h$ contains a facet of $\cap(\mathcal{P}+h)$. A facet is irredundant for some h, and not so for some other h, thus the face lattice of the intersection polytope changes depending on the position of h. The following lemma, however, shows that the face lattice is uniquely determined by the topological vertex set.

Lemma 2.5 *Two intersection polytopes having the same topological vertex set $\mathcal{T}$ have the same (isomorphic) face lattice.*

Proof. Suppose that $\cap(\mathcal{P}+h)$ and $\cap(\mathcal{P}+h')$ have the topological vertex set equal to $\mathcal{T}$. We will show that the sets of irredundant facets are the same for h and h'. This implies that two intersection polytopes have the isomorphic vertex-facet incidence, thus the topological vertex sets of two intersection polytopes are the same. Then, by Property 2.1, we may conclude that the face lattices are isomorphic.

Let $\mathcal{F}$ be the union of all topological representations in $\mathcal{T}$. Let $F \in \mathcal{F}$ be a redundant facet. Since it is redundant, there is an irredundant facet F' in $\mathcal{F}$ such that $\{T \in \mathcal{T} | F \in T\}$ is properly contained in $\{T \in \mathcal{T} | F' \in T\}$. This in fact shows that one can detect all irredundant facets in $\mathcal{F}$ by looking only at $\mathcal{T}$. This completes the proof. □

$\boxed{3}$ Convexity on the Intersection Volume of Polytopes

Our first result is the following theorem.

Theorem 3.1 *For any two convex bodies A and B in $\mathbb{R}^d$, the function $(vol(\cap(\{A, B+h\})))^{1/d}$ is concave in h over the region of nonempty intersection, $\Omega = \{h \in \mathbb{R}^d | \cap(\{A, B+h\}) \neq \emptyset\}$.*

Notice that the region Ω of nonempty intersection is convex. To see this, we look at different representations as follows:

$$\begin{aligned}\Omega &= \{h \in \mathbb{R}^d | \cap(\{A, B+h\}) \neq \emptyset\}\\ &= \{h \in \mathbb{R}^d | x = y + h \text{ for some } x \in A \text{ and } y \in B\}\\ &= \{h \in \mathbb{R}^d | h = x - y \text{ for some } x \in A \text{ and } y \in B\}\\ &= A + (-B) = A - B.\end{aligned}$$

The last line says the region of nonempty intersection is simply the Minkowski sum of A and the negative copy of B. The third equation above shows that Ω is the orthogonal projection of a convex body in the space of x, y and h onto the h space. In general, when there are k convex bodies $P_1, \ldots, P_k$, the corresponding region $\Omega = \{h \mid \cap(\{P_1, P_2+h_2, \ldots, P_k+h_k\}) \neq \emptyset\}$ is easily seen to be an orthogonal projection of a convex body and thus convex.

The special case $d = 2$ of Theorem 3.1 is proved by [6]. Our proof is a natural extension of their proof for $d = 2$. In particular, we use the Brunn-Minkowski theorem below. Let us denote by $H(z, d)$ the hyperplane in $\mathbb{R}^d$ given by $x_d = z$.

Theorem 3.2 (Brunn-Minkowski, see [4]) *Let P be any convex body in $\mathbb{R}^d$. Then, the function $(vol(\cap(\{P, H(z,d)\})))^{1/(d-1)}$ is concave in z over the region $\Omega = \{z \in \mathbb{R}^d | \cap(\{P, H(z,d)\}) \neq \emptyset\}$.*

Proof (of Theorem 3.1). Let $h, h' \in \mathbb{R}^d$ be any vectors such that both $\cap(\{A, B+h\})$ and $\cap(\{A, B+h'\})$ are non-empty. It suffices to show that the function is concave over the line segment connecting h and h'. Let

$$\begin{aligned}\bar{A} &= \{(x_1, \ldots, x_d, \lambda) \mid 0 \leq \lambda \leq 1, (x_1, \ldots, x_d) \in A\},\\ \bar{B} &= \{(x_1, \ldots, x_d, \lambda) \mid 0 \leq \lambda \leq 1, (x_1, \ldots, x_d) \in B + (\lambda h + (1-\lambda)h'))\}.\end{aligned}$$

