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ON GENERIC NONDIFFERENTIABILITY OF MAXIMUM TYPE FUNCTIONS AT A POINT OF MINIMUM

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Abstract: In this paper we study a class of maximum type functions satisfying a growth condition. This class is identified with a complete metric space of pairs of continuously differentiable functions defined on a real line. It is shown that a set of all pairs (f, g) for which there is a point of minimum of the corresponding maximum type function, where this function is differentiable and where f and g possess the same value, is a closed nowhere dense subset of the whole space of pairs.

Key words: *complete metric space, minimax problem, open everywhere dense set*

Mathematics Subject Classification: *49J35, 54E52*

1 Introduction

In this paper we study differential properties for a class of maximum type functions $\max\{f(x), g(x)\},\ x \in R^1$ satisfying a growth condition where $f, g \in C^1(R^1)$. We show that for most functions the following property holds:

If z is a point of minimum and $f(z) = g(z)$, then $f'(z) \neq g'(z)$ and the maximum type function is nondifferentiable at z.

When we say here that most elements of a complete metric space Y enjoy a certain property, we mean that the set of points which have this property contains an open, everywhere dense subset of Y. In particular, this property holds generically.

For each $f : R^1 \to R^1$ define $\inf(f) = \inf\{f(x) :\ x \in R^1\}$. For each pair $f, g : R^1 \to R^1$ define a function $\max\{f, g\} : R^1 \to R^1$ by

$$(\max\{f, g\})(x) = \max\{f(x), g(x)\},\ x \in R^1 \tag{1.1}$$

and for each $\psi \in C^1(R^1)$ set

$$||\psi||_1 = \sup\{|\psi(z)|,\ |\psi'(z)| :\ z \in R^1\}.$$

Denote by $\mathcal{M}$ the set of all pairs of real valued functions (f, g), where $f, g \in C^1(R^1)$ satisfy

$$\lim_{|x|\to\infty} \max\{f(x), g(x)\} = \infty. \tag{1.2}$$

For any two pairs $(f_1, g_1),\ (f_2, g_2) \in \mathcal{M}$ define

$$\tilde{d}((f_1, g_1), (f_2, g_2)) = \max\{||f_1 - f_2||_1,\ ||g_1 - g_2||_1\}, \tag{1.3}$$

$$d((f_1, g_1), (f_2, g_2)) = \tilde{d}((f_1, g_1), (f_2, g_2))(1 + \tilde{d}((f_1, g_1), (f_2, g_2)))^{-1}. \tag{1.4}$$

(Here we use the convention that $\infty/\infty = 1$.)

It is not difficult to see that $(\mathcal{M}, d)$ is a complete metric space. By (1.2) for each $(f, g) \in \mathcal{M}$ the minimization problem

$$\max\{f(x), g(x)\} \to \min, \ x \in R^1$$

has a solution.

Denote by $\mathcal{M}_0$ the set of all pairs $(f, g) \in \mathcal{M}$ for which there is $\bar{x} \in R^1$ such that

$$f(\bar{x}) = g(\bar{x}) = \inf(\max\{f, g\}). \tag{1.5}$$

Denote by G the set of all pairs $(f, g) \in \mathcal{M}$ for which there is $\bar{x} \in R^1$ such that (1.5) holds and

$$f'(\bar{x}) = g'(\bar{x}). \tag{1.6}$$

In [4, Proposition 1.1] we proved the following useful result.

Proposition 1.1. *$\mathcal{M}_0$ and G are closed subsets of the metric space $(\mathcal{M}, d)$.*

The next theorem is our main result.

Theorem 1.1. *$\mathcal{M}_0 \setminus G$ is an open everywhere dense subset of the metric space $(\mathcal{M}_0, d)$ and $\mathcal{M} \setminus G$ is an open everywhere dense subset of $(\mathcal{M}, d)$.*

Note that our main result implies that G is nowhere dense.

Theorem 1.1 extends the main result of [4] which was obtained for a subclass of maximum type functions with f, g in $C^2(R^1)$. It should be mentioned that in [4] the assumption that $f, g \in C^2(R^1)$ plays a crucial role.

By Proposition 1.1 the set $\mathcal{M} \setminus G$ is an open subset of $(\mathcal{M}, d)$ and $\mathcal{M}_0 \setminus G$ is an open subset of $(\mathcal{M}_0, d)$. In order to prove the theorem it is sufficient to show that $\mathcal{M} \setminus G$ is an everywhere dense subset of $(\mathcal{M}, d)$ and $\mathcal{M}_0 \setminus G$ is an everywhere dense subset of $(\mathcal{M}_0, d)$.