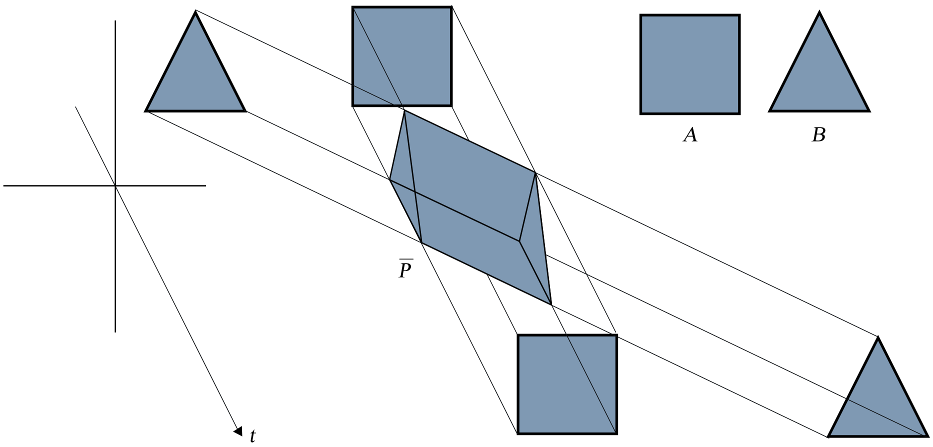

Figure 3: Polytopes $\bar{A}$, $\bar{B}$, and $\bar{P}$

Since both sets are convex, $\bar{P} = \bar{A} \cap \bar{B}$ is also convex, see Figure 3. Consequently, the intersection of $H(\lambda, d+1)$ and $\bar{P}$ is

$$\begin{aligned} H(\lambda, d+1) \cap \bar{P} &= \{(x_1, \ldots, x_d, \lambda) \mid (x_1, \ldots, x_d) \in A, (x_1, \ldots, x_d) \in B + (\lambda h + (1-\lambda)h')\} \\ &= \{(x_1, \ldots, x_d, \lambda) \mid (x_1, \ldots, x_d) \in A \cap (B + (\lambda h + (1-\lambda)h'))\}. \end{aligned}$$

Thus, the intersection is a lifted copy of $\cap(\{A, B + (\lambda h + (1-\lambda)h')\})$. The theorem then follows directly from the Brunn-Minkowski Theorem. □

The theorem above can be easily extended to the intersection of several convex bodies.

Theorem 3.3 *For any set $\mathcal{P}$ of k convex bodies $P_1, \ldots, P_k$ in $\mathbb{R}^d$, the function $vol(\cap(\{P_1, P_2 + h_2, \ldots, P_k + h_k\})))^{1/d}$ is concave in $h = (h_2, \ldots, h_m)$ over $\Omega = \{h \mid \cap(\{P_1, P_2 + h_2, \ldots, P_k + h_k\}) \neq \emptyset\}$.*

It follows from Theorem 3.1 that any locally maximum solution of the intersection maximization problem is a global maximum solution. This also implies that the volume function is unimodal. Moreover, it is semistrictly quasiconcave. A function $f(x)$ is called *semistrictly quasiconcave* [3] if $f(y) < f(\lambda x + (1-\lambda)y)$ holds for any x and y with $f(x) > f(y)$, and $0 < \lambda < 1$. Since the dth power of any non-negative concave function satisfies this condition, the volume function is also a semistrictly quasiconcave function.

Corollary 3.4 *For any set $\mathcal{P}$ of k convex bodies in $\mathbb{R}^d$, $vol(\cap(\mathcal{P} + h))$ is semistrictly quasiconcave in h over $\Omega = \{h \mid \cap(\mathcal{P} + h) \neq \emptyset\}$.*

$\boxed{4}$ Decomposing the Domain into Equivalence Regions

In this section, we will explore the structure of the volume function in depth, and show that it is a continuous piecewise-polynomial function. In particular, we introduce a decomposition of the region Ω of nonempty intersection (defined in Section 3) into full dimensional convex polytopes (called *pieces*) in such a way that the volume function is a polynomial function in the translation vector h within each of the pieces. This result will be used when we present efficient algorithms for the intersection maximization problem in the following sections.

First, we show that the volume function is continuous.

Lemma 4.1 *For any sequence $\mathcal{P}$ of polytopes $P_1, \ldots, P_k$, $vol(\cap(\mathcal{P}+h))$ is continuous in h.*

Proof. Let $h = (h_2, \ldots, h_k)$ and $x = (x_2, \ldots, x_k)$ be arbitrary vectors in $\mathbb{R}^{d\times(k-1)}$, and $V_i, i = 2, \ldots, k$ be the volume of the polytope obtained by projecting P_i to the hyperplane normal to x_i. Then, we can see that for any $\epsilon > 0$, the difference between $vol(h)$ and $vol(h + \epsilon x)$ is bounded by $\sum \epsilon V_i ||x_i||$. Hence, $vol(h) - vol(h + \epsilon x)$ converges to zero when $\epsilon \rightarrow 0$. Therefore $vol(h)$ is continuous. $\square$

Note that the volume function is continuous but not differentiable, as we can see in Figure 1; When the half of the triangle is included in the square, the derivative of the volume function changes from positive to negative without visiting zero.

Next we investigate the form of the volume function. For this, it is useful to review how the volume of a polytope can be computed.