Let $\epsilon > 0$ and

$$(f, g) \in G. \tag{1.7}$$

We will show that there exists a pair of functions which will be denoted in the sequel as (f_2, g_2) such that $(f_2, g_2) \in \mathcal{M}_0 \setminus G$ and

$$d((f, g), (f_2, g_2)) \le \epsilon.$$

By (1.7) there exists $\bar{x} \in R^1$ such that (1.5) and (1.6) hold. We will construct the pair (f_2, g_2) which is close to (f, g) in $(\mathcal{M}, d)$ and such that $f_2(\bar{x}) = g_2(\bar{x})$, $f_2'(\bar{x}) \ne g_2'(\bar{x})$ and that $\bar{x}$ is a unique point of minimum of the function $\max\{f_2, g_2\}$.

It is not difficult to perturb the functions f and g in such a way that $\bar{x}$ remains a point of local minimum of the maximum of the perturbed functions while their derivatives at $\bar{x}$ are not equal. But we need that the point $\bar{x}$ will be *a unique global minimizer* of the maximum of the perturbed functions. It makes the problem more difficult. (Note that in general the function $\max\{f, g\}$ does not have a unique point of minimum.)

First in Section 2 we construct a pair $(f_1, g_1) \in G$ which is close to (f, g) in $(\mathcal{M}, d)$ such that

$$f_1(\bar{x}) = f(\bar{x}) = g(\bar{x}) = g_1(\bar{x}), \ f_1'(\bar{x}) = f'(\bar{x}) = g'(\bar{x}) = g_1'(\bar{x})$$

and $\bar{x}$ is a unique point of minimum of the function $\max\{f_1, g_1\}$. The functions f_1 and g_1 are defined as $f_1 = f + c_0\psi$, $g_1 = g + c_0\psi$ where c_0 is a positive constant and ψ is a function defined in Section 2. The further construction of the pair (f_2, g_2) depends on the behavior of the functions f, g on the interval $[\bar{x} - c_2, \bar{x} + c_2]$ where c_2 is a small positive parameter chosen in Section 2.

The construction of the pair (f_2, g_2) will be done in Section 3.

It should be mentioned that the study of minimization problems with maximum type objective functions is one of central topics in optimization theory. See, for example, [1-3] and the references mentioned therein.

2 Construction of the Pair (f_1, g_1) and Auxiliary Results

In the sequel we use the following auxiliary result [1].

Lemma 2.1. *Let $(f, g) \in G$, $x \in R^1$ and let $f(x) = g(x) = \inf(\max\{f, g\})$ and $f'(x) = g'(x)$. Then $f'(x) = 0$.*

Lemma 2.1 implies that

$$f'(\bar{x}) = g'(\bar{x}) = 0. \tag{2.1}$$

In view of (1.2) there is $d_0 > 1$ such that

$$\max\{f(x), g(x)\} \geq f(\bar{x}) + 8 \text{ for each } x \in R^1$$

$$\text{satisfying } |x - \bar{x}| \geq d_0. \tag{2.2}$$

There is a function $\phi : R^1 \to [0, 1]$ such that $\phi \in C^\infty(R^1)$,

$$\phi(t) = 1 \text{ if } |t| \leq 1/2, \ \phi(t) = 0 \text{ if } |t| \geq 1, \tag{2.3}$$

$$0 < \phi(t) < 1 \text{ if } 2^{-1} < |t| < 1.$$

Choose a positive number

$$c_1 \leq (2d_0)^{-1} \tag{2.4}$$

and set

$$\psi(x) = (x - \bar{x})^2 \phi((x - \bar{x})c_1), \ x \in R^1. \tag{2.5}$$

Clearly,

$$\psi(x) = 0 \text{ if } |x - \bar{x}| \geq c_1^{-1}, \tag{2.6}$$

$$\psi(x) = (x - \bar{x})^2 \text{ if } |x - \bar{x}| \leq (2c_1)^{-1}. \tag{2.7}$$

By (2.4) and (2.7)

$$\psi(x) = (x - \bar{x})^2 \text{ if } |x - \bar{x}| \leq d_0. \tag{2.8}$$

Choose a positive number c_0 such that

$$c_0 ||\psi||_1 < \epsilon/16 \tag{2.9}$$

and define

$$f_1(x) = f(x) + c_0\psi(x), \ g_1(x) = g(x) + c_0\psi(x), \ x \in R^1. \tag{2.10}$$

Thus we have constructed the pair (f_1, g_1). Now we study some properties of this pair which will be used in the sequel.

It is not difficult to see that

$$(f_1, g_1) \in \mathcal{M}. \tag{2.11}$$

Relations (1.5), (1.6), (2.1), (2.7), (2.8) and (2.10) imply that

$$f_1(\bar{x}) = f(\bar{x}) = g(\bar{x}) = g_1(\bar{x}), \tag{2.12}$$

$$f_1'(\bar{x}) = f'(\bar{x}) = g'(\bar{x}) = g_1'(\bar{x}) = 0. \tag{2.13}$$

By (1.3), (2.9) and (2.10)

$$\tilde{d}((f,g),(f_1,g_1)) \le c_0||\psi||_1 < \epsilon/16. \tag{2.14}$$

We show that for each $x \in R^1 \setminus \{\bar{x}\}$,

$$\max\{f_1(x), g_1(x)\} > \max\{f_1(\bar{x}), g(\bar{x})\}. \tag{2.15}$$

Let $x \in R^1 \setminus \{\bar{x}\}$. There are two cases: $|x - \bar{x}| \ge d_0$; $\quad d_0 > |x - \bar{x}| > 0$.

Consider the first case with

$$|x - \bar{x}| \ge d_0. \tag{2.16}$$

By (1.5), (2.2), (2.5), (2.10), (2.12) and (2.16)

$$\begin{aligned}\max\{f_1(x), g_1(x)\} &\ge \max\{f(x), g(x)\}\\ &\ge 8 + \max\{f(\bar{x}), g(\bar{x})\} = 8 + \max\{f_1(\bar{x}), g_1(\bar{x})\}.\end{aligned} \tag{2.17}$$

Consider the second case with

$$0 < |x - \bar{x}| < d_0. \tag{2.18}$$

In view of (2.8) and (2.18)

$$\psi(x) = (x - \bar{x})^2. \tag{2.19}$$

It follows from (1.5), (2.10), (2.12) and (2.19) that

$$\begin{aligned}\max\{f_1(x), g_1(x)\} &= \max\{f(x) + c_0\psi(x),\ g(x) + c_0\psi(x)\}\\ &= \max\{f(x) + c_0(x - \bar{x})^2,\ g(x) + c_0(x - \bar{x})^2\}\\ &= \max\{f(x), g(x)\} + c_0|x - \bar{x}|^2 \ge \max\{f(\bar{x}), g(\bar{x})\} + c_0|x - \bar{x}|^2\\ &= \max\{f_1(\bar{x}), g_1(\bar{x})\} + c_0|x - \bar{x}|^2 > \max\{f_1(\bar{x}), g_1(\bar{x})\}.\end{aligned} \tag{2.20}$$

Relations (2.17) and (2.20) imply that (2.15) holds in both cases.

Choose a positive number c_2 such that

$$\begin{aligned}&c_2 < \min\{d_0/2,\ 1/16\},\\ &|f_1(t) - f_1(\bar{x})|,\ |g_1(t) - g_1(\bar{x})| \le \epsilon/64,\\ &|f_1'(t) - f_1'(\bar{x})|,\ |g_1'(t) - g_1'(\bar{x})| \le \epsilon/64 \text{ for each } t \in [\bar{x} - c_2, \bar{x} + c_2].\end{aligned} \tag{2.21}$$

We show that at least one of the following properties hold:

(P1) $f(x) \ge f(\bar{x})$ for all $x \in [\bar{x} - c_2, \bar{x} + c_2]$;

(P2) $g(x) \ge g(\bar{x})$ for all $x \in [\bar{x} - c_2, \bar{x} + c_2]$;

(P3) there are $x_1 \in (\bar{x}, \bar{x} + c_2]$, $x_2 \in [\bar{x} - c_2, \bar{x})$ such that $f(x_1) \ge f(\bar{x})$ and $g(x_2) \ge g(\bar{x})$;

(P4) there are $x_1 \in (\bar{x}, \bar{x}+c_2]$, $x_2 \in [\bar{x}-c_2, \bar{x})$ such that $g(x_1) \geq g(\bar{x})$ and $f(x_2) \geq f(\bar{x})$.

Assume that (P1)-(P4) do not hold. Then since (P1) does not hold there is $y_1 \in [\bar{x} - c_2, \bar{x}+c_2]$ such that

$$f(y_1) < f(\bar{x}). \tag{2.22}$$

By (1.5) and (2.22)

$$g(y_1) \geq f(\bar{x}) = g(\bar{x}). \tag{2.23}$$

We consider the case

$$y_1 \in (\bar{x}, \bar{x}+c_2]. \tag{2.24}$$

Since (P4) does not hold it follows from (2.23) and (2.24) that

$$f(x) < f(\bar{x}) \text{ for all } x \in [\bar{x}-c_2, \bar{x}). \tag{2.25}$$

By (1.5) and (2.25)

$$g(x) \geq g(\bar{x}) \text{ for all } x \in [\bar{x}-c_2, \bar{x}). \tag{2.26}$$

Since (P3) does not hold it follows from (2.26) that $f(z) < f(\bar{x})$ for all $z \in (\bar{x}, \bar{x}+c_2]$. Combined with (1.5) this implies that $g(z) \geq g(\bar{x})$ for all $z \in (\bar{x}, \bar{x}+c_2]$. Combined with (2.26) this implies that $g(z) \geq g(\bar{x})$ for all $z \in [\bar{x}-c_2, \bar{x}+c_2]$ and (P2) holds, a contradiction.