There are several methods to compute the volume of a d-polytope $P \subseteq \mathbb{R}^d$. Here we use the following simple recursive method, proposed by Cohen and Hickey [9]. The method is to compute a triangulation of P, and compute the sum of the volume of the simplices in the triangulation. The volume of a d-simplex S with vertices $v_0, ..., v_d$ is given by

$$vol(S) = \frac{|\det(v_1 - v_0, v_2 - v_0, \ldots, v_d - v_0)|}{d!},$$

where $\det(v_1 - v_0, v_2 - v_0, \ldots, v_d - v_0)$ is the determinant of the matrix composed of column vectors $v_1 - v_0, \ldots, v_d - v_0$.

To obtain a triangulation of P, we triangulate faces recursively. Namely, we choose any vertex v of P, and triangulate every facet of P not containing v recursively. Then the collection of convex hulls of v and resulting simplices in the triangulations of the chosen facets is a triangulation of P.

For an intersection polytope $\cap(\mathcal{P}+h)$, we consider the expression of the volume with the use of the functional representations of vertices instead of their positions, given as the sum of the volumes of d-simplices in a triangulation. We denote this function by $V_h : \mathbb{R}^{(k-1)\times d} \rightarrow \mathbb{R}$ and call it a *volume function with respect to h*. Note that this function is a polynomial of degree at most d.

For two vectors h and h', $V_h(h')$ may not be equal to $vol(h')$. The following theorem characterizes the case when they are equal. This is extremely important for the design of our algorithms.

Theorem 4.2 *For two vectors h and h', $V_h(h') = vol(h')$ if the topological vertex sets of $\cap(\mathcal{P} + h)$ and $\cap(\mathcal{P} + h')$ are equal.*

To verify the theorem, we first observe the following fact.

Lemma 4.3 *The Cohen-Hickey algorithm behaves identically for all intersection polytopes having the same topological vertex set.*

Proof. This comes from that the Cohen-Hickey algorithm looks at only the structure of the face lattice. Thus, by Lemma 2.5, the algorithm behaves identically for the polytopes with the same topological vertex set. $\square$

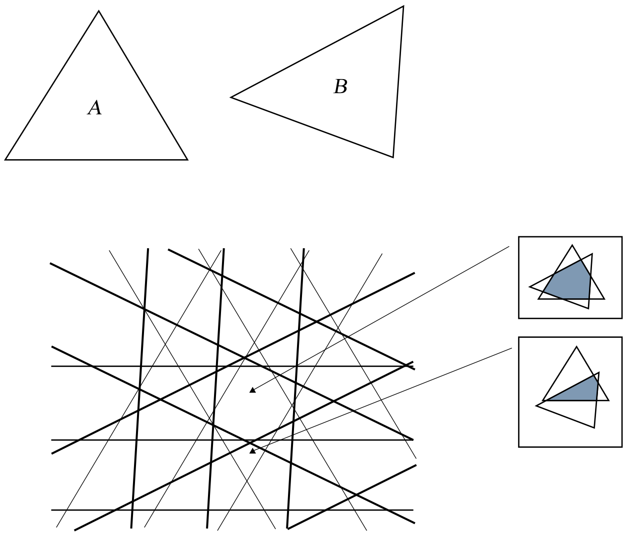

Figure 4: Arrangement of the hyperplanes induced by the facets of A and B

Proof (of Theorem 4.2). By Lemma 4.3, the volume function with respect to h, obtained by the Cohen-Hickey algorithm is equivalent to that of h'. This completes the proof. □

Now we can see that the volume functions of two intersection polytopes are identical if their topological vertex sets are the same. Thus, $\mathbb{R}^{d\times(k-1)}$ is decomposed into maximal regions in which the topological vertex sets are identical. We call such a region an *equivalence region*. See a two dimensional example in Figure 4.

Lemma 4.4 *A set $\mathcal{F}$ of facets in $\mathcal{F}(\mathcal{P})$ is the topological representation of a vertex v of $\cap(\mathcal{P}+h)$ if and only if the following conditions hold;*

(a) *$\cap(\mathcal{F}+h)$ is a point,*

(b) *$\cap(\mathcal{F}+h)$ is contained in $\cap(\mathcal{P}+h)$, and*

(c) *no facet F in $\mathcal{F}(\mathcal{P})\setminus\mathcal{F}$ satisfies $v\in F+h$.*

Proof. The only-if part of the statement is obvious. Thus we prove the if part. Suppose that $\mathcal{F}$ satisfies the conditions (a), (b) and (c). Let $\{v\}=\cap(\mathcal{F}+h)$, and x be any vector in $\mathbb{R}^d$ satisfying $x\neq 0$. Since $\cap(\mathcal{F}+h)$ is a point, there is at least one facet in $\mathcal{F}$ such that its normal vector is not orthogonal to x. Thus, at most one of $v+x$ and $v-x$ can be included in $\cap(\mathcal{P}+h)$. This together with (b) implies that v is a vertex of $\cap(\mathcal{P}+h)$. From (c), $\mathcal{F}$ is the topological representation of v. □