Consider now the case with

$$y_1 \in [\bar{x}-c_2, \bar{x}). \tag{2.27}$$

Since (P3) does not hold it follows from (2.23) and (2.27) that

$$f(x) < f(\bar{x}) \text{ for all } x \in (\bar{x}, \bar{x}+c_2]. \tag{2.28}$$

By (1.5) and (2.28)

$$g(x) \geq g(\bar{x}) = f(\bar{x}) \text{ for all } x \in (\bar{x}, \bar{x}+c_2]. \tag{2.29}$$

Since (P4) does not hold it follows from (2.29) that

$$f(z) < f(\bar{x}) \text{ for all } z \in [\bar{x}-c_2, \bar{x}). \tag{2.30}$$

In view of (1.5) and (2.30)

$$g(z) \geq g(\bar{x}) = f(\bar{x}) \text{ for all } z \in [\bar{x}-c_2, \bar{x}). \tag{2.31}$$

Relations (2.29) and (2.31) imply that $g(z) \geq g(\bar{x})$ for all $z \in [\bar{x}-c_2, \bar{x}+c_2]$ and (P2) holds, a contradiction. Thus in both cases we have a contradiction. Therefore our assumption is not true and at least one of the properties of (P1)-(P4) holds.

We will consider four different cases separately. By (2.8) and (2.21) for each $x \in [\bar{x} - c_2, \bar{x}+c_2]$ equality (2.19) holds. It follows from (2.10) and (2.19) that for each $x \in [\bar{x} - c_2, \bar{x}+c_2] \setminus \{\bar{x}\}$

$$f_1(x) > f(x), \; g_1(x) > g(x). \tag{2.32}$$

Properties (P1)-(P4) describe the behavior of the functions f_1, g_1 on the interval $[\bar{x} - c_2, \bar{x}+c_2]$. The following four useful lemmas provide some additional information about the derivatives f_1', g_1' on the interval $[\bar{x}-c_2, \bar{x}+c_2]$.

Lemma 2.2. *Assume that $x_1 \in (\bar{x}, \bar{x}+c_2]$ and $f(x_1) \geq f(\bar{x})$. Then there is $x_2 \in (\bar{x}, \bar{x}+c_2]$ such that*

$$f(x_2) \geq f(\bar{x}), \; f_1'(x_2) > f'(x_2) \geq 0. \tag{2.33}$$

Proof. If $f'(x_1) \geq 0$, then we set $x_2 = x_1$. By (2.10) and (2.19)

$$f_1'(x_2) = f'(x_2) + c_0\psi'(x_2) = f'(x_2) + 2c_0(x_2 - \bar{x}) > f'(x_2) = f'(x_1) \geq 0$$

and the assertion of the lemma holds.

Consider the case with

$$f'(x_1) < 0. \tag{2.34}$$

Set

$$\Omega = \{z \in (\bar{x}, x_1) : \ f'(y) < 0 \text{ for all } y \in [z, x_1]\}. \tag{2.35}$$

By (2.34) and continuity of f'

$$\Omega \neq \emptyset. \tag{2.36}$$

Set

$$x_* = \inf \Omega. \tag{2.37}$$

Clearly,

$$\bar{x} \leq x_* < x_1 \tag{2.38}$$

and f is strictly decreasing in $(x_*, x_1]$. Then

$$f(x_*) > f(x_1) \geq f(\bar{x}) \tag{2.39}$$

and

$$x_* > \bar{x}. \tag{2.40}$$

Clearly

$$f'(x) < 0 \text{ for all } x \in (x_*, x_1] \tag{2.41}$$

and

$$f'(x_*) \leq 0. \tag{2.42}$$

If $f'(x_*) < 0$ then there is $\delta > 0$ such that

$$x_* - \delta > \bar{x}, \ f'(z) < 0 \text{ for all } z \in [x_* - \delta, x_*]$$

and $x_* - \delta \in \Omega$, a contradiction. Therefore $f'(x_*) \geq 0$. Combined with (2.42) this implies that

$$f'(x_*) = 0. \tag{2.43}$$

By (2.38) and (2.40)

$$x_* \in (\bar{x}, \bar{x} + c_2]. \tag{2.44}$$

It follows from (2.10), (2.19), (2.43) and (2.44) that

$$f_1'(x_*) = f'(x_*) + c_0\psi'(x_*) = f'(x_*) + 2c_0(x_* - \bar{x}) > f'(x_*) = 0.$$

Thus the assertion of Lemma 2.2 holds with $x_2 = x_*$. □

Analogously to Lemma 2.2 we can prove the following auxiliary results.

Lemma 2.3. *Assume that $x_1 \in (\bar{x}, \bar{x} + c_2]$ and $g(x_1) \geq g(\bar{x})$. Then there is $x_2 \in (\bar{x}, \bar{x} + c_2]$ such that $g(x_2) \geq g(\bar{x})$ and $g_1'(x_2) > g'(x_2) > 0$.*

Lemma 2.4. *Assume that $x_1 \in [\bar{x} - c_2, \bar{x})$ and $f(x_1) \geq f(\bar{x})$. Then there is $x_2 \in [\bar{x} - c_2, \bar{x})$ such that $f(x_2) \geq f(\bar{x})$ and $f_1'(x_2) < f'(x_2) \leq 0$.*

Lemma 2.5. *Assume that $x_1 \in [\bar{x} - c_2, \bar{x})$ and $g(x_1) \geq g(\bar{x})$. Then there is $x_2 \in [\bar{x} - c_2, \bar{x})$ such that $g(x_2) \geq g(\bar{x})$ and $g_1'(x_2) < g'(x_2) \leq 0$.*

3 Construction of the Pair (f_2, g_2).

It is clear that in our construction of the pair (f_2, g_2) it is sufficient to consider only the cases with the properties (P1) and (P3).

Assume that the property (P1) holds. Now we define the functions $f_2, g_2 : R^1 \to R^1$. Set

$$f_2 = f_1 \tag{3.1}$$

and

$$\psi_0(x) = (x - \bar{x})\phi(c_2^{-1}(x - \bar{x})),\ x \in R^1. \tag{3.2}$$

By (2.3) and (3.2) for each $x \in R^1 \setminus (\bar{x} - c_2, \bar{x} + c_2)$

$$\psi_0(x) = 0. \tag{3.3}$$

By (2.3) and (3.2) for each $x \in [\bar{x} - 2^{-1}c_2, \bar{x} + 2^{-1}c_2]$

$$\psi_0(x) = (x - \bar{x}). \tag{3.4}$$

Choose a positive number d_1 such that

$$d_1||\psi_0||_1 < \epsilon/16 \tag{3.5}$$

and set

$$g_2(x) = g_1(x) + d_1\psi_0(x),\ x \in R^1. \tag{3.6}$$

Clearly $g_2 \in C^1(R^1)$. For each $x \in R^1\setminus(\bar{x}-c_2, \bar{x}+c_2)$ it follows from (3.1), (3.3) and (3.6) that $(f_2(x), g_2(x)) = (f_1(x), g_1(x))$. Since $(f_1, g_1) \in \mathcal{M}$ we conclude that $\max\{f_2(x), g_2(x)\} \to \infty$ as $|x| \to \infty$ and $(f_2, g_2) \in \mathcal{M}$. By (3.1), (3.5) and (3.6)

$$\tilde{d}((f_1, g_1), (f_2, g_2)) \le d_1||\psi_0||_1 < \epsilon/16.$$

Combined with (2.14) this inequality implies that

$$\tilde{d}((f, g), (f_2, g_2)) \le \epsilon/8. \tag{3.7}$$

Relations (2.12), (3.1), (3.4) and (3.6) imply that

$$g_2(\bar{x}) = g_1(\bar{x}) + d_1\psi_0(\bar{x}) = g_1(\bar{x}) = f_1(\bar{x}) = f_2(\bar{x}). \tag{3.8}$$

In view of (2.13), (3.1), (3.4) and (3.6)

$$g_2'(\bar{x}) = g_1'(\bar{x}) + d_1\psi_0'(\bar{x}) = d_1 = d_1 + f_1'(\bar{x}) = f_2'(\bar{x}) + d_1$$

and

$$g_2'(\bar{x}) \ne f_2'(\bar{x}). \tag{3.9}$$

We show that for all $x \in R^1 \setminus \{\bar{x}\}$,

$$\max\{f_2(x), g_2(x)\} > \max\{f_2(\bar{x}), g_2(\bar{x})\}. \tag{3.10}$$

Assume that $x \in R^1 \setminus \{\bar{x}\}$. If $|x - \bar{x}| \le c_2$, then it follows from (2.21), (2.8), (2.10), (3.1), property (P1), (2.12), (3.4), (3.6) that

$$\max\{f_2(x), g_2(x)\} \ge f_2(x) = f_1(x) = f(x) + c_0\psi(x) = f(x) + c_0|x - \bar{x}|^2$$
$$> f(x) \ge f(\bar{x}) = \max\{f_1(\bar{x}), g_1(\bar{x})\} = \max\{f_2(\bar{x}), g_2(\bar{x})\}.$$

If $|x-\bar{x}|>c_2$, then it follows from (2.15), (3.1), (3.3), (3.6) and (3.8) that

$$\max\{f_2(x),g_2(x)\}=\max\{f_1(x),g_1(x)+d_1\psi_0(x)\}$$
$$=\max\{f_1(x),g_1(x)\}>\max\{f_1(\bar{x}),g_1(\bar{x})\}=\max\{f_2(\bar{x}),g_2(\bar{x})\}.$$

Thus (3.10) is valid for all $x\in R^1\setminus\{\bar{x}\}$. By (3.8), (3.9) and (3.10)

$$(f_2,g_2)\in\mathcal{M}_0\setminus G. \tag{3.11}$$

Thus (f_2,g_2) satisfies (3.7) and (3.11).