For any topological representation $\mathcal{T}(v)$, we define the *feasible region* of $\mathcal{T}(v)$ by the set of h such that $\cap(\mathcal{P}+h)$ has a vertex whose topological representation is $\mathcal{T}(v)$. From Lemma 4.4, the closure of the feasible region R of $\mathcal{T}(v)$ is given by the intersection of the halfspaces $S_v(F):=\{h\mid\emptyset\neq\cap(\mathcal{T}(v)+h)\subseteq H^+(F+h)\}$, for any $F\in\mathcal{F}(\mathcal{P})$. Note that the set $S_v(F)$ can be represented as the set of solutions to a system of one linear inequality and linear equations in h. More explicitly, the equations determine the unique point specified in the condition (a) of Lemma 4.4, while the inequality represents the condition (b) expressing the vertex being on the feasible side of each facet F. The condition (c) does not appear in the representation of the closure of the feasible region. The boundary of $S_v(F)$ is the affine subspace $\{h \mid \emptyset \neq \cap(\mathcal{T}(v) + h) \subseteq H(F + h)\}$. We call this subspace the *boundary subspace* induced by a facet F and a topological representation $\mathcal{T}(v)$. When this subspace is a hyperplane, we call it the *boundary hyperplane*. Any facet of the feasible region R is contained in a boundary subspace. Consequently the feasible region is the set of relative interior points of a polyhedron.

Lemma 4.5 *Let Q be the equivalence region in which the topological vertex set of the intersection polytope is $\mathcal{T}$. Then, Q is the intersection of the feasible regions of the topological representations of vertices in $\mathcal{T}$, thus is the relative interior of a polyhedron in $\mathbb{R}^{d\times(k-1)}$.*

Since the volume function is continuous, we have the following lemma.

Lemma 4.6 *The volume function V_h with respect to h of an equivalence region R represents the volume function in the closure of R.*

In the following, we will show that the equivalence regions can be decomposed into some regions of a hyperplane arrangement. We here introduce a notation. A set $\mathcal{F}$ of facets in $\mathcal{F}(\mathcal{P})$ is called an *independent facet set* if **for any** h, the intersection of the affine hyperplanes of $F + h$ over all $F \in \mathcal{F}$, that is $\cap(\{H(F+h) \mid F \in \mathcal{F}\})$, is a point, and for any i, $\cap(\mathcal{F} \cap \mathcal{F}_\rangle) = \emptyset$ if and only if $\mathcal{F} \cap \mathcal{F}_\rangle = \emptyset$. From this definition, we can see the following lemma.

Lemma 4.7 *The topological vertex set of any full dimensional equivalence region is composed only of independent facet sets.*

Proof. Let v be a vertex of some intersection polytope such that $\mathcal{T}(v)$ is not an independent facet set. Then, there is a vector h' such that $\cap(\{H(F + h') \mid F \in \mathcal{T}(v)\})$ is not a single point. Let h'' be any point in $S := \{h \mid \emptyset \neq \cap(\mathcal{T}(v) + h)\}$. Since v is a vertex, $\cap(\mathcal{T}(v) + h'') = \{v\}$ holds, and in particular, the normal vectors of $\mathcal{T}(v)$ span $\mathbb{R}^d$. It follows that $\cap(\{H(F + h') \mid F \in \mathcal{T}(v)\}) = \emptyset$ and consequently $\cap(\mathcal{T}(v) + h') = \emptyset$. Then, one can easily see that $\cap(\mathcal{T}(v) + (1 - \lambda)h'' + \lambda h') = \emptyset$ for any $0 < \lambda \leq 1$. This means that no point in S has a full-dimensional neighbor in S, and thus S is not full dimensional. Since S contains the equivalence region of the intersection polytope, the equivalence region cannot be full dimensional. $\square$

For any $\mathcal{F} \subseteq \mathcal{F}(\mathcal{P})$, we denote by $\mathcal{F}_i$ the set of its facets taken from $P_i \in \mathcal{P}$.

Lemma 4.8 *Let $\mathcal{F} \subseteq \mathcal{F}(\mathcal{P})$ be an independent facet set. Then, the following statements hold.*

(1) *$\cap(\{H(F) \mid F \in \mathcal{F}\})$ is a point,*

(2) *for each i, $\mathcal{F}_i = \emptyset$ or $\cap(\mathcal{F}_i)$ is a face of P_i, and*

(3) *the sum of the dimensions of $\cap(\mathcal{F}_i)$ over all $i = 1, \dots, k$ equals $(k-1)d$ where we consider the dimension of $\cap(\mathcal{F}_i)$ as d when $\mathcal{F}_i = \emptyset$.*

Proof. (1) and (2) are obvious from the definition. If (3) is violated, then the affine hulls of $\mathcal{F}_i$'s are dependent, thus for some h, it has no intersection. □

A facet of the closure of an equivalence region is given by a facet of the feasible region of some topological representation. This together with Lemma 4.4 implies the following lemma.