Assume now that property (P3) holds. In this case the construction of (f_2,g_2) becomes more complicated. By Lemma 2.2 there is

$$x_1\in(\bar{x},\bar{x}+c_2] \tag{3.12}$$

such that

$$f(x_1)\geq f(\bar{x}),\ f_1'(x_1)>f'(x_1)\geq 0 \tag{3.13}$$

and by Lemma 2.5 there is

$$x_2\in[\bar{x}-c_2,\bar{x}) \tag{3.14}$$

such that

$$g(x_2)\geq g(\bar{x}),\ g_1'(x_2)<g'(x_2)\leq 0. \tag{3.15}$$

By (2.8), (2.10), (2.21), (3.12) and (3.14)

$$f_1(x_1)>f(x_1),\ g_1(x_2)>g(x_2). \tag{3.16}$$

We will define the function f_2 such that

$$f_2(x)=f_1(x)\text{ for each }x\in(-\infty,x_2]\cup[x_1,\infty),$$
$$f_2(x)=f_1(\bar{x})+\int_{\bar{x}}^{x}\xi_1(t)dt,\ x\in[\bar{x},x_1],$$
$$f_2(x)=f_1(x_2)+\int_{x_2}^{x}\xi_2(t)dt,\ x\in[x_2,\bar{x}],$$

where ξ_1 and ξ_2 are continuous functions defined below.

Let us construct continuous functions $\xi_1:[\bar{x},x_1]\to R^1$, $\xi_2:[x_2,\bar{x}]\to R^1$. Choose a positive number h such that

$$h<\min\{f_1'(x_1),\ -g_1'(x_2)\}/8 \tag{3.17}$$

and a positive number Δ_1 such that

$$6\Delta_1\leq x_1-\bar{x},\ f_1(x_1)-f_1(\bar{x})\geq 2\Delta_1[2h+f_1'(x_1)]. \tag{3.18}$$

Define a number H_1 as

$$H_1=2[(x_1-\bar{x}-3\Delta_1)^{-1}(f_1(x_1)-f_1(\bar{x})-\Delta_1[3/2h+2^{-1}f_1'(x_1)])]. \tag{3.19}$$

Clearly H_1 is well defined. By (3.13), (3.19) and (3.18)

$$H_1>0. \tag{3.20}$$

By (3.13), (3.16), (3.18) and (3.19)

$$\begin{aligned} H_1 &\le 2(x_1 - \bar{x} - 3\Delta_1)^{-1}(f_1(x_1) - f_1(\bar{x})) \\ &\le 2[f_1(x_1) - f_1(\bar{x})((x_1 - \bar{x})/2)^{-1}] \le 4\sup\{|f_1'(t)| :\ t \in [\bar{x}, x_1]\}. \end{aligned}$$

Combined with (2.13), (2.21) and (3.12) this implies that

$$H_1 \le 4\sup\{|f_1'(t)| :\ t \in [\bar{x}, x_1]\} \le \epsilon/16. \tag{3.21}$$

Set

$$\begin{aligned} \xi_1(t) =& h,\ t \in [\bar{x}, \bar{x} + \Delta_1], \qquad (3.22) \\ \xi_1(\bar{x} + \Delta_1 + t) =& h - t\Delta_1^{-1}h,\ t \in [0, \Delta_1]. \\ \xi_1(t) =& 2[t - (\bar{x} + 2\Delta_1)](x_1 - \bar{x} - 3\Delta_1)^{-1}H_1, \\ & t \in [\bar{x} + 2\Delta_1, 2^{-1}(\bar{x} + x_1 + \Delta_1)], \\ \xi_1(t) =& H_1 - (t - 2^{-1}(\bar{x} + x_1 + \Delta_1))(x_1 - \bar{x} - 3\Delta_1)^{-1}2H_1, \\ & t \in [2^{-1}(\bar{x} + x_1 + \Delta_1), x_1 - \Delta_1], \\ \xi_1(t) =& (t - (x_1 - \Delta_1))\Delta_1^{-1}f_1'(x_1),\ t \in [x_1 - \Delta_1, x_1]. \end{aligned}$$

Clearly ξ_1 is well defined, is continuous on $[\bar{x}, x_1]$,

$$\xi_1(t) \ge 0 \text{ for all } t \in [\bar{x}, x_1], \tag{3.23}$$

$$\xi_1(t) = 0 \text{ if and only if } t \in \{\bar{x} + 2\Delta_1, x_1 - \Delta_1\}. \tag{3.24}$$