Lemma 4.9 *Let Q be a full dimensional equivalence region. Then, every facet of Q is included in a boundary hyperplane $H(F, \mathcal{F}) := \{h \mid \cap(\mathcal{F} + h) \in H(F + h)\}$ for some facet $F \in \mathcal{F}(\mathcal{P})$ and some independent facet set $\mathcal{F}$.*

Let $\mathcal{H}(\mathcal{P})$ be the set of all such boundary hyperplanes. Since every boundary hyperplane of an equivalence region belongs to $\mathcal{H}(\mathcal{P})$, we have the following lemma.

Lemma 4.10 *The relative interior of any cell of the hyperplane arrangement $\mathcal{H}(\mathcal{P})$ is contained in an equivalence region.*

We now have the following theorem characterizing the volume function in detail.

Theorem 4.11 *For any sequence $\mathcal{P}$ of polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^d$, the volume function $vol(\cap(\mathcal{P} + h))$ is a continuous semistrictly quasiconcave piecewise-polynomial function of degree at most d on $\{h \mid \cap(\mathcal{P} + h) \neq \emptyset\}$, where "piecewise-polynomial" means that $\mathbb{R}^{d\times(k-1)}$ is decomposed into pieces which are full dimensional polyhedra, and in each piece the function is a polynomial in h. Moreover, every facet of each piece is contained in some boundary hyperplane $H(F, \mathcal{F})$.*

Proof. We already showed that the volume function is continuous and semistrictly quasiconcave in Lemma 4.1 and Theorem 3.3. In an equivalence region, the volume function is a polynomial function with some absolute operators, which makes the determinants nonnegative. We here prove that one can replace some absolute-value operators by constants, $+1$'s and -1's, according to the sign of the value of determinants. Let h and h' be interior points of a full dimensional equivalence region, and V'_h be the polynomial function obtained from V_h by replacing each absolute operator by a constant $+1$ or -1. Lemma 4.3 says that when we continuously move h directed to h', the topological structure of the triangulation does not change, and there is no simplex in the triangulation such that its volume touches 0 during the move. It means that the signs of the determinants of the simplices used to compute V'_h do not change during the move, thereby $V'_h(h') = V_{h'}(h')$. Lemma 4.9 shows the correctness of the representation of the facets of the regions. □

$\boxed{5}$ Ellipsoid Method for Higher Dimensions

The ellipsoid method is a fundamental algorithm for solving convex programming problems [5, 11]. For a given point x and a function f having a maximum solution, a hyperplane is called *separating hyperplane* if it separates x from the set of the optimal solutions. For any concave function, an ellipsoid method computes a point maximizing the function by finding polynomially many separating hyperplanes [5, 11]. In this section, we show that we can compute a separating hyperplane for any h, thus the intersection maximization problem can be solved in oracle polynomial time where oracle is to answer the volume of a polytope and the gradient of volume function with respect to a given point.

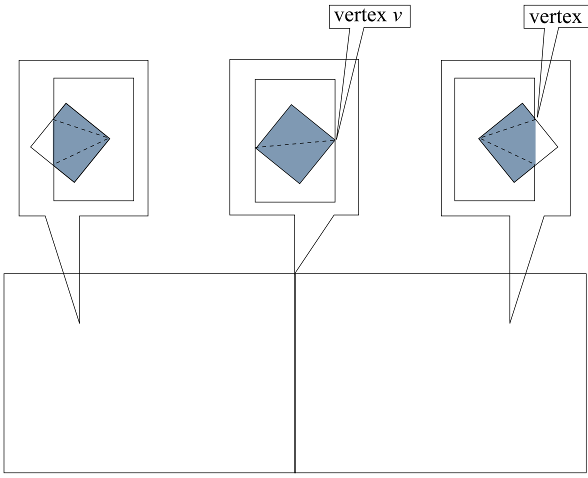

Figure 5: Triangulations in three equivalence regions

It is known [5] that for any non-differentiable continuous concave function and a point x, a separating hyperplane is induced by its subgradient at x. Thus, here we present an algorithm to obtain a subgradient of the volume function by computing the gradient of the volume function with respect to given h.

Computing a subgradient of the dth root of the volume function is not a trivial task. The derivative of dth root of the volume function with respect to h is possibly not a subgradient at h when h is on the boundary of a full dimensional equivalence region. See Figure 5. There are two full dimensional equivalence regions corresponding to the intersection polytopes drawn above, and their boundary is the vertical line on the center. In both two full dimensional regions, the triangulation of the intersection polytope is composed of three simplices. On the other hand, on the center line, the triangulation is composed of two simplices. One simplex is missing on the center. The functional representation of vertex v is the intersection point of three facets including v. Thus, the functional representation has no solution when we move to the right-hand region. If we use the functional representation of vertex u instead of that of v, the sum of the volume of the two simplices is smaller than the volume of the intersection polytope. Thus, the derivative of the dth root of the volume function with respect to h is possibly not a subgradient at h when h is on the boundary of a full dimensional equivalence region.

Suppose that we are given a vector $h \in \mathbb{R}^{d\times(k-1)}$ and going to compute a subgradient at h. If h is an interior point of a full dimensional equivalence region, then the volume function is differentiable at h, thus a subgradient at h can be obtained easily. Otherwise, we perturb h slightly, by adding a vector $(\epsilon_1, \ldots, \epsilon_{d\times(k-1)})$ such that each ϵ_i is a positive number smaller than any positive real number, and $\epsilon_i >> \epsilon_{i+1}$ holds for any i. We denote the result of the addition by h'. We can take such ϵ's by introducing a lexicographic ordering, i.e., we represent each real number r by $r = a_0\epsilon_0 + a_1\epsilon_1 + \cdots + a_{d\times(k-1)}\epsilon_{d\times(k-1)}$, where $\epsilon_0 = 1$, and consider that $r > q$ holds for $q = b_0\epsilon_0 + b_1\epsilon_1 + \cdots + b_{d\times(k-1)}\epsilon_{d\times(k-1)}$ if there is an index j such that $a_i = b_i$ holds for any $i < j$ and $a_i > b_i$ holds for $i = j$.

Since each ϵ_i is sufficiently small, h' is an interior point of a full dimensional region. From Lemma 4.6, the volume function $V_{h'}$ with respect to h' satisfies that $V_{h'}(h) = vol(h)$, thus the derivative of $(V_{h'})^{1/d}$ gives a subgradient at h.

For a convex programming problem over a feasible region Ω (which is a convex body), let $R \in \mathbb{R}$ be a number such that there is a ball of radius R containing Ω, and $r \in \mathbb{R}$ be a number such that there is a ball of radius r contained in Ω, and f_{max} be the maximum function value on Ω, and f_{min} is the minimum function value. Then, an ellipsoid method finds an ϵ-optimal solution by finding $O(d^2(\log R + \log(f_{max} - f_{min}) - \log r - \log \epsilon))$ separation hyperplanes [5]. Here an ϵ-optimal solution has an objective value f such that $f_{max} - f \leq \epsilon$. For the intersection maximization problem, f_{max} is the optimal value, and f_{min} is zero.

The ellipsoid method in [5] needs both r and R to solve the optimization problem. The standard method is an algorithmic version of the Löwner-John theorem, e.g. see [11, Section 4.6] which computes r, R and an affine transformation so that the transformed body is sandwiched between balls of radii r and R with $R/r = O(d^{3/2})$ (i.e. $\log R - \log r$ is of order $\log d$), in oracle polynomial time. Here the oracle is a weak separation oracle for the feasible region Ω. In our setting, the feasible region can be written as an orthogonal projection of a higher dimensional convex polytope as explained in Section 3. Therefore, a weak separation oracle is realizable by solving a polynomial-size LP in either case when input polytopes P_i's are H-polytopes or V-polytopes.

Since $f_{max} \leq R^d$, and $f_{min} \geq 0$, both $\log f_{max}$ and $\log f_{min}$ are bounded by a polynomial in the input size. Combining all observations above, we obtain the following theorem.

Theorem 5.1 *For the problem of maximizing the volume of the intersection of k polytopes in d dimension by translation without rotation, one can compute a solution having objective value no less than the optimal value minus ϵ by $O(d^2(\log R + \log(f_{max} - f_{min}) - \log r - \log \epsilon))$ oracle calls where the oracle is to return the volume and the gradient of the volume function with respect to a point. In particular, $d^2(\log R + \log(f_{max} - f_{min}) - \log r - \log \epsilon)$ is bounded by a polynomial in the input size.*

$\boxed{6}$ Algorithm for Two Dimensional Case

In this section, we assume that $d = 2$ and $k = 2$. Thus we address the intersection maximization problem of two polytopes in the plane. For the problem, we present an enumeration based algorithm running in $O(n^4)$ time with $O(n^2)$ space for non-convex polytopes with at most n vertices. In the following, we will show that at most $O(n^2)$ hyperplanes are irredundant in $\mathcal{H}(\mathcal{P})$, and they are classified into $O(n)$ groups in each of which hyperplanes are parallel. Here a hyperplane is *irredundant* if it includes a facet of a full dimensional equivalence region. Note that $\cap(\mathcal{P} + h) = \cap(\{P_1, P_2 + h\})$ since $k = 2$.