It follows from (3.19) and (3.22) that

$$\begin{aligned} \int_{\bar{x}}^{x_1} \xi_1(t)dt &= \int_{\bar{x}}^{\bar{x}+\Delta_1} \xi_1(t)dt + \int_{\bar{x}+\Delta_1}^{\bar{x}+2\Delta_1} \xi_1(t)dt + \int_{\bar{x}+2\Delta_1}^{x_1-\Delta_1} \xi_1(t)dt + \int_{x_1-\Delta_1}^{x_1} \xi_1(t)dt \\ &= h\Delta_1 + h\Delta_1/2 + 2^{-1}(x_1 - \bar{x} - 3\Delta_1)H_1 + f_1'(x_1)\Delta_1/2 = f_1(x_1) - f_1(\bar{x}). \end{aligned} \tag{3.25}$$

Choose a positive number Δ_2 such that

$$16\Delta_2 < \bar{x} - x_2, \tag{3.26}$$

$$16\Delta_2(3h + |f_1'(x_2)|) < \epsilon(\bar{x} - x_2). \tag{3.27}$$

Set

$$H_2 = 2[f_1(\bar{x}) - f_1(x_2) - 2^{-1}\Delta_2(3h + f_1'(x_2))](\bar{x} - x_2 - 3\Delta_2)^{-1}. \tag{3.28}$$

Clearly H_2 is well defined. It follows from (3.26), (3.27) and (3.28), the mean value theorem that

$$\begin{aligned} |H_2| &\le [2|f_1(\bar{x}) - f_1(x_2)| + \Delta_2(3h + |f_1'(x_2)|](\bar{x} - x_2)^{-1})2 \\ &\le 2(\bar{x} - x_2)^{-1}[2|f_1(\bar{x}) - f_1(x_2)| + 16^{-1}\epsilon(\bar{x} - x_2)] \\ &= 8^{-1}\epsilon + 4|f_1(\bar{x}) - f_1(x_2)|(\bar{x} - x_2)^{-1} = 8^{-1}\epsilon + 4\sup\{|f_1'(t)| :\ t \in [x_2, \bar{x}]\}. \end{aligned}$$

Combined with (2.12), (2.21) and (3.14) this relation implies that

$$|H_2| \le 8^{-1}\epsilon + 16^{-1}\epsilon = 3\epsilon/16. \tag{3.29}$$

Set

$$\begin{aligned}
\xi_2(t) =& f_1'(x_2) - (t - x_2)\Delta_2^{-1} f_1'(x_2),\ t \in [x_2, x_2 + \Delta_2], \qquad (3.30)\\
\xi_2(t) =& (t - (x_2 + \Delta_2))2(\bar{x} - x_2 - 3\Delta_2)^{-1} H_2,\\
& t \in [x_2 + \Delta_2, (\bar{x} + x_2 - \Delta_2)/2],\\
\xi_2(t) =& H_2 - (t - (\bar{x} + x_2 - \Delta_2)/2)2(\bar{x} - x_2 - 3\Delta_2)^{-1} H_2,\\
& t \in [(\bar{x} + x_2 - \Delta_2)/2, \bar{x} - 2\Delta_2],\\
\xi_2(t) =& (t - (\bar{x} - 2\Delta_2))\Delta_2^{-1} h,\ [\bar{x} - 2\Delta_2, \bar{x} - \Delta_2],\\
\xi_2(t) =& h,\ t \in [\bar{x} - \Delta_2, \bar{x}].
\end{aligned}$$

Clearly, ξ_2 is well defined and continuous. It follows from (3.28) and (3.30) that

$$\begin{aligned}
\int_{x_2}^{\bar{x}} \xi_2(t)dt &= \int_{x_2}^{x_2+\Delta_2} \xi_2(t)dt + \int_{x_2+\Delta_2}^{\bar{x}-2\Delta_2} \xi_2(t)dt + \int_{\bar{x}-2\Delta_2}^{\bar{x}-\Delta_2} \xi_2(t)dt + \int_{\bar{x}-\Delta_2}^{\bar{x}} \xi_2(t)dt\\
&= f_1'(x_2)\Delta_2/2 + 2^{-1}(\bar{x} - x_2 - 3\Delta_2)H_2 + h\Delta_2/2 + h\Delta_2 = f_1(\bar{x}) - f_1(x_2).
\end{aligned} \tag{3.31}$$

Set

$$\begin{aligned}
f_2(x) =& f_1(x) \text{ for each } x \in (-\infty, x_2] \cup [x_1, \infty), \qquad (3.32)\\
f_2(x) =& f_1(\bar{x}) + \int_{\bar{x}}^{x} \xi_1(t)dt,\ x \in [\bar{x}, x_1],\\
f_2(x) =& f_1(x_2) + \int_{x_2}^{x} \xi_2(t)dt,\ x \in [x_2, \bar{x}].
\end{aligned}$$

By (3.25), (3.31), (3.32), f_2 is well defined. Relations (3.22), (3.30) and (3.32) imply that $f_2 \in C^1(R^1)$. By (3.23), (3.24) and (3.32)

$$f_2(x) > f_2(\bar{x}) = f_1(\bar{x}) \text{ for all } x \in (\bar{x}, x_1]. \tag{3.33}$$

If $x \in (-\infty, x_2] \cup [x_1, \infty)$ then

$$f_1(x) = f_2(x),\ f_1'(x) = f_2'(x). \tag{3.34}$$

Assume that

$$x \in (x_2, x_1) = (x_2, \bar{x}] \cup [\bar{x}, x_1). \tag{3.35}$$

We show that $|f_2'(x)| \le 3\epsilon/16$. Assume that

$$x \in [\bar{x}, x_1). \tag{3.36}$$

In view of (3.32) and (3.36) $f_2'(x) = \xi_1(x)$. By this equation, (2.13), (2.21), (3.12), (3.17), (3.21) and (3.22)

$$|f_2'(x)| = |\xi_1(x)| \le \max\{h, H_1, f_1'(x_1)\} \le \max\{\epsilon/16, f_1'(x_1)\} = \epsilon/16. \tag{3.37}$$

Assume that

$$x \in (x_2, \bar{x}). \tag{3.38}$$

Relations (3.32) and (3.38) imply that $f_2'(x) = \xi_2(x)$. By this equation, (2.13), (2.21), (3.12), (3.14), (3.17), (3.29), (3.30) and (3.38)

$$|f_2'(x)| \le \max\{|f_1'(x_2)|, |H_2|, h\} \le \max\{|f_1'(x_2)|, f_1'(x_1)/8, 3\epsilon/16\} \le 3\epsilon 16^{-1}. \tag{3.39}$$

Thus we have that for all $x \in (x_2, x_1)$

$$|f_2'(x)| \le 3\epsilon/16. \tag{3.40}$$

It follows from (2.13), (2.21), (3.12), (3.14) that for all $x \in (x_2, x_1)$

$$|f_1'(x)| \le \epsilon/64. \tag{3.41}$$

Relations (3.40) and (3.41) imply that for each $x \in (x_2, x_1)$

$$|f_1'(x) - f_2'(x)| \le 3\epsilon/16 + \epsilon/64 = 13\epsilon/64. \tag{3.42}$$

By (2.21), (3.12), (3.14), (3.32), (3.40) and (3.41) for each $x \in (x_1, x_2)$

$$\begin{aligned} |f_2(x) - f_1(x)| &\le |f_2(x) - f_2(\bar{x})| + |f_1(x) - f_1(\bar{x})| \\ &\le |x - \bar{x}|(\epsilon/64 + 3\epsilon/64) \le (\epsilon/4)c_2 \le \epsilon/64. \end{aligned} \tag{3.43}$$

In view of (3.34), (3.42) and (3.43) for each $x \in R^1$

$$|f_1'(x) - f_2'(x)| \le 13\epsilon/64, \ |f_1(x) - f_2(x)| \le \epsilon/64. \tag{3.44}$$

By (3.22), (3.30), (3.32)

$$f_2'(\bar{x}) = h. \tag{3.45}$$

Analogously we can construct $g_2 \in C^1(R^1)$ such that

$$g_2(x) > g_2(\bar{x}) = g_1(\bar{x}) \text{ for all } x \in [x_2, \bar{x}), \tag{3.46}$$

$$g_2(x) = g_1(x), \ g_1'(x) = g_2'(x) \text{ for each } x \in (-\infty, x_2] \cup [x_1, \infty),$$

$$|g_1'(x) - g_2'(x)|, \ |g_1(x) - g_2(x)| \le 13\epsilon/64 \text{ for all } x \in R^1$$

and that

$$g_2'(\bar{x}) = -h. \tag{3.47}$$

Clearly $(f_2, g_2) \in \mathcal{M}$ and $\tilde{d}((f_1, g_1), (f_2, g_2)) \le \epsilon/4$. Combined with (2.14) this implies that $\tilde{d}((f, g), (f_2, g_2)) \le \epsilon/2$. To complete the proof of the theorem it is sufficient to show that (3.11) holds. By (2.12), (3.32), (3.45), (3.46) and (3.47) in order to meet this goal it is enough to show that for each $x \in R^1 \setminus \{\bar{x}\}$ (3.10) is valid. If $x \in R^1 \setminus [x_2, x_1]$, then (3.10) follows from (2.15), (3.32), (3.46). If $x \in (\bar{x}, x_1]$ then by (3.33), (3.46), $\max\{f_2(x), g_2(x)\} \ge f_2(x) > f_2(\bar{x}) = g_2(\bar{x})$. If $x \in [x_2, \bar{x})$, then by (3.33), (3.46), $\max\{f_2(x), g_2(x)\} \ge g_2(x) > g_2(\bar{x}) = f_2(\bar{x})$. Thus (3.10) holds for any $x \in R^1 \setminus \{\bar{x}\}$. This completes the proof of the theorem.

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