Lemma 6.1 *For any two polytopes $\mathcal{P} = \{P_1, P_2\}$ of at most n vertices, (a) there are at most $O(n^2)$ irredundant hyperplanes in $\mathcal{H}(\mathcal{P})$, and (b) these hyperplanes can be classified into $O(n)$ groups such that any two hyperplanes in a group are parallel.*

Proof. Let Q be a full dimensional equivalence region. Then, each facet of the closure of Q is contained in a facet of the closure of the feasible region of a topological representation $\mathcal{F}$ of a vertex. We denote the closure by R. We note that R is a polyhedron. From Lemmas 4.7 and 4.8, $\mathcal{F}$ is composed of two facets, say F_1 and F_2. From Lemma 4.9, F_1 and F_2 satisfy that

(1) both F_1 and F_2 are facets of P_1, and $F_1 \cap F_2$ is a vertex of P_1,

(2) both F_1 and F_2 are facets of P_2, and $F_1 \cap F_2$ is a vertex of P_2, or

(3) F_1 and F_2 are facets of P_1 and P_2, respectively.

By Lemma 4.4, $R = \{h \mid \cap(\{F_1+h, F_2+h\}) \neq \emptyset, \cap(\{F_1+h, F_2+h\}) \in \cap(\{P_1, P_2+h\})\}$. Thus, in case (1), any facet of R is given by an inequality that the vertex $F_1 \cap F_2$ is in $H^+(F+h)$ for a facet F in P_2. Similarly, in case (2), any facet of R is given by an inequality that the vertex $(F_1 \cap F_2) + h$ is in $H^+(F)$ for a facet F in P_1. In case (3), if F_1 and $F_2 + h$ intersect, then $F_1 \cap (F_2+h)$ is always included in $\cap(\{P_1, P_2+h\})$. Let u_1 and u_2 be the vertices of F_1, and v_1 and v_2 be the vertices of F_2. From the condition of two segments intersecting, one can see that F_1 and $F_2 + h$ intersect if and only if $u_1 \in H^+(F_2 + h), u_2 \in H^-(F_2 + h)$ (or $u_1 \in H^-(F_2 + h), u_2 \in H^+(F_2 + h)$), and $v_1 + h \in H^+(F_1), v_2 + h \in H^-(F_1)$ (or $v_1 + h \in H^-(F_1), v_2 \in H^+(F_1)$).

In all cases, any facet is given by a hyperplane $\{h \mid v_i + h \in H(F_1)\}$, or a hyperplane $\{h \mid u_i \in H(F_2 + h)\}$. Since the number of pairs of a vertex and a facet is $O(n^2)$, there are $O(n^2)$ hyperplanes which induce a facet of full dimensional equivalence regions. Moreover, the hyperplanes given by a facet F of $\mathcal{F}(\{P_1, P_2\})$ are parallel to each other, hence we can classify the facets into $O(n)$ parallel classes. $\square$

Here we denote by $\mathcal{H}'(\mathcal{P})$ the set of hyperplanes described in the above proof, i.e., the hyperplanes given by $\{h \mid v \in H(F + h)\}$ for a vertex v of P_1 and a facet F of P_2, or given by $\{h \mid u + h \in H(F)\}$ for a vertex u of P_2 and a facet F of P_1.

Suppose that we are given two non-convex polygons P_1 and P_2 composed of at most n edges. From the discussion above, we can see that the volume function is no longer semistrictly quasiconcave, but still a continuous piecewise-polynomial function. Any piece is a polyhedron given by the union of the regions in the arrangement of the hyperplanes in $\mathcal{H}'(\mathcal{P} = \{P_1, P_2\})$. Here we consider the edge of P_1 and P_2 as facets of them, and define $\mathcal{H}'(\mathcal{P})$ by the set of hyperplanes including an edge of P_1 or P_2. Then, one can see that Lemma 6.1 also holds for non-convex case.

Thus, in each region of the arrangement, the intersection maximization problem is a non-convex non-concave quadratic programming problem with two variables. It can be solved by evaluating the values of the objective function on the vertices, the edges, and the points satisfying that the gradient is zero. It can be done in linear time in the number of edges of the region. We note that the sum of the number of edges in each region is $O(n^4)$, since each edge is included in just two regions. Enumerating all regions in the arrangement of $O(n^2)$ lines takes $O(n^4)$ time by a topological sweep method [10].

Suppose that h and h' are vectors in equivalence regions adjacent to each other via facets such that there are facets of the regions that the affine hull of their intersection is $(d-1)$-dimensional. We can observe that the topological vertex set of $\cap(\mathcal{P} + h)$ and that of $\cap(\mathcal{P} + h')$ differ by at most a constant size. Thus, we obtain the volume function of one by adding the volumes of several triangles to the other. From the fact we obtain the volume function corresponding to a region by slight "update."

Let us see the detail of the update. A topological sweep method finds the regions of a hyperplane arrangement by sweeping the plane by a topological line which intersects each hyperplane at most once. Thus, keeping all the regions touching the sweeping line, we can get a neighboring region of a newly output region R. It can be done by looking at each edge of R, and check whether there is a region including the edge in the memory, or not. This takes constant time for each edge. Thus, we can get a neighboring region in linear time of the number of edges of R. The sum of the numbers of edges in the polytopes touching the sweeping line is $O(n^2)$.

Next we consider the update of the volume function. Suppose that regions R and R' are adjacent via a hyperplane induced by a vertex v of P_1 and an edge e of P_2. Without loss of generality, the boundary hyperplane given by e and v overlaps no other boundary hyperplane. Let e_1 and e_2 be the edges incident to v, u_1 and u_2 are the intersection points of e and e_1, and that of e and e_2, respectively. We denote the volume functions with respect to R and R' by V and V'. When we translate P_2 so that e contains v, there are two cases. The first case is that $H^+(e)$ (or $H^-(e)$) contains both e_1 and e_2, and the second case is that $H^+(e)$ contains one of e_1 and e_2. In the first case, the difference between V and V' is the volume of the triangle given by v, u_1, and u_2. This can be computed in constant time. The difference is also the triangle given by v, u_1 and u_2 in the second case. In any case, the computation time for the update is $O(1)$.

Now, we obtain the following result.

Theorem 6.2 *The problem of maximizing the volume of the intersection of two non-convex polygons in the plane can be solved in $O(n^4)$ time and $O(n^2)$ space.*

When the number of polytopes is a fixed constant k, we can also solve the problem of non-convex cases in strongly polynomial time. When we are given k non-convex polygons, the size of $\mathcal{H}'(\mathcal{P} = \{P_1, \ldots, P_k\})$ is $O(k^3n^3)$ since any independent facet set is given by two edges taken from some polygons. The number of regions in a hyperplane arrangement of $O(k^3n^3)$ hyperplanes in $\mathbb{R}^{2k-2}$ is $O((k^3n^3)^{2k-2})$. The regions are enumerated by a reverse search algorithm [2] in $O((k^3n^3)L(2k-2, k^3n^3))$ time for each, where $L(2k-2, k^3n^3)$ is the time to solve a linear programming problem with $2k-2$ variables and $O(k^3n^3)$ inequalities. In each full dimensional region, we compute the derivative of the volume function, and solve it to obtain the points in the region with the zero gradient. Since the volume function is a quadratic function with $2k-2$ variables, we can find a point x where the derivative vanishes by solving a linear programing problem of $2k-2$ variables with $O(k^3n^3)$ inequalities, which takes $O(L(2k-2, k^3n^3))$ time. In total, we can solve the problem in $O(L(2k-2, k^3n^3)(k^3n^3)^{2k-2})$ time. Note that since k is fixed, $L(2k-2, k^3n^3)$ is $O(n^3)$.

Theorem 6.3 *One can solve the intersection maximization problem of k non-convex polygons in the plane in $O(L(2k-2, k^3n^3)(k^3n^3)^{2k-2})$ time, where $L(2k-2, k^3n^3)$ is the time to solve a linear programming problem of $2k-2$ variables with $O(k^3n^3)$ inequalities.*

In the d-dimensional cases, the size of $\mathcal{H}'(\mathcal{P})$ is bounded by $O((kn)(n^{dk})) = O(kn^{dk+1})$ for polytopes with at most n facets, since each hyperplane is given by a combination of faces taken from polytopes in $\mathcal{P}$, and a facet in $\mathcal{F}(\mathcal{P})$. Thus the number of regions in the arrangement is bounded by $O((kn^{dk+1})^d)$. For each region, we need to solve a polynomial system with degree at most d. By considering this task as an oracle, we obtain the following theorem.

Theorem 6.4 *One can solve the intersection maximization problem of k non-convex polytopes in $\mathbb{R}^d$ by solving polynomial systems of degrees d in a polytope at most $O((kn^{dk+1})^d)$ times.*

7 Conclusion

In this paper, we addressed the problem of maximizing the volume of the intersection of polytopes. We proved that the dth root of the objective function of the problem is concave in any dimension d, and thus the objective function is continuous and semistrictly quasiconcave. Using these, we showed that the problem can be solved by an ellipsoid method in oracle polynomial time, where oracle is to return the volume of a given polytope and its gradient on the point at which the function is differentiable. Moreover, it was shown that the objective function is a piecewise-polynomial function of degree at most d, and each piece is a polytope. The piece is decomposed into regions of the arrangements associated with the faces of the polytopes. For the two dimensional case, we presented an enumeration based algorithm running in $O(n^4)$ time for two non-convex polygons with at most n edges.

It is interesting to study the enumeration problem of the equivalent regions in which the set of vertices of the intersection is equivalent, which enables us to perform an efficient enumeration based algorithm for solving this problem. We can also consider the problem of minimizing the union of two polytopes, or the convex hull of two polytopes. We believe that these problems have similar structures to the intersection maximization problem, but it is not clear whether the objective functions have any sort of concavity. We shall leave these as open problems to be investigated in the future.

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